

MODES OF VIBRATION OF THE BEAMS UNDER THE INFLUENCE OF DISCONTINUITY IN FOUNDATION

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Abstract *Vibrations of the Timoshenko beams resting on the Winkler and Pasternak elastic foundation with discontinuity are investigated in this paper. A p-version finite element method that accounts for shear deformation is used. This p-element has special displacement shape functions that make it particularly appropriate for dealing with problems with discontinuities such as those introduced in the foundation. A set of ordinary differential equations is derived; geometrical non-linearity is considered in these equations for the sake of generality and for future use. Natural frequencies and mode shapes of vibration (composed by transverse displacements and rotations of cross sections) of the shear deformable beam are presented for diverse sizes and location of the discontinuity in the foundation. Results of the present approach are compared with the ones computed via established finite element software for various stiffness of the elastic support of the Winkler and Pasternak type.*

Key Words: *P-version FEM, Mode shape, Natural frequency, Foundation, Discontinuity, Vibrations*

1. INTRODUCTION

Vibrations of the beams on elastic foundations are of a wide practical interest involving applications such as analyses of roads, rail tracks and foundations of diverse structures. There have been a large number of publications related to this problem considering different types of foundation such as Winkler, Pasternak, elastic or viscoelastic, linear or non-linear (Refs. [1-10]). For example, Mamandi et al. [4] have studied nonlinear effects of the elastic foundation in frequency domain. Investigations in the nonlinear regime, with the beam actuated by moving loads are carried out in [5]. Kim and Cho [6] explore the vibration of a shear beam-column, resting on an elastic foundation when the system is subjected to moving loads of either constant amplitude or harmonic amplitude variation.

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In either case, the load moves with a constant advance velocity; a good insight of the linear dynamical behavior of the beam on elastic foundations is given. A more complex linear model [7] is also of great interest for understanding the problems of response of the beams resting on visco-elastically damped foundation under moving SDoF oscillators with a number of internal variables introduced and with the aim of representing the frequency-dependent behavior of the viscoelastic foundation.

With larger beam deflections, geometric nonlinearities lead to motions which are not predictable by linear formulations. Chang and Liu [8] have investigated deterministic and random vibrations of a nonlinear beam on an elastic foundation, subjected to a moving load. The nonlinear system of differential equations is solved by an implicit direct integration method. Rotary inertia and shear deformation are neglected, while the effects of longitudinal deflections and inertia are considered, based on the Bernoulli–Euler hypothesis. Sapountzakis and Kampitsis [3] have developed a boundary element method (BEM) for the geometrically nonlinear response of shear deformable beams traversed by moving loads, resting on tensionless nonlinear three-parameter viscoelastic foundation; they have shown that the BEM may be advantageous for exploring nonlinear effects on this kind of problems. Nonlinear foundation effects are also investigated in a different type of structures. Malekzadeh and Vosoughi [9] have investigated problems of composite thin beams using an efficient and accurate differential quadrature (DQ) method for a large amplitude free vibration analysis. Demeio and Lenci [10] have used the Multiple Time Scales (MTS) method to study nonlinear resonances of a semi-infinite cable resting on a nonlinear elastic foundation.

The finite-element method (FEM) is based on approximating the solution of a problem by means of admissible functions. In the p -version of the FEM, accuracy is improved by increasing the number of shape functions over the elements, without introducing more elements in the mesh; it has been often found that the p -version FEM is an efficient approach to the study of mechanical systems vibrations [11-21]. Recently, new shape functions have been proposed to be used when the p -version finite element method is applied to problems with steep changes in the domain [22]. In the later work, shear deformable beams with discontinuity in the cross section, due to a notch, are analyzed. The two shape functions for p -version beam elements lead to a significant improvement of the efficiency of the p -version FEM in the presence of notches.

This work explores the influence of discontinuity of foundations on the natural frequencies and mode shapes of beams, for various sizes and locations of the discontinuity. In the real world, one example of occurrence of large discontinuity in a foundation is provided by a railway crossing a bridge; in a simple model, the beam with discontinuity in the foundation would represent the bridge. A second type of real world example, this time with a discontinuity of small length, is an anomaly in the subsoil, which occurs very often. Two types of foundation (Winkler and Pasternak) are considered. The first order shear deformation theory is used because it provides more accurate results than Bernoulli-Euler formulations, particularly in the case of thick beams. The equations of motion are obtained by the principle of virtual work [23]. Although only linear analyses are performed in the numerical tests, the presented model includes geometrical non-linear effects for future use. The model is validated by comparison with finite element software ANSYS [25] in the computation of natural frequencies.

2. MATHEMATICAL MODEL AND EQUATIONS OF MOTION

The physical model is defined by beam's length L , width b and thickness h . The discontinuity in the foundation is defined by l_1 and l_2 visualized in Fig. 1. The beam material is assumed to be elastic, homogeneous and isotropic. The foundation is represented by an elastic layer of the Winkler's and Pasternak type (neglecting the shear layer reduces the foundation to Winkler type). More details on the two types of foundation are given in Ref. [3]. Co-ordinate axis x and z are also shown in Fig. 1.

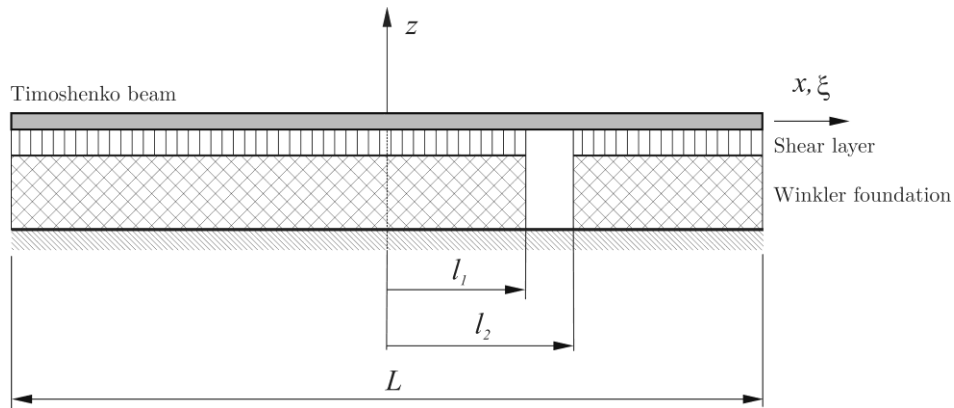


Fig. 1 Timoshenko beam supported by elastic foundation with discontinuity

Non-dimensional co-ordinate, ζ , represented in Fig. 1 is the local coordinate typical of the finite element method. Adopting the first order shear deformation theory [29], the displacement field of the model is given by:

$$u(x, z, t) = u^0(x, t) + z\theta^0(x, t), \tag{1}$$

$$w(x, z, t) = w^0(x, t), \tag{2}$$

where superscript “0” indicates axis x , which crosses the cross section centroids and t represents time. Letters u and w represent, respectively, the displacement components along axes x and z . The independent rotation of cross-sections about the axis parallel to y is given by $\theta^0(x, t)$. For the sake of generality, geometrical non-linearity will be considered in the formulation. The longitudinal and shear strains are, therefore, written as:

$$\varepsilon_x(x, t) = u_{,x}^0(x, t) + \frac{1}{2}(w_{,x}^0(x, t))^2 + z\theta_{,x}^0(x, t), \tag{3}$$

$$\gamma_{xz}(x, t) = w_{,x}^0(x, t) + \theta^0(x, t). \tag{4}$$

A comma in subscript, followed by a variable, represents partial derivation with respect to the latter.

Vector $\mathbf{d}_0(\zeta, t)$, which is formed by the components of displacement, is written as the product of shape functions by the generalized coordinates:

$$\mathbf{d}_0(\xi, t) = \mathbf{N}(\xi)\mathbf{q}(t), \quad (5)$$

$$\begin{Bmatrix} u^0(\xi, t) \\ w^0(\xi, t) \\ \theta^0(\xi, t) \end{Bmatrix} = \begin{bmatrix} \mathbf{N}^u(\xi)^T & 0 & 0 \\ 0 & \mathbf{N}^w(\xi)^T & 0 \\ 0 & 0 & \mathbf{N}^\theta(\xi)^T \end{bmatrix} \begin{Bmatrix} \mathbf{q}_u(t) \\ \mathbf{q}_w(t) \\ \mathbf{q}_\theta(t) \end{Bmatrix}. \quad (6)$$

In the equation above, $\mathbf{N}^u(\xi)$, $\mathbf{N}^w(\xi)$ and $\mathbf{N}^\theta(\xi)$ are, respectively, the in-plane, the out-of-plane and rotation shape functions, which together form the matrix of shape functions $\mathbf{N}(\xi)$. Time dependent generalized displacements are represented by: $\mathbf{q}_u(t)$ (generalized longitudinal displacements); $\mathbf{q}_w(t)$ (generalized transverse displacements); $\mathbf{q}_\theta(t)$ (generalized rotations about y axis). The shape functions associated to the discontinuity are represented by superscript “d”. Consequently, the row vectors of longitudinal, transverse and rotation shape functions, are respectively, the following:

$$\mathbf{N}^u(\xi)^T = \left[f_1^d \left(L, \frac{l_1+l_2}{2}, \xi \right) f_2^d \left(L, \frac{l_1+l_2}{2}, \xi \right) \mid f_1(\xi) f_2(\xi) f_3(\xi) \dots f_{pu}(\xi) \right], \quad (7)$$

$$\mathbf{N}^w(\xi)^T = \left[f_1^d \left(L, \frac{l_1+l_2}{2}, \xi \right) f_2^d \left(L, \frac{l_1+l_2}{2}, \xi \right) \mid f_1(\xi) f_2(\xi) f_3(\xi) \dots f_{pw}(\xi) \right], \quad (8)$$

$$\mathbf{N}^\theta(\xi)^T = \left[f_1^d \left(L, \frac{l_1+l_2}{2}, \xi \right) f_2^d \left(L, \frac{l_1+l_2}{2}, \xi \right) \mid f_1(\xi) f_2(\xi) f_3(\xi) \dots f_{p\theta}(\xi) \right], \quad (9)$$

The total numbers of longitudinal, transverse and rotational shape functions employed are, respectively, pu , pw , and $p\theta$ plus two (due to functions f_1^d and f_2^d). The last two functions are here associated to the discontinuity of the foundation and are explained in detail in Ref. [22]. In the p -version finite element method, the number of elements is essentially defined by the geometry of the structure to analyze. For example, in Ref. [14] a portal plane frame constituted by three straight beams is analyzed using three p -version beam elements. But localized steep variations, as the ones introduced by discontinuity, are thought to advocate the use of several elements, diminishing the interest of p -elements. Shape functions f_1^d and f_2^d provide an answer to this disadvantage of the p -version approach. Polynomial of types $f_1(\xi) - f_4(\xi)$ represent Hermit cubics [18].

The constitutive equation of an isotropic beam is:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \Leftrightarrow \begin{Bmatrix} \sigma_x \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} E & 0 \\ 0 & \lambda G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix}, \quad (10)$$

where E is the Young modulus and $G = E/[2(1+\nu)]$ is the shear modulus of elasticity, with ν representing Poisson ratio. In Eq. (10) \mathbf{D} represents matrix of elastic constants, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, respectively, the non-zero stresses and the strains in form of vectors. Shear correction factor λ employed is $\lambda = (5+5\nu)/(6+5\nu)$, as given in Ref. [26], because of the good agreement with the experimentally obtained results [27]. The longitudinal strain is:

$$\varepsilon_x = \left[1 \quad z \right] \left(\begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} + \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right), \quad (11)$$

where ε_0^p and $z\varepsilon_0^b$ represent longitudinal and bending strains, and ε_L^p geometrically non-linear longitudinal strain. These strains are:

$$\varepsilon_0^p = \mathbf{N}_{,\xi}^u \mathbf{q}_u \frac{2}{L}, \varepsilon_0^b = \mathbf{N}_{,\xi}^{\theta} \mathbf{q}_\theta \frac{2}{L}, \varepsilon_L^p = \frac{2}{L^2} \mathbf{q}_w \mathbf{T} \mathbf{N}_{,\xi}^w \mathbf{N}_{,\xi}^{wT} \mathbf{q}_w, \gamma_{xz} = \frac{2}{L} \left[\mathbf{N}_{,\xi}^w \mathbf{T} \mathbf{N}^{\theta T} \right] \begin{Bmatrix} \mathbf{q}_w \\ \mathbf{q}_\theta \end{Bmatrix}. \quad (12)$$

Integrating the normal stress, the shear stress and the moment of the normal stress we obtain:

$$\begin{Bmatrix} T \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} + \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right), Q_x = \frac{Eh\lambda}{2(1+\nu)} \gamma_{xz}, \quad (13)$$

where:

$$A, B, D = \int_z (1, z, z^2) E dz \Rightarrow A = Eh, B = 0, D = \frac{Eh^3}{12}. \quad (14)$$

The equations of motion are achieved by the principle of virtual work, according to which:

$$\delta W_{in} + \delta W_V + \delta W_{ex} = 0, \quad (15)$$

where δW_{in} , δW_V and δW_{ex} are, in this order, virtual works done by inertia, internal and external forces due to a virtual displacement with components δu , δw and $\delta \theta$. These components form vector $\delta \mathbf{d}$, as follows:

$$\delta \mathbf{d} = \begin{Bmatrix} \delta u \\ \delta w \\ \delta \theta \end{Bmatrix} = \mathbf{N} \delta \mathbf{q}. \quad (16)$$

The virtual work of internal forces is:

$$\delta W_V = - \int_V \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV = -b \int_L \left(\begin{Bmatrix} \delta \varepsilon_0^p \\ \delta \varepsilon_0^b \end{Bmatrix} + \begin{Bmatrix} \delta \varepsilon_L^p \\ 0 \end{Bmatrix} \right)^T \begin{bmatrix} A & B \\ B & D \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_0^p \\ \varepsilon_0^b \end{Bmatrix} + \begin{Bmatrix} \varepsilon_L^p \\ 0 \end{Bmatrix} \right) dx = \delta \mathbf{q}^T \mathbf{K}(\mathbf{q}) \mathbf{q}, \quad (17)$$

and the virtual work of the inertia forces is:

$$\delta W_{in} = - \int_V \rho \delta \mathbf{d}^T \ddot{\mathbf{d}} dV = -b \int_{-h/2-L/2}^{h/2-L/2} \int \rho (\delta u \ddot{u} + \delta w \ddot{w}) dx dz = \delta \mathbf{q}^T \mathbf{M} \ddot{\mathbf{q}}, \quad (18)$$

where ρ is the mass per unit volume, \mathbf{M} the mass matrix and $\ddot{\mathbf{d}} = d^2 \mathbf{d} / dt^2$. The virtual work of external forces is given by:

$$\begin{aligned} \delta W_{ex} = & \int_L \left[P_w^j(t) \delta(x - x_j) + P_w^d(x, t) - k_w w(x, t) \right] \delta w(x, t) + \left[k_p \frac{\partial^2 w(x, t)}{\partial x^2} \right] \delta w(x, t) + \\ & + [P_u^j(t) \delta(x - x_j) + P_u^d(x, t)] \delta u(x, t) + [M^j(t) \delta(x - x_j) + M^d(x, t)] \delta \theta(x, t) \end{aligned} \quad (19)$$

$$= \{ \delta q_u \delta q_w \delta q_\theta \}^T \begin{Bmatrix} \mathbf{F}_u^E \\ \mathbf{F}_w^E \\ \mathbf{M}^E \end{Bmatrix}$$

where $\{\mathbf{F}_u^E(t), \mathbf{F}_w^E(t), \mathbf{M}^E(t)\}$ represents the vector of generalized external forces and $\delta(x-x_j)$ represents Dirac delta function. $P^l(t)$ and $M^l(t)$ are concentrated forces or moments acting at point $x=x_j$, $P^d(t)$ and $M^d(t)$ are distributed forces or moments. In the Eq. (19) k_w and k_p represent, respectively, the Winkler and Pasternak stiffness and shear foundation moduli. Using the virtual work principle, including the foundation effect, the following equations of motion are obtained:

$$\begin{aligned} \begin{bmatrix} \mathbf{M}^l & 0 & 0 \\ 0 & \mathbf{M}^b & 0 \\ 0 & 0 & \mathbf{M}^r \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_u \\ \ddot{\mathbf{q}}_w \\ \ddot{\mathbf{q}}_\theta \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^l & 0 & 0 \\ 0 & \mathbf{K}^{\gamma 11} + \mathbf{K}^W - \mathbf{K}^P & \mathbf{K}^{\gamma 12} \\ 0 & \mathbf{K}^{\gamma 21} & \mathbf{K}^b + \mathbf{K}^{\gamma 22} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_u \\ \mathbf{q}_w \\ \mathbf{q}_\theta \end{Bmatrix} + \\ + \begin{bmatrix} 0 & \mathbf{K}_n^2(\mathbf{q}_w) & 0 \\ \mathbf{K}_n^3(\mathbf{q}_w) & \mathbf{K}_n^4(\mathbf{q}_w) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{q}_u \\ \mathbf{q}_w \\ \mathbf{q}_\theta \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_u^E \\ \mathbf{F}_w^E \\ \mathbf{M}^E \end{Bmatrix} \end{aligned} \quad (20)$$

Matrices of type \mathbf{M} and \mathbf{K} are constant matrices that originate linear terms in the equations of motion, of the latter type, matrices \mathbf{K}^W and \mathbf{K}^P represent linear influences of the foundation with discontinuity and are not introduced in [22]. The matrices that depend on $\mathbf{q}_w(t)$ - \mathbf{K}_n^2 , \mathbf{K}_n^3 and matrix \mathbf{K}_n^4 - lead to non-linear terms. The superscripts l, b, r and γ represent, respectively, longitudinal, bending, cross section rotation and shear effects. Adding Rayleigh-type damping, with damping coefficients α_r and β_r , one obtains the equations of motion (in a more condensed notation) :

$$\mathbf{M}\dot{\mathbf{q}}(t) + \alpha_r(\mathbf{K})\dot{\mathbf{q}}(t) + \beta_r(\mathbf{M})\dot{\mathbf{q}}(t) + (\mathbf{K} + \mathbf{K}_n(\mathbf{q}(t)))\mathbf{q}(t) = \mathbf{F}(t). \quad (21)$$

The mass and stiffness matrices have the following forms:

$$\mathbf{M}^l = \frac{\rho h b l}{2} \int_{-1}^{\frac{2l_c}{L}} \mathbf{N}^{u1} \mathbf{N}^{u1T} d\xi + \frac{\rho h b l}{2} \int_{\frac{2l_c}{L}}^1 \mathbf{N}^{u2} \mathbf{N}^{u2T} d\xi, \quad (22)$$

$$\mathbf{M}^b = \frac{\rho h b l}{2} \int_{-1}^{\frac{2l_c}{L}} \mathbf{N}^{w1} \mathbf{N}^{w1T} d\xi + \frac{\rho h b l}{2} \int_{\frac{2l_c}{L}}^1 \mathbf{N}^{w2} \mathbf{N}^{w2T} d\xi, \quad (23)$$

$$\mathbf{M}^r = \frac{\rho h b l}{2} \int_{-1}^{\frac{2l_c}{L}} \mathbf{N}^{\theta 1} \mathbf{N}^{\theta 1T} d\xi + \frac{\rho h b l}{2} \int_{\frac{2l_c}{L}}^1 \mathbf{N}^{\theta 2} \mathbf{N}^{\theta 2T} d\xi, \quad (24)$$

$$\mathbf{K}^l = \frac{2Ehb}{L} \int_{-1}^{\frac{2l_c}{L}} \frac{d\mathbf{N}^{u1}}{d\xi} \frac{d\mathbf{N}^{u1T}}{d\xi} d\xi + \frac{2Ehb}{L} \int_{\frac{2l_c}{L}}^1 \frac{d\mathbf{N}^{u2}}{d\xi} \frac{d\mathbf{N}^{u2T}}{d\xi} d\xi, \quad (25)$$

$$\mathbf{K}^{\gamma_{11}} = \frac{2\lambda Ghb}{L} \int_{-1}^{\frac{2l_c}{L}} \frac{d\mathbf{N}^{w1}}{d\xi} \frac{d\mathbf{N}^{w1T}}{d\xi} d\xi + \frac{2\lambda Ghb}{L} \int_{\frac{2l_c}{L}}^1 \frac{d\mathbf{N}^{w2}}{d\xi} \frac{d\mathbf{N}^{w2T}}{d\xi} d\xi, \quad (26)$$

$$\mathbf{K}^{\gamma_{12}} = \lambda Ghb \int_{-1}^{\frac{2l_c}{L}} \frac{d\mathbf{N}^{w1}}{d\xi} \mathbf{N}^{\theta 1T} d\xi + \lambda Ghb \int_{\frac{2l_c}{L}}^1 \frac{d\mathbf{N}^{w2}}{d\xi} \mathbf{N}^{\theta 2T} d\xi, \quad (27)$$

$$\mathbf{K}^{\gamma_{21}} = \lambda Ghb \int_{-1}^{\frac{2l_c}{L}} \mathbf{N}^{\theta 1} \frac{d\mathbf{N}^{w1T}}{d\xi} d\xi + \lambda Ghb \int_{\frac{2l_c}{L}}^1 \mathbf{N}^{\theta 2} \frac{d\mathbf{N}^{w2T}}{d\xi} d\xi, \quad (28)$$

$$\mathbf{K}^{\gamma_{22}} = \frac{\lambda GhbL}{2} \int_{-1}^{\frac{2l_c}{L}} \mathbf{N}^{\theta 1T} \mathbf{N}^{\theta 1} d\xi + \frac{\lambda GhbL}{2} \int_{\frac{2l_c}{L}}^1 \mathbf{N}^{\theta 2T} \mathbf{N}^{\theta 2} d\xi, \quad (29)$$

$$\mathbf{K}^W = \frac{L}{2} k_w \int_{-1}^{\frac{2l_c}{L}} \mathbf{N}^{w1} \mathbf{N}^{w1T} d\xi + \frac{L}{2} k_w \int_{\frac{2l_c}{L}}^1 \mathbf{N}^{w2} \mathbf{N}^{w2T} d\xi, \quad (30)$$

$$\mathbf{K}^P = \frac{2}{L} k_p \int_{-1}^{\frac{2l_c}{L}} \frac{d\mathbf{N}^{w1}}{d\xi} \frac{d\mathbf{N}^{w1T}}{d\xi} d\xi + \frac{2}{L} k_p \int_{\frac{2l_c}{L}}^1 \frac{d\mathbf{N}^{w2}}{d\xi} \frac{d\mathbf{N}^{w2T}}{d\xi} d\xi, \quad (31)$$

$$\mathbf{K}_n^2(\mathbf{q}_w) = \frac{2Ebh}{L^2} \int_{-1}^{\frac{2l_c}{L}} \frac{dw_0}{d\xi} \frac{d\mathbf{N}^{u1}}{d\xi} \frac{d\mathbf{N}^{u1T}}{d\xi} d\xi + \frac{2Ebh}{L^2} \int_{\frac{2l_c}{L}}^1 \frac{dw_0}{d\xi} \frac{d\mathbf{N}^{u2}}{d\xi} \frac{d\mathbf{N}^{u2T}}{d\xi} d\xi, \quad (32)$$

$$\mathbf{K}_n^3(\mathbf{q}_w) = 2\mathbf{K}_n^2(\mathbf{q}_w), \quad (33)$$

$$\mathbf{K}_n^4(\mathbf{q}_w) = \frac{4Ebh}{L^3} \int_{-1}^{\frac{2l_c}{L}} \left(\frac{dw_0}{d\xi} \right)^2 \frac{d\mathbf{N}^{u1}}{d\xi} \frac{d\mathbf{N}^{u1T}}{d\xi} d\xi + \frac{4Ebh}{L^3} \int_{\frac{2l_c}{L}}^1 \left(\frac{dw_0}{d\xi} \right)^2 \frac{d\mathbf{N}^{u2}}{d\xi} \frac{d\mathbf{N}^{u2T}}{d\xi} d\xi, \quad (34)$$

$$\mathbf{K}_n^2(\mathbf{q}_w) = \frac{2Ebh}{L^2} \int_{-1}^{\frac{2l_c}{L}} \frac{dw_0}{d\xi} \frac{d\mathbf{N}^{u1}}{d\xi} \frac{d\mathbf{N}^{u1T}}{d\xi} d\xi + \frac{2Ebh}{L^2} \int_{\frac{2l_c}{L}}^1 \frac{dw_0}{d\xi} \frac{d\mathbf{N}^{u2}}{d\xi} \frac{d\mathbf{N}^{u2T}}{d\xi} d\xi, \quad (35)$$

where $l_c = (l_1 + l_2)/2$.

3. NATURAL FREQUENCIES AND MODE SHAPES

The p -FEM will be applied to derive a model for a clamped-clamped beam with geometric and material properties based on an example from Ref. [3]. Used data about the

beam are: $L=10\text{m}$, $E=210\text{GPa}$, $G=77\text{GPa}$, $I=30.55\cdot 10^{-6}\text{m}^4$, $A=30.55\cdot 10^{-6}\text{m}^2$, $\rho=7850\text{kg/m}^3$, $k_{w1}=20\text{MPa}$, $k_{p1}=69\text{kN}$, $k_{w2}=35\text{MPa}$, $k_{p2}=200\text{kN}$.

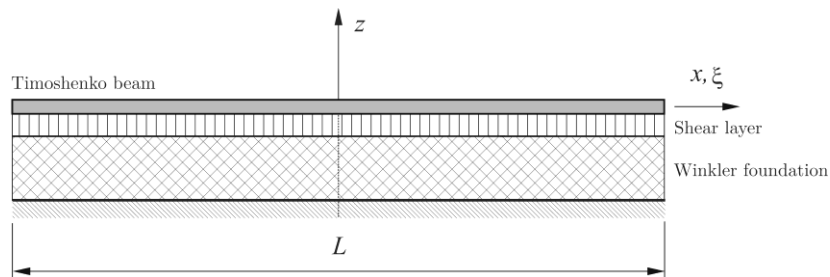


Fig. 2 Timoshenko beam supported by the elastic foundation without discontinuity (Case 1)

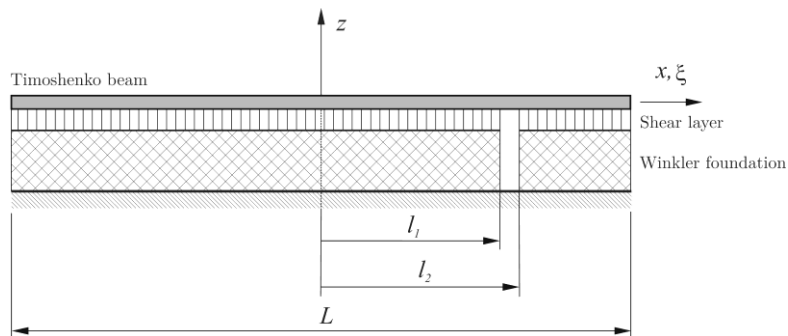


Fig. 3 Timoshenko beam supported by the elastic foundation with discontinuity $l_1=1.67\text{m}$, $l_2=1.85\text{m}$ (Case 2)

Four cases of the shear deformable beam on elastic foundation are shown on Figs. 2, 3, 4 and 5. For the presented cases and two types of foundation, natural frequencies are given in Tables 1-8. For the case without discontinuity in foundation, natural frequencies for simply supported beam can be easily obtained from the frequency equation (27) of Ref. [24], for the Timoshenko theory, by setting $m=1$.

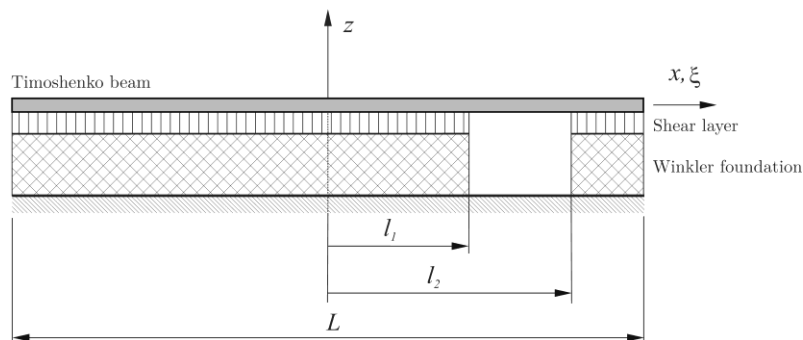


Fig. 4 Timoshenko beam supported by the elastic foundation with discontinuity $l_1=1.67\text{m}$, $l_2=2.6\text{m}$ (Case 3)

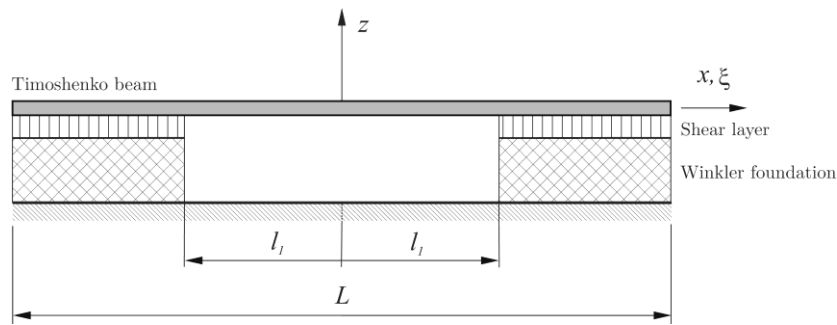


Fig. 5 Timoshenko beam supported by the elastic foundation with discontinuity $l_1=L/6$ (Case 4)

The Ansys *h*-version element used is element BEAM 189, which has 3 nodes with six degrees of freedom at each node (but to analyze vibrations in a plane only 3 degrees of freedom per node are required). BEAM189 is an element based on the Timoshenko beam theory and suitable for analyzing slender to moderately thick beam structures. The current results from Ansys are obtained with 100 elements.

Table 1 Natural frequencies [Hz] of a clamped-clamped beam Case 1 for k_{wI} and k_{pI}

Mode	Winkler		Pasternak	
	ANSYS BEAM189 100 elements	<i>p</i> -FEM (30sf)	ANSYS BEAM189 100 elements	<i>p</i> -FEM (30sf)
1	92.5	92.33	92.4	92.35
2	97.2	96.90	97.1	96.97
3	111.3	110.45	111.2	110.58
4	137.4	136.66	137.3	136.84

Table 2 Natural frequencies [Hz] of a clamped-clamped beam Case 2 for k_{wI} and k_{pI}

Mode	Winkler		Pasternak	
	ANSYS BEAM189 100 elements	<i>p</i> -FEM (30sf)	ANSYS BEAM189 100 elements	<i>p</i> -FEM (30sf)
1	90.7	90.55	90.7	90.57
2	95.8	95.59	95.9	95.65
3	110.5	110.27	110.6	110.40
4	136.5	136.27	136.6	136.45

Table 3 Natural frequencies [Hz] of a clamped-clamped beam Case 3 for k_{w1} and k_{p1}

Mode	Winkler		Pasternak	
	ANSYS BEAM189 100 elements	p -FEM (30sf)	ANSYS BEAM189 100 elements	p -FEM (30sf)
1	77.2	77.07	77.3	77.08
2	94.3	94.06	94.4	94.11
3	108.5	108.26	108.6	108.38
4	136.1	135.85	136.2	136.02

Table 4 Natural frequencies [Hz] of a clamped-clamped beam Case 4 for k_{w1} and k_{p1}

Mode	Winkler		Pasternak	
	ANSYS BEAM189 100 elements	p -FEM (30sf)	ANSYS BEAM189 100 elements	p -FEM (30sf)
1	35.3	35.08	35.3	35.06
2	59.6	59.30	59.6	59.30
3	77.5	77.31	77.5	77.33
4	104.5	104.33	104.5	104.42

Changes in the natural frequencies have a tendency of decreasing as the discontinuity in the foundation increases. This model is important for analysis because of the easy generalization of the discontinuity in the foundation. Non-linear matrices are just obtained and presented in the work and can be used for a further nonlinear analysis in the time domain.

Table 5 Natural frequencies [Hz] of a clamped-clamped beam Case 1 for k_{w2} and k_{p2}

Mode	Winkler		Pasternak	
	ANSYS BEAM189 100 elements	p -FEM (30sf)	ANSYS BEAM189 100 elements	p -FEM (30sf)
1	122.1	121.74	122.2	121.78
2	126.5	125.20	126.6	125.36
3	136.1	135.91	136.2	136.22
4	158.1	157.90	158.2	158.35

Table 6 Natural frequencies [Hz] of a clamped-clamped beam Case 2 for k_{w2} and k_{p2}

Mode	Winkler		Pasternak	
	ANSYS BEAM189 100 elements	p -FEM (30sf)	ANSYS BEAM189 100 elements	p -FEM (30sf)
1	119.2	118.84	119.3	118.90
2	124.1	123.81	124.2	123.94
3	135.9	135.68	136.1	135.98
4	157.6	157.33	157.7	157.77

Table 7 Natural frequencies [Hz] of a clamped-clamped beam Case 3 for k_{w2} and k_{p2}

Mode	Winkler		Pasternak	
	ANSYS BEAM189 100 elements	<i>p</i> -FEM (30sf)	ANSYS BEAM189 100 elements	<i>p</i> -FEM (30sf)
1	97.2	97.09	97.2	97.10
2	123.1	122.88	123.1	122.96
3	133.7	133.54	133.7	133.83
4	156.9	156.66	156.9	157.10

To understand the effect of the foundation on the vibration modes, natural frequencies are not enough. It is important to present the changes in the natural mode shapes. In Figs. 6-11 natural mode shapes are presented for all the cases of discontinuity in the foundation. It is evident that the effect of the discontinuity results in the change in natural transverse and rotation mode shape. It is important to underline that the asymmetrical mode shapes appear when the discontinuity exists on one side of the beam.

Table 8 Natural frequencies [Hz] of a clamped-clamped beam Case 4 for k_{w2} and k_{p2}

Mode	Winkler		Pasternak	
	ANSYS BEAM189 100 elements	<i>p</i> -FEM (30sf)	ANSYS BEAM189 100 elements	<i>p</i> -FEM (30sf)
1	39.9	39.71	39.9	39.65
2	59.5	59.30	59.5	59.30
3	92.5	92.37	92.5	92.38
4	117.9	117.75	117.9	117.75

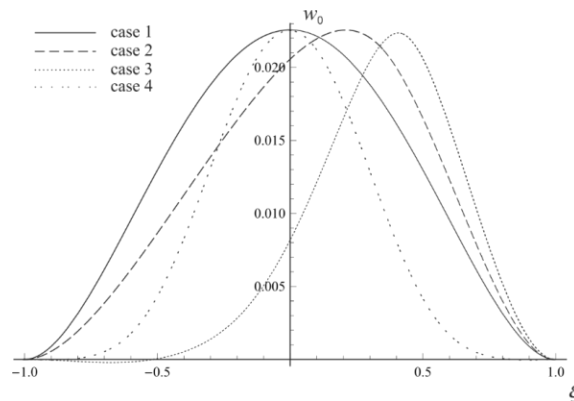


Fig. 6 Transverse components of mode shapes - mode 1

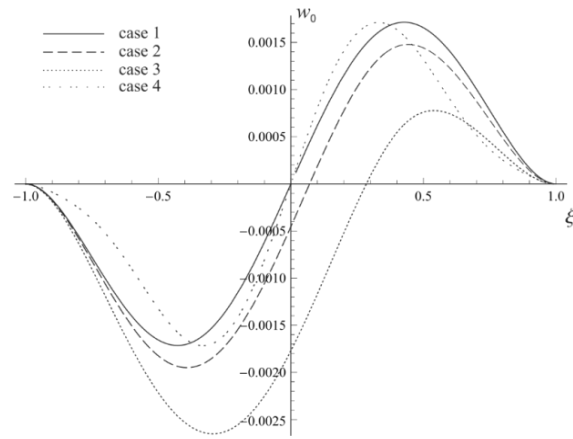


Fig. 7 Transverse components of mode shapes - mode 2

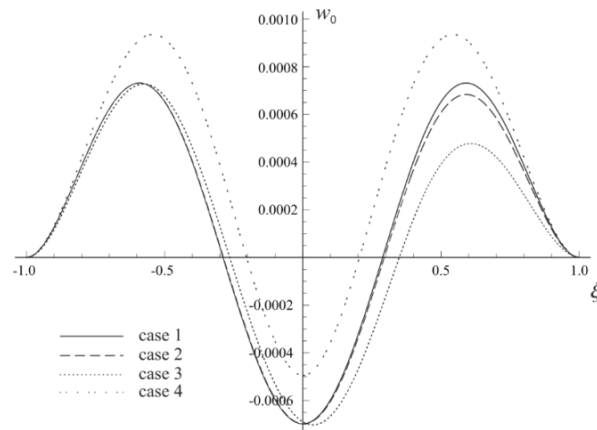


Fig. 8 Transverse components of mode shapes - mode 3

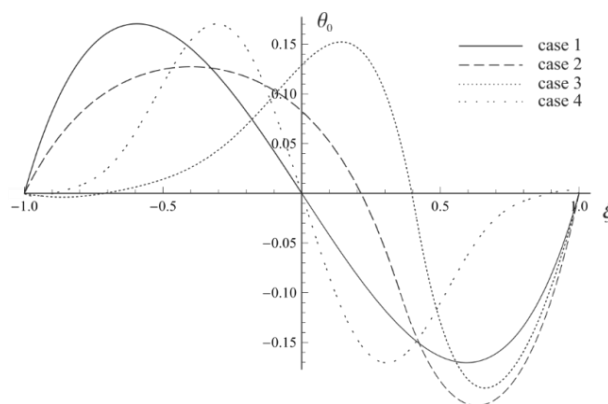


Fig. 9 Cross section rotation components of mode shapes - mode 1

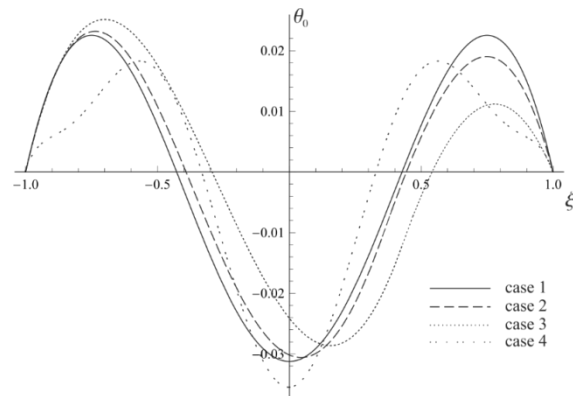


Fig. 10 Cross section rotation components of mode shapes - mode 2

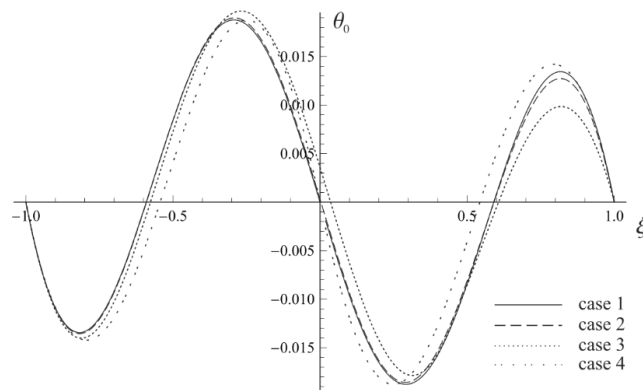


Fig. 11 Cross section rotation components of mode shapes - mode 3

For the cases 2 and 3 the obtained results have a significant importance in understanding the vibrations of the geometrically asymmetrical model. The advantages of the p -FEM model are better approximations of the solutions in comparison with commercial software ANSYS (convergence occurs from above in the p -version FEM, and our values are lower than the ones of ANSYS), with a lower number degrees of freedom. In a number of cases, the values computed via ANSYS for the natural frequencies of vibration considering the Pasternak and Winkler type foundation were the same. This rarely occurs with the p -version model, which generally allows to detect the effect of the shear layer.

5. CONCLUSIONS

Vibrations of shear deformable beams, elastically connected to a foundation with discontinuity, are investigated in this paper. A p -version finite element based on the Timoshenko theory for bending is applied. The frequencies obtained agree with the ones computed via well know finite element software. The developed matrices are easily applicable to any size of discontinuity in the foundation, simply taking into account changes

in the boundaries of the discontinuity. Various discontinuity cases (when the centre of the discontinuity is not under the middle of the beam) lead to asymmetrical vibrations. This is made evident by the mode shapes presented. The p -version finite element method developed gives the possibility for a further non-linear analysis of the vibrations of the beams resting on a foundation with discontinuity.

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MODOVI OSCILACIJA NOSAČA POD UTICAJEM PROMENLJIVOG DISKONTINUITETA U PODLOZI

Oscilacije nosača Timoshenko-vog tipa na Winkler-ovoj i Pasternak-ovoj podlozi sa promenljivim diskontinuitetom razmatrane su u ovom radu. Razvijena je p-verzija metode konačnih elemenata za oscilacije deformabilnih nosača na elastičnoj podlozi. U studiji je korišćen p- element koji je proizašao upotrebom posebno razvijenih oblika funkcija primenjenih na nosačima sa oštećenjem i upotrebljen na modelu sa osnovom koja sadrži diskontinuitet. Novina ove studije predstavlja laku generalizaciju pristupa pri određivanju prirodnih frekvencija, opštih oblika oscilovanja (transverzalnih i rotacija poprečnih preseka) nosača za proizvoljno izabrane veličine i lokacije diskontinuiteta. Izveden je sistem parcijalnih diferencijalnih jednačina koji omogućava dalje istraživanje u nelinearnom vremenskom domenu oscilovanja. U radu su prikazana poređenja rezultata sa različitim vrednostima krutosti nelinearne elastične osnove Winkler-ovog i Pasternak-ovog tipa.

Ključne reči: *p-verzija MKE, osnovni oblik oscilovanja, prirodna frekvencija, diskontinuitet, oscilacije.*