

## A MODEL OF PLANETARY GEAR TRANSMISSION

*Ljiljana Veljović<sup>1</sup>, Vera Nikolić-Stanojević<sup>2</sup>, Dragan Milosavljević<sup>3</sup>, Gordana Bogdanović<sup>4</sup>, Aleksandar Radaković<sup>5</sup>*

UDC: 531.391:62-342

### INTRODUCTION

Because of its compactness, and high transmission ratios, planetary gears have a great application in modern engineering systems as a replacement for the conventional manual transmission complex. Planetary gears have substantial advantages over parallel shaft drives, including large torque to weight ratio and high efficiency to transfer power. They are widely used for transmissions of automobiles, aircrafts, heavy machinery and marine vehicles. In most high precision reduction of an industrial robot, planetary gears are used in the first stage of gear reducer. Despite the advantages of planetary gears, noise and vibration have been major concerns in their applications.

Generally, a single stage planetary gear-set consists of a sun gear, a ring gear, a carrier, and several planets. Planetary gears possess complicate structure. When modelling planetary gears, both the inertia and the supporting condition should be considered for the sun gear, the ring gear, the carrier and the planets. For this reason, the dynamic analysis of a planetary gear is difficult. Dynamic loads cause damage to the gears, bearings and other elements of the transmission. Precise study of the dynamic behaviour of planetary gear is often a difficult mathematical problem, because there are no adequate models.

Dynamic analyses of the planetary gear system have been investigated by many researchers. First papers on the dynamic behaviour of gears in use, contain a great simplification, such as that all changes have linear character. Experimental studies have shown that this approach is not realistic. The dynamic behaviour of gears is influenced by many factors that cannot be described by linear relationships. The simplest models are found in a number of textbooks used in education in this field. So, the teeth in meshing action can be modelled as an oscillatory system [7], etc. This model consists of concentrated masses connected with elastic and dump element. Each mass represent one gear. For different analysis purposes, there are several modelling choices such as a simple dynamic factor model, compliance tooth model, torsion model, and geared rotor dynamic model, [8]. In order to obtain better results, it is possible to model the elastic element as a nonlinear spring. Natural frequencies and vibration modes are critical parameters that are essential for almost all dynamic investigations. Those parameters may be calculated by using the free vibration

---

<sup>1</sup> *Ljiljana Veljović, Assistant professor, Faculty of Engineering, University of Kragujevac, Sestre Janjić 6, Kragujevac, veljovicljilja@yahoo.co.uk*

<sup>2</sup> *Vera Nikolic-Stanojevic, Professor emeritus, State University of Novi Pazar, Vuka Karadzica bb, 36300 Novi Pazar, Serbia, veranikolic1@gmail.com*

<sup>3</sup> *Dragan Milosavljevic, Full professor, Faculty of Engineering, University of Kragujevac, Sestre Janjić 6, Kragujevac, dmilos@kg.ac.rs*

<sup>4</sup> *Gordana Bogdanovic, Assistant professor, Faculty of Engineering, University of Kragujevac, Sestre Janjić 6, Kragujevac, gocab@kg.ac.rs*

<sup>5</sup> *Aleksandar Radakovic, Teaching assistant, State University of Novi Pazar, Vuka Karadzica bb, 36300 Novi Pazar, Serbia, sale@1311gmailt.com*

analysis. The free vibration properties are very useful for further analyses of planetary gear dynamics, including eigen-sensitivity to design parameters, natural frequency veering, planet mesh phasing, and parametric instabilities from mesh stiffness variations [9].

The dynamic characteristics of the planetary model considering nonlinear time-variable parameters are studied extended it to a three-dimensional model and examined the influence of planet phasing on dynamic response [6]. The planet gears in planetary gear system are fixed at the carrier, so the motion of planets is considered along with the dynamics characteristics of the carrier. In other words, the motion of the planet gears depends on a translation and rotation of the carrier as well as the deflection of the planet gear bearings. Motion of the carrier is considered with the deflections due to the bearings of the planetary gears, because the rigid-body motion of the carrier influences the mesh stiffness between the sun, planetary and ring gears. The revolutions of planets due to the carrier rotation are analysed using polar coordinates [1]. The equations of motion which considered a gyroscopic effect with respect to a rotation are derived also. In recent years, many researchers have used that dynamic model to analyse a planetary gear system.

In the latest research, light fractional order coupling element, is used to describe the dynamic behaviour of gears and set of constitutive relationships, so the fractional calculus can be successfully applied to obtain results [4].

A dynamic model of a planetary gear system in this paper represents a new dynamic model of the fractional order dynamics of the planetary gears with four degrees of freedom. Based on this model, the equations of motion are derived by using Lagrange's equation. The analytical expressions for the corresponding fractional order modes like one frequency eigen vibration modes are obtained.

Applying the Math CAD time integration method to the derived equations, time responses for a planetary gear are calculated. From the computed responses, the dynamic characteristics of the planetary gear system are analysed.

## EQUATION OF MOTIONS

Consider the motions of the sun gear, the ring gear, the carrier and the planets gears. It is assumed in this paper that all components of the planetary gear system have the planar motion which is described by translations and rotations.

This model consists of reduced masses of the gear with elastic and damping connections [7]. Contact between two teeth is constructed by standard light element with constitutive stress – strain state relations which can be expressed by fractional order derivatives. In the paper [2] standard light coupling elements of negligible mass in the form of axially stressed rod without bending, with the ability to resist deformation under static and dynamic conditions is analysed in details.

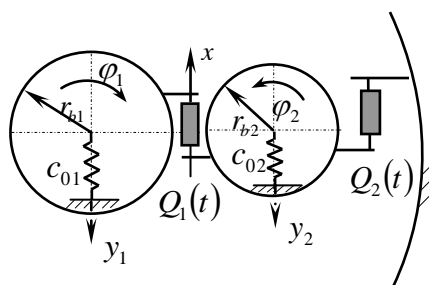


Figure 1 The model of the planetary gear with viscoelastic fractional order tooth coupling

The motion of the sun gear and the ring gear is given by translations that is expressed as  $y_i$ ,  $i=1,2$  and rotations that is expressed as  $\varphi_i$ ,  $i=1,2$ , (Figure 1). The kinetic energy  $E_K$  of the planetary stage can be written in a form

$$E_K = E_{K1} + E_{K2}$$

The kinetic energy for the system is represented by:

$$E_K = \frac{1}{2} m_1 \dot{y}_1^2 + \frac{1}{2} J_1 \dot{\varphi}_1^2 + \frac{1}{2} m_2 \dot{y}_2^2 + \frac{1}{2} J_2 \dot{\varphi}_2^2 \tag{1}$$

Where  $m_1$  is mass of the sun gear and  $m_2$  is mass of ring gear,  $J_1$  is mass moment of the inertia of sun gear,  $J_2$  is mass moment of the inertia of ring gear,  $\dot{y}_1$  is velocity of mass center of the sun gear and  $\dot{y}_2$  is velocity of ring gears mass center;  $\dot{\varphi}_1$  is angular velocity of the sun gear and  $\dot{\varphi}_2$  is angular velocity of the ring gear.

Sun gear is supported with bearing which is modelled as linear spring  $c_1$ , but the meshes of sun gear-planet gear and ring gear-planet gear are described by standard light fractional element with stiffness  $c_{01}$  and  $c_{02}$ . Thus the potential energies of the bearings are:

$$E_p = \frac{1}{2} c_{01} y_1^2 + \frac{1}{2} c_{02} y_2^2 \tag{2}$$

The potential energy due to gear mesh between the sun gear and the planet gear is

$$E_p = \frac{1}{2} c_1 [(y_2 + r_{b2} \varphi_2) - (y_1 + r_{b1} \varphi_1)]^2 \tag{3}$$

The potential energy due to gear mesh between the ring gear and the planet gear is

$$E_p = \frac{1}{2} c_2 (y_2 - r_{b2} \varphi_2)^2 \tag{4}$$

The equations of motion for the planetary gear are derived from Lagrange's equation given by

$$\frac{d}{dt} \left( \frac{\partial E_K}{\partial \dot{q}_j} \right) - \frac{\partial E_K}{\partial q_j} + \frac{\partial E_p}{\partial q_j} = Q_j^* - \frac{\partial \Phi}{\partial \dot{q}_j}, \quad j = 1, 2, \dots, 4 \tag{5}$$

Where  $q_j$  are generalized coordinates (for the given system generalized coordinates are:  $y_1, y_2, \varphi_1$  and  $\varphi_2$ ).

Light standard creep constraint element between sun gear and planet gear is strained for  $x_1 = y_2 - y_1 + r_{b2} \varphi_2 - r_{b1} \varphi_1$  and light standard creep constraint element between planet gear and ring gear is strained for  $x_2 = y_2 - r_{b2} \varphi_2$ .

So, due to the constitutive stress-strain relation of the standard light fractional order coupling elements the restitution forces as a function of elongation of elements are

$$Q_1^* = -c_1 x_1 - c_\alpha D_\alpha' [x_1] = -c_1 [(y_2 + r_{b2} \varphi_2) - (y_1 + r_{b1} \varphi_1)] - c_\alpha D_\alpha' [(y_2 + r_{b2} \varphi_2) - (y_1 + r_{b1} \varphi_1)] \tag{6}$$

$$Q_2^* = -c_2 x_2 - c_\alpha D'_\alpha [x_2] = -c_2 [y_2 - r_{b2} \varphi_2] - c_\alpha D'_\alpha [y_2 - r_{b2} \varphi_2]$$

Substitution equations (6) into equation (1) the Lagrange equations of motion can be expressed as:

$$\begin{aligned} m_1 \ddot{y}_1 + c_{01} y_1 + c_1 [(y_1 + r_{b1} \varphi_1) - (y_2 + r_{b2} \varphi_2)] &= c_\alpha D'_\alpha [(y_2 + r_{b2} \varphi_2) - (y_1 + r_{b1} \varphi_1)] \\ J_1 \ddot{\varphi}_1 + c_1 [(y_1 + r_{b1} \varphi_1) - (y_2 + r_{b2} \varphi_2)] r_{b1} &= c_\alpha D'_\alpha r_{b1} [(y_2 + r_{b2} \varphi_2) - (y_1 + r_{b1} \varphi_1)] \\ m_2 \ddot{y}_2 + c_{02} y_2 + c_1 [(y_2 + r_{b2} \varphi_2) - (y_1 + r_{b1} \varphi_1)] + c_2 [(y_2 - r_{b2} \varphi_2)] &= c_\alpha D'_\alpha [(y_1 + r_{b1} \varphi_1) - (y_2 + r_{b2} \varphi_2)] - \\ - c_\alpha D'_\alpha [(y_2 - r_{b2} \varphi_2)] & \quad (7) \\ J_2 \ddot{\varphi}_2 + c_1 [(y_2 + r_{b2} \varphi_2) - (y_1 + r_{b1} \varphi_1)] r_{b2} + c_2 [r_{b2} \varphi_2 - y_2] r_{b2} &= c_\alpha D'_\alpha [(y_1 + r_{b1} \varphi_1) - (y_2 + r_{b2} \varphi_2)] + \\ + c_\alpha D'_\alpha [(y_2 - r_{b2} \varphi_2)] & \end{aligned}$$

The matrix form of equations is well known as:

$$\mathbf{M} \{\ddot{q}\} + \mathbf{C} \{q\} = Q_j^* - \frac{\partial \Phi}{\partial \dot{q}_j}, \quad j = 1, 2, \dots, 4 \quad (8)$$

with matrix  $\mathbf{M}$  as diagonal inertia matrix in a form:

$$\mathbf{M} = \begin{bmatrix} m_1 & & & \\ & J_1 & & \\ & & m_2 & \\ & & & J_2 \end{bmatrix} \quad (9)$$

and the matrix  $\mathbf{C}$  as stiffness matrix that is in a form:

$$\mathbf{C} = \begin{bmatrix} c_{01} + c_1 & c_1 r_{b1} & -c_1 & -c_1 r_{b2} \\ c_1 r_{b1} & c_1 r_{b1}^2 & -c_1 r_{b1} & -c_1 r_{b1} r_{b2} \\ -c_1 & -c_1 r_{b1} & c_{02} + c_1 + c_2 & (c_1 - c_2) r_{b2} \\ -c_1 r_{b2} & -c_1 r_{b1} r_{b2} & (c_1 - c_2) r_{b2} & (c_1 + c_2) r_{b2}^2 \end{bmatrix} \quad (10)$$

## MODAL ANALYSIS

### *Eigenvalue problem*

The proposed solutions are in the form:

$$\{q\} = \{A\} \cos(\omega t + \varepsilon) \quad (11)$$

and it can be written as:

$$(\mathbf{C} - \lambda \mathbf{M}) \{q\} = 0 \quad (12)$$

The matrix on the left side is singular in aim to obtain non-trivial solutions. It follows that the determinant of the matrix must be equal to 0, so:

$$\begin{bmatrix} (c_{01} + c_1) - \lambda m_1 & c_1 r_{b1} & -c_1 & -c_1 r_{b2} \\ c_1 r_{b1} & (c_1 r_{b1}^2) - \lambda J_1 & -c_1 r_{b1} & -c_1 r_{b1} r_{b2} \\ -c_1 & -c_1 r_{b1} & (c_{02} + c_1 + c_2) - \lambda m_2 & (c_1 - c_2) r_{b2} \\ -c_1 r_{b2} & -c_1 r_{b1} r_{b2} & (c_1 - c_2) r_{b2} & ((c_1 + c_2) r_{b2}^2) - \lambda J_2 \end{bmatrix} = 0 \quad (13)$$

Solving this determinant four eigen circular frequencies  $\omega_j = \sqrt{\lambda_j}, j = 1,2,3,4$ , can be obtained.

The solution of basic linear differential equation is:

$$\{q(t)\} = \mathbf{R}\{C_s \cos(\omega_s t + \varepsilon_s)\} \quad (14)$$

where  $\mathbf{R}$  is modal matrix defined by the corresponding cofactors and  $\xi_s = C_s \cos(\omega_s t + \varepsilon_s), s = 1,2,3,4$  are main coordinates of the linear system. The system of the fractional differential equations (7) can be transformed in the form [2]:

$$\ddot{\xi}_s + \omega_s^2 \xi_s = -\omega_{\alpha s}^2 D'_\alpha [\xi_s], s = 1,2,3,4 \quad (15)$$

Using the approach presented in [2] the solution of the basis system (7) can be expressed in the following form:

$$\begin{aligned} \xi_s(t) = & \xi_{0s} \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha s}^{2k} t^{2k} \sum_{j=0}^k \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha s}^{2j} t^{-\alpha j}}{\omega_s^{2j} \Gamma(2k + 1 - \alpha j)} + \\ & + \dot{\xi}_{0s} \sum_{k=0}^{\infty} (-1)^k \omega_{\alpha s}^{2k} t^{2k+1} \sum_{j=0}^k \binom{k}{j} \frac{(\mp 1)^j \omega_{\alpha s}^{-2j} t^{-\alpha j}}{\omega_s^{2j} \Gamma(2k + 2 - \alpha j)}, s = 1,2,3,4 \end{aligned} \quad (16)$$

Where  $\xi_s(0) = \xi_{0s}$  and  $\dot{\xi}_s(0) = \dot{\xi}_{0s}$  are initial values of main coordinates defined by initial conditions.

**Computation observation**

Eigen solutions of a sample system with four degrees of freedom are evaluated numerically to expose the modal properties.

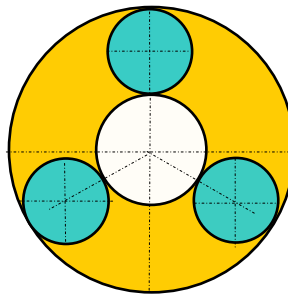


Figure 2 The initial position of planet gear

For the Sun base radius  $r_{b1}=24$  mm, Planet base radius  $r_{b2}=16$  mm, Radial bearing stiffnesses  $c_{01} = 0.5 \times 10^9$  N/m and  $c_{02} = 0.5 \times 10^9$  N/m, Stiffness of teeth  $c_1 = 2.91 \times 10^8$  N/m and  $c_2 = 1.81 \times 10^8$  N/m, Mass  $m_1 = 0.3$  kg,  $m_2 = 0.3$  kg, Rotational inertia,  $J_1 = 10 \times 10^{-3}$  and  $J_2 = 100 \times 10^{-6}$  kgm<sup>2</sup> some So,

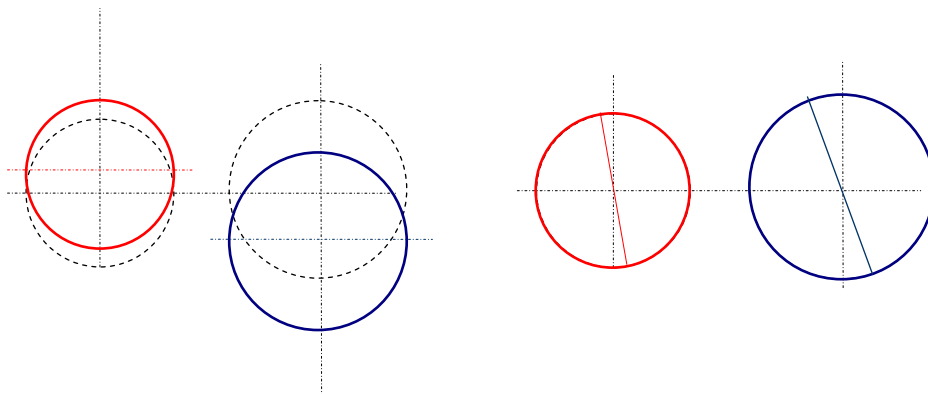


Figure 3 Translational (a) and angular (b) displacement modes (215,564 Hz) mesh of sun-gear planet gear defined in [8]

For one planetary gear, eigen fractional modes are obtained and visualization is presented on Figure 3 by using MathCAD .

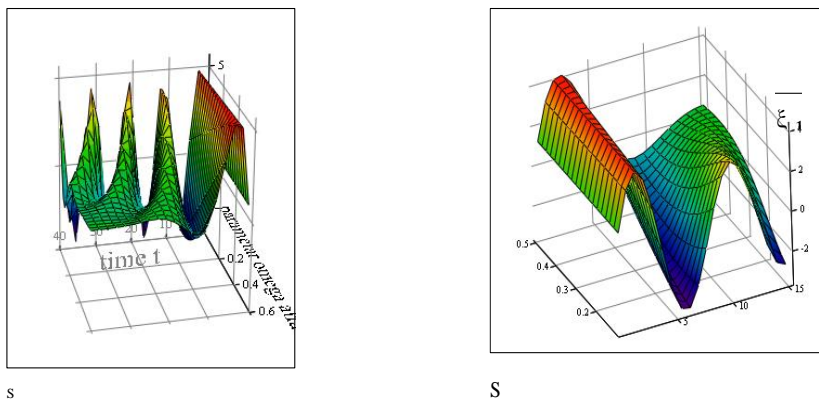


Figure 4 First main coordinates defined by initial conditions;  $\xi_s(0) = \xi_{0s}$  and  $\dot{\xi}_s(0) = \dot{\xi}_{0s}$

The first main coordinate is decreasing and increasing for changing of parameter  $\alpha$

### DYNAMIC RESPONSE ANALYSIS

The mechanical systems presented in this paper contain several simplifications. The planetary gear system is modelled as a mass-less shaft, bearings are assumed to be linear as linear elastic springs and the gears are assumed to be rigid.

However, the results in paper [4] and paper [5] indicate that the models with these simplifications acceptably predict the system characteristics.

Dynamic responses of the planetary gear system are computed from equation (16), using the Math CAD software.

Based on equation (18), the first normal mode corresponds to both masses moving in the opposite direction while angular displacements are in the same direction.

The numerical simulations indicate that the first, second and third fourth eigen frequencies are different from zero but the fourth eigen frequency is equal to zero for the presented values. Also, one can see 4-5 peaks of first main coordinate presented on Figure 4. The first main coordinate is decreasing and increasing unbalance for an increase of parameter  $\alpha$  (Figure 4a). According to Figure 4b one can see that the first main coordinate is changing with increasing of parameter  $\omega_\alpha$ . Parameters  $\alpha$  and  $\omega_\alpha$  are parameters that define the derivation of fractional order differential operator [3].

## CONCLUSIONS

The dynamic characteristics of planetary gear system are analysed considering the motion of carrier which influences the translation and rotation motions of the planetary gears. The equations of motion for the planetary gear system are derived by applying the Lagrange equation. Based upon the derived equations, the time responses are computed using the Math CAD. The dynamic behaviours of the planetary gear system are investigated with the time responses computed from the equations of motion. In addition, the motions of the components are also studied when they are in a steady state. The new model of the fractional order dynamic planetary gear here presented can be applied to study the real behaviour of the planetary gear. With this simple model, it is possible to research the nonlinear dynamics of the planetary gear and nonlinear phenomena in free and forced dynamics. The model is suitable to explain source of vibrations and big noise, as well as no stability in planetary gear.

In this paper a new method is used for the obtaining of the eigen values and for analysis results by MATCAD software.

## ACKNOWLEDGMENTS

Parts of this research were supported by the Ministry of Sciences of Republic Serbia through Mathematical Institute SANU Belgrade Grants No. ON 174001 “Dynamics of hybrid systems with complex structures. Mechanics of materials” and also through the Faculty of Mechanical Engineering University of Niš and the State University of Novi Pazar.

## REFERENCES

- [1] Cheon-Jae, B., Parker, G. R.: “Analytical Solution for the Nonlinear Dynamics of Planetary Gears”, *Journal of Computational and Nonlinear Dynamics* APRIL 2011, Vol. 6 / 021007-1 - 021007-15., 2011, New York City, p. 298-319
- [2] Goroško, O. A. i Hedrih (Stevanović), K.: “Analitička dinamika (mehanika) diskretnih naslednih sistema, (Analytical Dynamics (Mechanics) of Discrete Hereditary Systems)”, *Monograph*, 2001, University of Niš, p. 426
- [3] Hedrih K.: “Dynamics of multi-pendulum systems with fractional order creep elements”, *Journal of Theoretical and Applied Mechanics*, vol. 46, no. 3, 2008, pp. 483–509

- [4] Hedrih (Stevanovic), K., Nikolic-Stanojevic, V.: "A model of gear transmission fractional order system dynamics". *Mathematical Problems in Engineering* from: <http://www.hindawi.com/journals/mpe/aip972873.html> accessed on 2010-05-06.
- [5] Lin, J., Parker, G.R.: "Analytical characterization of the unique properties of planetary gear free vibration. *Transactions of the ASME*", vol. 121, 1999, New York City, p. 316-321.
- [6] Nikolic, V.: "Machine elements, theory, calculation, examples", Faculty of Mechanical Engineering, Prizma, 2004. Kragujevac, p. 671
- [7] Nikolic, V. : "Mechanical analysis of gears", Faculty of Mechanical Engineering, Cimpes, 1999, Kragujevac, p. 189
- [8] Nikolic-Stanojevic V., Veljovic Lj., Dolicanin C.: "A new Model of the Fractional order Dynamics of the Planetary gears", *Mathematical Problems in Engineering*, Volume, 2013 Article ID 932150, 14 pages <http://dx.doi.org/10.1155/2013/932150>
- [9] Vinayak, H., Singh, R.: "Multi-body dynamics and modal analysis of compliant gear bodies", *Journal of Sound and Vibration*, Vol. 210, no. 2, 1998, Great Britain, p. 171-214.