



DEVELOPMENT OF ANDREWS-GARVAN-LIANG'S SELF-CONJUGATE S-PARTITIONS

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Abstract:

In 2013, Andrews, Garvan and Liang defined Self-conjugate S-partitions. In 2011, Andrews stated the definition of $spt(n)$. This paper shows how to find the Self-conjugate S-partitions of 4 and 5 respectively, and proves the Corollary-1 that is 'The number of Self-conjugate S-partitions counted according to the weight w is congruent to the total number of smallest parts in all the partitions of n modulo 2'. This paper shows how to generate the generating functions for $M_{sc}(n)$ in different ways. This paper shows how to find the number of partitions of n with an odd number of smallest parts and a total number of even (respectively odd) parts, and shows how to find the number of partitions of n in odd parts without gaps, and also shows how to generate the generating functions for $(A_o(n) - A_e(n))$ and $(L_1(n) - L_3(n))$ respectively. This paper shows how to prove the further three Corollaries with the help of examples, and shows how to prove the three Theorems by easy algebraic method.

Keywords:

Crank, congruent to involution, product notations, Self-conjugate, $spt(n)$, weight.

1. INTRODUCTION

We give some related definitions of $P(n)$, Self-conjugate S-partitions, $M_{sc}(n)$, $A_e(n)$, $A_o(n)$, $L_1(n)$, $L_3(n)$, $(x)_\infty$, $(x; x^2)_\infty$, $(x^2; x^2)_\infty$, and $spt(n)$. We give two tables of the Self-conjugate S-partitions of 4 and 5 respectively and introduce Corollary-1 in terms of $M_{sc}(n)$ and $spt(n)$. We discuss the various generating functions for $M_{sc}(n)$, $(A_o(n) - A_e(n))$ and $(L_1(n) - L_3(n))$ and give some tables of the partitions of n with an odd number of smallest parts and of the partitions of n in odd parts without gaps. We discuss the number of Self-conjugate S-partitions counted according the weight w , and give further three Corollaries in terms of $(A_o(n) - A_e(n))$, $(L_1(n) - L_3(n))$ and $M_{sc}(n)$. Finally we prove the three Theorems with the help of various generating functions.

2. SOME RELATED DEFINITIONS

$P(n)[7]$: The number of partitions of n like 4, 3+1, 2+2, 2+1+1, 1+1+1+1 $\therefore P(4) = 5$

Self-conjugate S-partitions [3,5]:

Let D denote the set of partitions into distinct parts and P denote the set of partitions. The set of vector partitions V is defined the Cartesian product



$V = D \times P \times P$. If S is the subset of V ,
 $S = \{ \vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V : 1 \leq (\pi_1) < \infty \text{ and } s(\pi_1) \leq \min\{s(\pi_2), s(\pi_3)\} \}$. Here $s(\pi)$ is the smallest part in the partition with the convention that $s(\emptyset) = \infty$ for the empty partition. We call the vector partitions in S simply S -partitions for $\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in S$, we define the weight $w(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$, the Crank $\left(\vec{\pi} \right) = \#(\pi_2) - \#(\pi_3)$ and $|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3|$, When $|\pi_j|$ is the sum of the parts of π_j and $\#(\pi_j)$ denotes the number of parts of (π_j)

The map $T: S \rightarrow S$ given by,

$T(\vec{\pi}) = T(\pi_1, \pi_2, \pi_3) = T(\pi_1, \pi_3, \pi_2)$ is natural involution. An S -partition $\vec{\pi} = (\pi_1, \pi_3, \pi_2)$ is a fixed point of T if and only if $\pi_2 = \pi_3$. We call these fixed points “Self-conjugate S -partitions”. The number of Self-conjugate S -partitions counted according to the weight w is denoted by $M_{sc}(n)$, So that,

$$M_{sc}(n) = \sum_{\substack{\vec{\pi} \in S, \\ |\vec{\pi}| = n \\ T(\vec{\pi}) = \vec{\pi}}} w(\vec{\pi})$$

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- $A_e(n)$: The number of partitions of n with an odd number of smallest parts and a total number of even parts.
- $A_o(n)$: The number of partitions of n with an odd number of smallest parts, and a total number of odd parts.
- $L_1(n)$: The number of partitions of n in odd parts with no gaps, and the largest part is congruent to 1 mod 4.
- $L_3(n)$: The number of partitions of n in odd parts with no gaps, and the largest part is congruent to 3 mod 4.



Product notations [6]:

$$(x)_{\infty} = (1-x)(1-x^2)(1-x^3)\dots$$

$$(x; x^2)_{\infty} = (1-x)(1-x^3)(1-x^5)\dots$$

$$(x^2; x^2)_{\infty} = (1-x^2)(1-x^4)(1-x^6)\dots$$

$$(-x; x)_{\infty} = (1+x)(1+x^2)(1+x^3)\dots$$

$spt(n)$: $spt(n)$ The total numbers of appearances of the smallest parts in all the partitions of n like,

n		$spt(n)$
1	$\dot{1}$	1
2	$\dot{2}, \dot{1}+\dot{1}$	3
3	$\dot{3}, \dot{2}+\dot{1}, \dot{1}+\dot{1}+\dot{1}$	5
4	$\dot{4}, \dot{3}+\dot{1}, \dot{2}+\dot{2}, \dot{2}+\dot{1}+\dot{1}, \dot{1}+\dot{1}+\dot{1}+\dot{1}$	10
.....		

3. THERE ARE TWO TABLES OF THE SELF-CONJUGATE S-PARTITIONS

OF 4 AND 5: We get;

Table-1

Self-conjugate S-partition of 4	Weight $w(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1 = (4, \phi, \phi)$	+ 1	0
$\vec{\pi}_2 = (3+1, \phi, \phi)$	- 1	0
	$\sum w(\vec{\pi}) = 0$	

$\therefore M_{sc}(4) = 0$. Here we have used ϕ to indicate the empty partition. Again;

Table-2

Self-conjugate S-partition of 5	Weight $w(\vec{\pi})$	Crank $(\vec{\pi})$
$\vec{\pi}_1 = (5, \phi, \phi)$	+ 1	0
$\vec{\pi}_2 = (1, 2, 2)$	+ 1	0
$\vec{\pi}_3 = (1, 1+1, 1+1)$	+ 1	0
$\vec{\pi}_4 = (1+4, \phi, \phi)$	- 1	0



$\vec{\pi}_5 = (2 + 3, \phi, \phi)$	- 1	0
$\vec{\pi}_6 = (2 + 1, 1, 1)$	-1	0
	$\sum w(\vec{\pi}) = 0$	

$\therefore M_{sc}(5) = \sum_{i=1}^6 w(\pi_i) = 1 + 1 + 1 - 1 - 1 - 1 = 0.$

Corollary-1 $M_{sc}(n) \equiv spt(n) \pmod{2}$

Proof: From above table we get;

$M_{sc}(1) = 1, M_{sc}(2) = 1, M_{sc}(3) = 1, \dots$

$M_{sc}(3) = 1 \equiv 1 \pmod{2}, spt(3) = 5 \equiv 1 \pmod{2}$

$M_{sc}(4) = 0 \equiv 0 \pmod{2}, spt(4) = 10 \equiv 0 \pmod{2}$

.....

We can conclude that,

$M_{sc}(n) \equiv spt(n) \pmod{2}$. Hence the Corollary.

4. NOW WE DESCRIBE THE GENERATING FUNCTIONS

The generation functions for $M_{sc}(n)$ are given by

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{x^n (x^{n+1}; x)_{\infty}}{(x^{2n}; x^2)_{\infty}} &= \frac{x(x^2; x)_{\infty}}{(x^2; x^2)_{\infty}} + \frac{x^2(x^3; x)_{\infty}}{(x^4; x^2)_{\infty}} + \frac{x^3(x^4; x)_{\infty}}{(x^6; x^2)_{\infty}} \\ &= (x - x^4 - x^6 + \dots) + (x^2 - x^5 - x^7 + \dots) + (x^3 - x^7 + x^4 + x^5 + \dots) \\ &= (x + x^2 + x^3 + 0.x^4 + 0.x^5 + \dots) \\ &= M_{sc}(1)x + M_{sc}(2)x^2 + M_{sc}(3)x^3 + M_{sc}(4)x^4 + M_{sc}(5)x^5 + \dots \\ &= \sum_{n=1}^{\infty} M_{sc}(n)x^n. \quad \therefore \sum_{n=1}^{\infty} M_{sc}(n)x^n = \sum_{n=1}^{\infty} \frac{x^n (x^{n+1}; x)_{\infty}}{(x^{2n}; x^2)_{\infty}}. \end{aligned}$$

Again, we get;

$$\begin{aligned} &\frac{1}{(-x; x)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{(1-x^n)} \\ &= \frac{1}{(1+x)(1+x^2)\dots} \left\{ \frac{x}{1-x} + \frac{x^2(1+x)}{1-x^2} + \frac{x^3(1+x)(1+x^2)}{1-x^3} + \dots \right\} \\ &= (1-x-x^3+x^4-x^5+\dots)(x+x^3+x^4+x^5+x^6+\dots+x^2+x^3+x^4+\dots) \\ &= (1-x-x^3+x^4-x^5)(x+2x^2+3x^3+4x^4+5x^5+\dots) \\ &= x+x^2+x^3+0.x^4+0.x^5+\dots \\ &= M_{sc}(1)x + M_{sc}(2)x^2 + M_{sc}(3)x^3 + M_{sc}(4)x^4 + M_{sc}(5)x^5 + \dots \end{aligned}$$



$$= \sum_{n=1}^{\infty} M_{sc}(n)x^n .$$

$$\therefore \sum_{n=1}^{\infty} M_{sc}(n)x^n = \frac{1}{(-x; x)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{(1-x^n)} .$$

Again we get; $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n^2}}{(x; x^2)_n} = \frac{x}{(1-x)} - \frac{x^4}{(1-x)(1-x^3)} + \dots$

$$= x(1+x+x^2+x^3+x^4+x^5) - x^4(1+x+x^2+\dots)(1+x^3+\dots)$$

$$= x+x^2+x^3+0.x^4+0.x^5+\dots$$

$$= M_{sc}(1)x + M_{sc}(2)x^2 + M_{sc}(3)x^3 + M_{sc}(4)x^4 + M_{sc}(5)x^5 + \dots$$

$$= \sum_{n=1}^{\infty} M_{sc}(n)x^n . \therefore \sum_{n=1}^{\infty} M_{sc}(n)x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n^2}}{(x; x^2)_n}$$

Also we get; $\sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_n} \{(x)_{2n} - (x)_{\infty}\}$

$$= \left\{ 1 - (1-x)(1-x^2) \dots \right\} + \left\{ \frac{(1-x)(1-x^2) - (1-x)(1-x^2) \dots}{1-x^2} \right\}$$

$$+ \left\{ \frac{(1-x)(1-x^2)(1-x^3)(1-x^4) - (1-x)(1-x^2)(1-x^3) \dots}{(1-x^2)(1-x^4)} \right\} + \dots$$

$$= x+x^2+x^3+0.x^4+0.x^5+\dots$$

$$= M_{sc}(1)x + M_{sc}(2)x^2 + M_{sc}(3)x^3 + M_{sc}(4)x^4 + M_{sc}(5)x^5 + \dots$$

$$= \sum_{n=1}^{\infty} M_{sc}(n)x^n . \therefore \sum_{n=1}^{\infty} M_{sc}(n)x^n = \sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_n} \{(x)_{2n} - (x)_{\infty}\} .$$

The partitions of 5 with an odd number of smallest parts are in the table

Table-3

Partition (π) of 5	#(π)
5	1
4+1	2
3+2	2
2+2+1	3
2+1+1+1	4
1+1+1+1+1	5

We see that $A_o(5)=3$, and $A_e(5)=3$. $\therefore A_o(5) - A_e(5)=0$.

The partitions of 6 with an odd number of smallest parts are in the table

Table-4

Partition (π) of 6	#(π)
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6	1
5+1	2
4+2	2
3+2+1	3
3+1+1+1	4
2+2+2	3

We see that $A_0(6)=3, A_e(6)=3. \therefore A_0(6) - A_e(6)=0,$

Similar, we get;

$$A_0(1)- A_e(1)=1- 0=1$$

$$A_0(2)- A_e(2)=1- 0=1$$

$$A_0(3)- A_e(3)=1- 0=1$$

$$A_0(4)- A_e(4)=1-0=1$$

The generating function for $(A_0(n) - A_e(n))$ is given by

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{x^n}{(1-x^{2n})} \cdot \frac{1}{(-x^{n+1}; x)_{\infty}} \\ &= \frac{x}{(1-x^2)} \cdot \frac{1}{(1+x^2)(1+x^3)\dots} + \frac{x^2}{(1-x^4)(1+x^3)(1+x^4)\dots} + \frac{x^3}{(1-x^6)(1+x^4)(1+x^5)\dots} + \dots \\ &= x + x^2 + x^3 + 0.x^4 + 0.x^5 + 0.x^6 + -x^7 + \dots \\ &= \{A_0(1) - A_e(1)\}x + \{A_0(2) - A_e(2)\}x^2 + \{A_0(3) - A_e(3)\}x^3 + \dots \\ &= \sum_{n=1}^{\infty} \{A_0(n) - A_e(n)\}x^n. \end{aligned}$$

Corollary-2: $A_0(n)- A_e(n) = M_{sc}(n)$

Proof: We get, the generating function for $(A_0(n)- A_e(n))$ is

$$\begin{aligned} &\sum_{n=1}^{\infty} \{A_0(n) - A_e(n)\} x^n = \sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}} \cdot \frac{1}{(-x^{n+1}; x)_{\infty}} \\ &= \frac{x}{(1-x)(1+x^2)(1+x^4)\dots} + \frac{x^2}{(1-x^4)(1+x^3)(1+x^4)\dots} + \frac{x^3}{(1-x^6)(1+x^4)(1+x^5)\dots} + \dots \\ &= \frac{x}{(1-x)(1+x)(1+x^2)\dots} + \frac{x^2(1+x)}{(1-x^2)(1+x)(1+x^2)\dots} + \frac{x^3(1+x)(1+x^2)}{(1-x^3)(1+x)(1+x^2)\dots} + \dots \\ &= \frac{1}{(1+x)(1+x^2)(1+x^3)\dots} \left\{ \frac{x}{1-x} + \frac{x^2(1+x)}{1-x^2} + \frac{x^3(1+x)(1+x^2)}{1-x^3} + \dots \right\} \\ &= \frac{1}{(-x; x)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{1-x^n} = \sum_{n=1}^{\infty} M_{sc}(n)x^n \end{aligned}$$

Equation the co-efficient of x^n from both sides we get;



$A_0(n) - A_e(n) = M_{sc}(n)$. Hence the Corollary .
 The partitions of 5 in odd parts with no gaps are in the table

Table-5

Partition (π) of 5	Largest part
3+1+1	3
1+1+1+1+1	1

We see that $L_1(5) = 1, L_3(5) = 1. \therefore L_1(5) - L_3(5) = 1 - 1 = 0$.

The partitions of 6 in odd parts with no gaps are in the table

Table-6

Partition (π) of 6	Largest part
3+1+1+1	3
1+1+1+1+1+1	1

We set that $L_1(6) = 1, L_3(6) = 1. \therefore L_1(6) - L_3(6) = 1 - 1 = 0$.

Similarly we get;

$$L_1(1) - L_3(1) = 1 - 0 = 1$$

$$L_1(2) - L_3(2) = 1 - 0 = 1$$

$$L_1(3) - L_3(3) = 1 - 0 = 1$$

$$L_1(4) - L_3(4) = 1 - 1 = 0$$

.....
 The generating function for $(L_1(n) - L_3(n))$ is given by

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n^2}}{\left(x; x^2\right)_n} &= \frac{x}{1-x} - \frac{x^4}{(1-x)(1-x^3)} + \dots \\ &= x(1+x+x^2+x^3+x^4+x^5+\dots) - x^4(1+x+x^2+\dots)(1+x^3+\dots) + \dots \\ &= x+x^2+x^3+0.x^4+0.x^5+0.x^6-x^7+\dots \\ &= \{L_1(1) - L_3(1)\}x + \{L_1(2) - L_3(2)\}x^2 + \{L_1(3) - L_3(3)\}x^3 + \dots \\ &= \sum_{n=1}^{\infty} L_1(n) - L_3(n)x^n. \end{aligned}$$

Corollary-3 $A_0(n) - A_e(n) = L_1(n) - L_3(n) = M_{sc}(n)$

Proof: From above we get; $A_0(1) - A_e(1) = 1 = L_1(1) - L_3(1)$ and $M_{sc}(1) = 1$

$A_0(2) - A_e(2) = 1 = L_1(2) - L_3(2)$ and $M_{sc}(2) = 1$

$A_0(3) - A_e(3) = 1 = L_1(3) - L_3(3)$ and $M_{sc}(3) = 1$

.....
 We can conclude that,

$A_0(n) - A_e(n) = L_1(n) - L_3(n) = M_{sc}(n)$. Hence the Corollary .



Corollary-4 $A_0(n) - A_e(n) = L_1(n) - L_3(n) \equiv \text{spt}(n) \pmod{2}$

$$A_0(1) - A_e(1) = L_1(1) - L_3(1) = 1 \equiv 1 \pmod{2} \text{ and } \text{spt}(1) = 1 \equiv 1 \pmod{2}$$

$$A_0(4) - A_e(4) = L_1(4) - L_3(4) = 0 \equiv 0 \pmod{2} \text{ and } \text{spt}(4) = 10 \equiv 1 \pmod{2}$$

$$A_0(6) - A_e(6) = L_1(6) - L_3(6) = 0 \equiv 0 \pmod{2} \text{ and } \text{spt}(6) = 26 \equiv 0 \pmod{2}$$

.....
We can conclude that,

$A_0(n) - A_e(n) = L_1(n) - L_3(n) \equiv \text{spt}(n) \pmod{2}$. Hence the Corollary.

Theorem 1:
$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}} \frac{1}{(-x^{n+1}; x)_{\infty}} = \frac{1}{(-x; x)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{(1-x^n)}$$

Proof: Left hand side =
$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^{2n}} \cdot \frac{1}{(-x^{n+1}; x)_{\infty}}$$

$$= \frac{x}{(1-x^2)} \cdot \frac{1}{(1+x^2)(1+x^3)\dots} + \frac{x^2}{(1-x^4)(1+x^3)(1+x^4)\dots} + \frac{x^3}{(1-x^6)(1+x^4)(1+x^5)\dots} + \dots$$

$$= \frac{x}{(1-x)(1+x)(1+x^2)\dots} + \frac{x^2(1+x)}{(1-x^2)(1+x)(1+x^2)\dots} + \frac{x^3(1+x)(1+x^2)}{(1-x^3)(1+x)(1+x^2)\dots} + \dots$$

$$= \frac{1}{(1+x)(1+x^2)(1+x^3)\dots} \left\{ \frac{x}{1-x} + \frac{x^2(1+x)}{1-x^2} + \frac{x^3(1+x)(1+x^2)}{1-x^3} + \dots \right\}$$

$$= \frac{1}{(-x; x)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{1-x^n} = \text{Right hand side. Hence the Theorem.}$$

Theorem 2:
$$\frac{1}{(-x; x)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{(1-x^n)} = \sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_n} \{ (x)_{2n} - (x)_{\infty} \}$$

Proof: We get;
$$\frac{1}{(x^2; x^2)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{(1-x^n)}$$

$$= \frac{1}{(1-x^2)(1-x^4)\dots} \left[\frac{x}{1-x} + \frac{x^2(1+x)}{1-x^2} + \frac{x^2(1+x)(1+x^2)}{1-x^3} + \dots \right]$$

$$= \frac{1}{(1-x^2)(1-x^4)\dots} \left[\frac{x}{1-x} + \frac{x^2(1-x^2)}{(1-x)(1-x^2)} + \frac{x^3(1-x^2)(1-x^4)}{(1-x)(1-x^2)(1-x^3)} + \dots \right]$$

$$= \frac{x}{(1-x)(1-x^2)(1-x^4)\dots} + \frac{x^2}{(1-x)(1-x^2)(1-x^4)\dots} + \frac{x^3}{(1-x)(1-x^2)(1-x^3)(1-x^6)\dots} + \dots$$



$$\begin{aligned}
 &= \sum_{n=1}^{\infty} \frac{x^n}{(x)_n (x^{2n}; x^2)_{\infty}} = \sum_{n=1}^{\infty} \frac{x^n}{(x)_n} \sum_{k=0}^{\infty} \frac{x^{2nk}}{(x^{2n}; x^2)_k} \quad [[1], P.19] \\
 &= \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} \sum_{n=1}^{\infty} \frac{x^{n(2k+1)}}{(x)_n} = \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} \left[\frac{x^{2k+1}}{(1-x)} + \frac{x^{2(2k+1)}}{(1-x)(1-x^2)} + \dots \right] \\
 &= \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} [x^{2k+1} + x^{2k+2} + x^{2k+3} \dots] = \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} [1 + x^{2k+1} + x^{2k+2} + x^{2k+3} \dots - 1] \\
 &= \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} \left[\frac{1}{(1-x^{2k+1})(1-x^{2k+2}) \dots} - 1 \right]
 \end{aligned}$$

Multiplying both sides by $(x)_{\infty}$ We have;

$$\begin{aligned}
 \frac{(x)_{\infty}}{(x^2; x^2)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{(1-x^n)} &= \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} \left[\frac{(x)_{\infty}}{(1-x^{2k+1})(1-x^{2k+2}) \dots} - (x)_{\infty} \right] \\
 \frac{1}{(-x; x)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{(1-x^n)} &= \sum_{k=0}^{\infty} \frac{1}{(x^2; x^2)_k} [(x)_{2k} - (x)_{\infty}]
 \end{aligned}$$

[Since

$$\frac{(x)_{\infty}}{(x^2; x^2)_{\infty}} = \frac{(1-x)(1-x^2)(1-x^3)(1-x^4) \dots}{(1-x^2)(1-x^4)(1-x^6) \dots} = \frac{1}{(1-x)(1-x^2)(1-x^3) \dots} = \frac{1}{(-x; x)_{\infty}}$$

And $\frac{(1-x)(1-x^2) \dots}{(1-x)(1-x^2) \dots} + \frac{(1-x)(1-x^2)(1-x^3) \dots}{(1-x^3)(1-x^4) \dots} + \dots$

$$= 1 + (1-x)(1-x^2) + (1-x)(1-x^2)(1-x^3)(1-x^4) + \dots = \sum_{k=0}^{\infty} (x)_{2k}]$$

$$\frac{1}{(-x; x)_{\infty}} \sum_{n=1}^{\infty} \frac{x^n (-x; x)_{n-1}}{(1-x^n)} = \sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_n} \{(x)_{2n} - (x)_{\infty}\}.$$

∴ Left hand side = $\sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_n} \{(x)_{2n} - (x)_{\infty}\}$ = Right hand side. Hence the Theorem.

Theorem 3: $\sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_n} \{(x)_{2n} - (x)_{\infty}\} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n^2}}{(x; x^2)_n}$

Proof: Left hand side = $\sum_{n=0}^{\infty} \frac{1}{(x^2; x^2)_n} [(x; x)_{2n} - (x; x)_{\infty}]$

$$= \sum_{n=0}^{\infty} \left\{ \frac{(x; x)_{2n}}{(x^2; x^2)_n} - \frac{(x; x)_{\infty}}{(x^2; x^2)_n} \right\} = \sum_{n=0}^{\infty} \left\{ (x; x^2)_n - \frac{(x; x)_{\infty}}{(x^2; x^2)_n} \right\}$$



$$\begin{aligned}
 & [\text{Since } \frac{(1-x)(1-x^2)(1-x^3)(1-x^4)\dots}{(1-x^2)(1-x^4)\dots} = \sum_{n=0}^{\infty} \left[(x; x^2)_n - (x; x^2)_{\infty} + (x; x^2)_{\infty} - \frac{(x; x)_{\infty}}{(x^2; x^2)_n} \right] \\
 & = (1-x)(1-x^3)\dots\dots\dots] \\
 & = \sum_{n=0}^{\infty} [(x; x^2)_n - (x; x^2)_{\infty}] + \sum_{n=0}^{\infty} [(x; x^2)_{\infty} - \frac{(x; x)_{\infty}}{(x^2; x^2)_n}] \\
 & = -\sum_{n=1}^{\infty} \frac{(-1)^n x^{n^2}}{(x^2; x^2)_n} - (x; x^2)_{\infty} \sum_{n=1}^{\infty} \frac{x^{n^2}}{1-x^{2n}} + \sum_{n=0}^{\infty} \left\{ (x; x^2)_{\infty} - \frac{(x)_{\infty}}{(x; x^2)_n} \right\} [4] \\
 & \qquad \qquad \qquad \text{with } x \rightarrow x^2, a \rightarrow 0 \text{ and } t = x \\
 & = -\sum_{n=1}^{\infty} \frac{(-1)^n x^{n^2}}{(x^2; x^2)_n} - (x; x^2)_{\infty} \sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{2n}} + \sum_{n=0}^{\infty} \left\{ \frac{(x; x)_{\infty}}{(x^2; x^2)_{\infty}} - \frac{(x; x)_{\infty}}{(x^2; x^2)_n} \right\} \\
 & \qquad \qquad \qquad [\text{Since } (x; x^2)_{\infty} = (1-x)(1-x^3)\dots = \frac{(1-x)(1-x^2)\dots}{(1-x^2)(1-x^4)\dots} = \frac{(x; x)_{\infty}}{(x^2; x^2)_{\infty}}] \\
 & = -\sum_{n=1}^{\infty} \frac{(-1)^n x^{n^2}}{(x; x^2)_n} - \frac{(x; x)_{\infty}}{(x^2; x^2)_{\infty}} \sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{2n}} + (x; x)_{\infty} \sum_{n=0}^{\infty} \left\{ \frac{1}{(x^2; x^2)_{\infty}} - \frac{1}{(x^2; x^2)_n} \right\} \\
 & = -\sum_{n=1}^{\infty} \frac{(-1)^n x^{n^2}}{(x; x^2)_n} - \frac{(x; x)_{\infty}}{(x^2; x^2)_{\infty}} \sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{2n}} + (x; x)_{\infty} \frac{1}{(x^2; x^2)_{\infty}} \sum_{n=1}^{\infty} \frac{x^{2n}}{1-x^{2n}} [4] \\
 & \qquad \qquad \qquad \text{with } x \rightarrow x^2, a = b = c = 0 \\
 & = -\sum_{n=1}^{\infty} \frac{(-1)^n x^{n^2}}{(x; x^2)_n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n^2}}{(x; x^2)_n} = \text{Right hand side. Hence the Theorem.}
 \end{aligned}$$

5. CONCLUSION

We have shown all Self-conjugate S-partitions of 4 and 5 respectively, and have satisfied the Corollary-1 $M_{sc}(n) \equiv spt(n) \pmod{2}$ for any positive integral value of n. We have shown the partitions of n with an odd number of smallest parts for n = 5 and 6, and also have found the partitions of n in odd parts with no gaps for any positive integral value of n. We have introduced further three Corollaries in terms of $M_{sc}(n)$ and $spt(n)$ respectively and have proved three Theorems of Self-conjugate S-partitions with the help of various generating functions.

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