IMPACT: International Journal of Research in Engineering & Technology (IMPACT: IJRET)
ISSN(E): 2321-8843; ISSN(P): 2347-4599
Vol. 3, Issue 3, Mar 2015, 69-78

A CONVOLUTION STRUCTURE FOR EIGEN FUNCTION TRANSFORM

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ABSTRACT

Translation and convolution associated with Eigenfunction transform, studied by Zemanian, are defined and certain boundedness and continuity results are obtained. Convolution of a distribution and a test function, and that of two distributions are defined and their properties are investigated.

KEYWORDS: Eigen Function Transform, Convolution, Distributions

AMS Subject Classification: Primary 44A35; Secondary 46F12

INTRODUCTION

Eigenfunction transform investigated by Zemanian [15] is a unification of many transforms involving infinite series representations and having applications in solving various boundary value problems. Various properties of this transform have been investigated by [7-10, 13-15].

We first recall its definition from [15]. Let I denote any open interval $a < x < b$ on real line. Here $a = -\infty$ and $b = +\infty$ are permitted.

Then Eigen Function transform of $f \in L_2(I)$ is defined by

$$f^\wedge(n) := (f, \psi_n) := \int_a^b f(x)\overline{\psi_n(x)}\,dx; \qquad \psi_n \in L_2(I), \tag{1.1}$$

where $\overline{\psi_n(x)}$ denotes the complex conjugate of $\psi_n(x)$.

An important classical result [15, p.250] states that $\{\psi_n\}$ is complete if and only if, for every $f \in L_2(I)$, the coefficients (f, ψ_n) satisfy *Parseval's equation*:

$$\sum_{n=0}^{\infty} |(f, \psi_n)|^2 = \int_a^b |f(x)|^2\,dx. \tag{1.2}$$

Let $\Re$ denote the linear differential operator

$$\Re := \theta_0 D^{n_1} \theta_1 D^{n_2} \theta_2 \ldots D^{n_r} \theta_r, \tag{1.3}$$

where $D = \frac{d}{dx}$, the n_r are positive integers, and θ_r are smooth functions on I that are never equal to zero anywhere on I. Moreover, we assume that there exists a sequence $\{\lambda_n\}_{n=0}^{\infty}$ of real numbers called *eigenvalues* of $\Re$, and a sequence $\{\psi_n\}_{n=0}^{\infty}$ of smooth functions in $L_2(a,b)$, called *eigenfunctions* of

$\Re$, such that $|\lambda_n| \to \infty$ *as* $n \to \infty$ and

$$\Re\psi_n := \lambda_n\psi_n \ , \qquad n = 0, 1, 2 \ldots . \tag{1.4}$$

The zero function is not allowed as an eigenfunction. For various properties of eigenvalues, eigenfunctions and eigenfunction transforms we may refer to [4, 12, 18].

Next, we recall definition and properties of the testing function space *A* investigated by Zemanian [15, p.252]. The space *A* consists of all complex valued smooth functions ϕ on *I* such that

$$\alpha_K(\phi) := \left[\int_a^b |\Re^K \phi(x)|^2 \ dx\right]^{1/2} < \infty , \tag{1.5}$$

for each k = 0, 1, 2, ..., and for each $n, k \in \boldsymbol{N}_0$,

$$(\Re^k \phi, \psi_n) := (\phi, \Re^k \psi_n) . \tag{1.6}$$

A is a linear space. Moreover, it is a subspace of $L_2(I)$. The operator $\Re : A \to A$ is continuous and linear. The dual of *A* is denoted by A'. We also have

$$(\Re f , \phi) := (f, \Re \phi), \quad f \in A', \qquad \phi \in A . \tag{1.7}$$

Convolutions associated with certain special cases of the general Eigenfunction function transform have been investigated by [2,3,5,6,11].The aim of the present paper is to define translation and convolution associated with the general Eigenfunction transform and study their properties exploiting the technique of Glaeske [6] and Betancor et al [1]. Existence theorems for these translation and convolution are proved. Using Zemanian's theory It is shown that the Eigenfunction transform of the convolution of two distributions is a product of their transforms.

We shall use the following theorems due to Zemanian [15] in the proof of our results.

THEOREM 1.1 *If* $\phi \in A$, then

$$\phi = \sum_{n=0}^{\infty} (\phi, \psi_n)\, \psi_n \ , \quad \psi_n \in L_2(I), \tag{1.8}$$

where the series converges in *A*.

THEOREM 1.2. If $f \in A'$, then

$$f = \sum_{n=0}^{\infty} (f, \psi_n)\, \psi_n \ , \quad \psi_n \in L_2(I), \tag{1.9}$$

where the series converges in A'.

Let us define the generalized integral transform υ by

$$\upsilon f := F(n) := (f, \psi_n) \qquad f \in A', \qquad n = 0, 1, 2, 3, \ldots, \tag{1.10}$$

then the inverse mapping is given by (1.9), which we write as

$$f = \upsilon^{-1} F(n) = \sum_{n=0}^{\infty} F(n)\, \psi_n \quad , \tag{1.11}$$

where the series converges in A'.

A characterization of the convergence of the series (1.11) is given by the following theorem [15, p.261].

THEOREM 1.3

Let b_n denote complex numbers. Then $\sum_{n=0}^{\infty} b_n \psi_n$, converges in A' if and only if there exists a non- negative integer q such that $\sum_{\lambda_n \neq 0} |\lambda_n|^{-2q} |b_n|^2$, converges. Furthermore, if f denotes the sum in A' of (1), then $b_n = (f, \psi_n)$.

THE BASIC GENERALIZED FUNCTION $u(x,y;z)$

In this section we define a basic generalized function $u(x,y;z)$. In terms of this generalized function, translation and convolution associated with eigen function transform are defined. Various properties of the translation and convolution are investigated in the forthcoming sections.

In terms of eigen functions $\{\psi_n(x)\}$ let us define the basic generalized function $u(x,y;z)$ by

$$u(x,y;z) = \sum_{n=0}^{\infty} \overline{\psi_n(x)}\; \overline{\psi_n(y)}\; \psi_n(z) . \tag{2.1}$$

We show that the above series converges in A'. Let us assume that there exists a constant $H > 0$ and $p \in \mathbf{N}_0$, such that

$$\sup_{x \in I} |\psi_n(x)| \leq H \lambda_n^p . \tag{2.2}$$

Such estimates hold for many special cases of $\psi_n(x)$. Few examples are given below.

Example 1. I = (-1,1), $\Re = D(x^2-1)D$,

$$\psi_n(x) = (n+\tfrac{1}{2})^{\frac{1}{2}} P_n(x), \quad \lambda_n = n(n+1) \qquad [15, p.268].$$

Since $|P_n(x)| \leq 1$ for $-1 \leq x \leq 1$ [4, p.205],

we have

$$|\psi_n(x)| \leq (n+\tfrac{1}{2})^{\frac{1}{2}} \leq n(n+1) = \lambda_n \qquad \text{for } n \geq 1.$$

Example 2. I $= (-\infty, \infty)$, $\Re = e^{x^2/2} D e^{-x^2} D e^{x^2/2}$,

$$\psi_n(x) = \frac{e^{-x^2/2} H_n(x)}{(2^n n! \sqrt{\pi})^{\frac{1}{2}}}, \qquad \lambda_n = -2n \qquad [15, p. 267].$$

From [4, p.208], we know that there exists a positive constant H, $1 < H < 2$, such that

$$\left|e^{-x^2/2}H_n(x)\right|\le H2^{n/2}(n!)^{1/2}.$$

Hence

$$|\psi_n(x)|\le H\pi^{-1/4}\le H\pi^{-1/4}.2n=H\pi^{-1/4}|\lambda_n|, \quad \text{for} \quad n\ge 1.$$

Now, in view of estimate (2.2),

$$|b_n|:=\left|\overline{\psi_n(x)}\,\overline{\psi_n(y)}\right|\le H^2\lambda_n^{2p}.$$

Hence

$$\sum_{\lambda_n\ne 0}|\lambda_n|^{-2q}|b_n|^2\le\sum_{\lambda_n\ne 0}|\lambda_n|^{-2q}H^4|\lambda_n|^{4p}<\infty,$$

for some large q > 2p, because $\{\lambda_n\}$ is a increasing sequence and $\lambda_n\to\infty$ as $n\to\infty$ .Therefore, by Theorem 1.3,

$$\sum_{n=0}^{\infty}b_n\ \psi_n(z)=\sum_{n=0}^{\infty}(\overline{\psi_n(x)}\ \overline{\psi_n(y)})\psi_n(z)$$

converges in A', which is denoted by *u(x,y;z)*. Moreover, from Theorem 1.3, we also have

$$b_n=\overline{\psi_n(x)}\ \overline{\psi_n(y)}=(u(x,y;.),\psi_n(.)). \tag{2.3}$$

In case u(x,y; .) is a regular generalized function in A', (2.3) can be written as

$$\overline{\psi_n(x)}\ \overline{\psi_n(y)}=\int_a^b u(x,y;z)\psi_n(z)dz. \tag{2.4}$$

If we assume that $\psi_0(x)=1$ (this holds in many special cases), from (2.4) it follows that

$$\int_a^b u(x,y;z)dz=1. \tag{2.5}$$

TRANSLATION AND CONVOLUTION ON *A*

Using basic generalized function *u(x,y;z)* we define the generalized translation associated with the eigenfunction transform and investigate its properties.

DEFINITION 3.1. Translation associated with eigen function transform of a function $\phi\in A$, is defined by

$$(\tau_x\phi)(y):=\phi(x,y):=\langle u(x,y;z),\phi(z)\rangle. \tag{3.1}$$

THEOREM 3.2. Let $a < x, y, z < b$ and $\phi\in A$. Then

$$\tau_y[(\Re^{(k)}\phi)(z)] \;=\; \Re^{(k)}[(\tau_y\phi)(z)], \qquad k = 0, 1, 2\ldots \tag{3.2}$$

Proof : Using property (1.7) of $\Re$ we have

$$\tau_y(\Re\phi)(z)=\langle u(x,y;z),(\Re_z\phi)(x)\rangle$$

$$= \quad <\Re_z u(x,y;z),\phi(x)>$$

$$= <\sum_{n=0}^{\infty}[\psi_n(x)\overline{\psi_n(y)}\overline{\Re_z\psi_n(z)}],\phi(x)>$$

$$= \ <\sum_{n=0}^{\infty}\psi_n(x)\overline{\psi_n(y)}\overline{\lambda_n\psi_n(z)},\phi(x)>$$

$$= \ <\sum_{n=0}^{\infty}\overline{\lambda_n}\overline{\psi_n(x)}\overline{\psi_n(y)}\overline{\psi_n(z)},\phi(x)>$$

$$= \ <\sum_{n=0}^{\infty}\overline{\Re_x\psi_n(x)}\overline{\psi_n(y)}\psi_n(z),\phi(x)>$$

$$= \ <\sum_{n=0}^{\infty}\overline{\psi_n(x)}\overline{\psi_n(y)}\psi_n(z),\Re_x\phi(x)>$$

$$= \ \Re\ [(\tau_y\phi)\ (z)].$$

In general, we can prove that

$$[\tau_y\left(\Re^{(k)}\phi\right)(z)] \ = \ \Re^{(k)}\ [(\tau_y\phi)(z)].$$

THEOREM 3.3. Let $a < x, y, z < b$ and $\phi\in A$, then mapping $\phi\to\tau_y\phi$ is bounded and continuous from A into itself.

Proof. In view of definition (2.1), relation (3.2) gives

$$\Re^k(\tau_y\phi)(z)=\tau_y(\Re^k\phi)(z)$$

$$=< \sum_{n=0}^{\infty}\overline{\psi_n(x)}\ \overline{\psi_n(y)}\ \psi_n(z),\Re^k\phi(z)>$$

$$=\sum_{n=0}^{\infty}\overline{\psi_n(x)}\ \overline{\psi_n(y)}(\ \psi_n(z),\Re^k\phi(z)).$$

Therefore,

$$\left|\Re^k(\tau_y\phi)(z)\right|^2 = \sum_{n=0}^{\infty}\overline{\psi_n(x)}\ \overline{\psi_n(y)}(\ \psi_n(z),\Re^k\phi(z))\sum_{m=0}^{\infty}\psi_m(x)\psi_m(y)\ \overline{(\psi_m(z),\Re^k\phi(z))}.$$

Using orthonormality of $\{\psi_n\}$ we get

$$\int_a^b\left|\Re^k(\tau_y\phi)(z)\right|^2 dz = \sum_{n=0}^{\infty}\left|\psi_n(x)\right|^2\left|\psi_n(y)\right|^2\lambda_n^{-2q}\left|(\ \lambda_n^q\psi_n(z),\Re^k\phi(z))\right|^2$$

$$=\sum_{n=0}^{\infty}\left|\psi_n(x)\right|^2\left|\psi_n(y)\right|^2\lambda_n^{-2q}\left|(\ \psi_n(z),\Re^{k+q}\phi(z))\right|^2$$

$$\le\sum_{n=0}^{\infty}\left|\psi_n(x)\right|^2\left|\psi_n(y)\right|^2\lambda_n^{-2q}\left\|\Re^{k+q}\phi(z)\right\|_2^2.$$

Using estimate (2.2) and choosing q-large we can show that the above series converges. Hence there exists a constant $C > 0$ such that

$$\alpha_k(\tau_y\phi) \le C\alpha_{k+q}(\phi) \quad \text{for } k, q \in N_0; \tag{3.3}$$

from which the conclusion of the theorem follows.

CONVOLUTION OF A DISTRIBUTION AND A FUNCTION

In this section, we shall study the convolution of a distribution and a test function associated with the eigen function transform. For proving, existence theorem for this convolution, we shall use the expansion of $f \in A'$, given in Theorem 1.2 and generalized translation defined by (3.1).

THEOREM 4.1. Let $f \in A'$ and $\phi \in A$; define

$$(f*\phi)(y) := \langle f(z), (\tau_y\phi)(z) \rangle. \tag{4.1}$$

Then $f*\phi \in A$.

Proof. Since by Theorem 3.3, $(\tau_y\phi)(z) \in A$, the right-side expression is meaningful. Now, using Theorem 3.2,

$$\Re^k(f*\phi)(z) := \left\langle f(x), <\sum_{n=0}^{\infty} \overline{\psi_n(x)}\ \overline{\psi_n(y)}\ \psi_n(z), \Re^k(\tau_y\phi)(z)> \right\rangle.$$

$$= \sum_{n=0}^{\infty} (f, \psi_n)(\psi_n, \Re^k\phi)\overline{\psi_n(y)}.$$

Now, by the arguments used in the proof of Theorem 3.3,

$$\int_a^b |\Re^k(f*\phi)(z)|^2 dz = \sum_{n=0}^{\infty} |(f,\psi_n)|^2 |(\psi_n, \Re^k\phi)|^2$$

$$\le \sum_{n=0}^{\infty} |(f,\psi_n)|^2 \lambda_n^{-2q} \|\Re^{k+q}\phi\|_2^2;$$

so that

$$\alpha_k(f*\phi) \le C\alpha_{k+q}(\phi), \qquad \text{for some constant } C > 0.$$

THEOREM 4.2. Let $f \in A'$ *and* $\phi \in A$, then the following identity holds:

$$(f*\phi)^\wedge(m) = f^\sim(m)\ \phi^\wedge(m), \qquad m = 0,1,2,\ldots \tag{4.2}$$

where $f^\sim(m)$ denotes generalized integral transform of f in A'.

Proof. Since $f*\phi \in A$, its generalized eigen function transform exists. As in the proof of Theorem 4.1,

$$(f*\phi)(z) = \sum_{n=0}^{\infty} (f, \psi_n)(\overline{\psi_n}, \phi)\psi_n(z). \tag{4.3}$$

Using ortho normality condition, we get

$$(f*\phi)^{\wedge}(m)=\left((f*\phi)(z),\psi_m(z)\right)=(f,\psi_m)(\phi,\psi_m), \qquad m=0,1,2,\ldots\ .$$

$$= f^{\sim}(m)\phi^{\wedge}(m).$$

GENERALIZED CONVOLUTION OF TWO DISTRIBUTIONS

Let $f,g\in A'$ and $\phi\in A$. Then by Theorem 4.1, $g*\phi\in A$. Therefore, we can define $f*g$ by

$$\langle (f*g)(z),\ \phi(z)\rangle=\langle f(z),\ \langle g(y),\ \phi(y,z)\rangle\rangle$$

$$=\langle f(z),\ (g*\phi)(z)\rangle. \tag{5.1}$$

It can easily be shown that $f*g$ is a linear, continuous functional on A, so that $f*g\in A'$.

THEOREM 5.1. Let $f,g\in A'$; then

$$(f*g)^{\sim}(m)=F(m)G(m). \tag{5.2}$$

Proof: Let $\phi\in A$. Using (4.3) and (5.1) we have

$$\langle (f*g)(x),\ \phi(x)\rangle=\left\langle f(z),\ \sum_{m=0}^{\infty}(g,\psi_m)(\overline{\psi_m},\phi)\psi_m(z)\right\rangle$$

$$=\left\langle \sum_{m=0}^{\infty}(g,\psi_m)(f,\psi_m)\overline{\psi_m},\phi\right\rangle$$

$$=\left\langle \sum_{m=0}^{\infty}F(m)G(m)\psi_m(z),\ \phi(z)\right\rangle. \tag{5.3}$$

But in view of the fact that $f*g\in A'$, we have

$$\langle (f*g)(z),\ \phi(z)\rangle=\left\langle \sum_{m=0}^{\infty}(f*g)^{\sim}(m)\ \psi_m(z),\ \phi(z)\right\rangle. \tag{5.4}$$

Hence, by uniqueness of generalized integral transform, (5.3) and (5.4), yield

$$(f*g)^{\sim}(m)=F(m)G(m).$$

THEOREM 5.2 Let f, g, $h\in A'$; then the following holds:

(i) $$(f*g)(x)=(g*f)(x), \tag{5.5}$$

(ii) $$(f*g)*h=f*(g*h). \tag{5.6}$$

Proof. By Theorem 5.1, we have

$$\left(f*g\right)^{\sim}(m) = F(m)G(m) = G(m)F(m) = (g*f)^{\sim}(m) \quad , m=0,1,2,\ldots .$$

Uniqueness property of the transform gives (i). Next, in view of (5.1) for $\phi\in A$, we have

$$\langle [(f*g)*h](x), \phi(x)\rangle = \langle (f*g)(x), (h*\phi)(x) \rangle$$

$$= \langle f(x), [g*(h*\phi)](x)\rangle . \tag{5.7}$$

But

$$[g*(h*\phi)](x) = \langle g(y), \langle h(z), (\tau_y\phi)(z)\rangle\rangle$$

$$= \langle (g*h)(y), \phi(z,y)\rangle . \tag{5.8}$$

From (5.7) and (5.8), we get

$$\langle [(f*g)*h](x), \phi(x)\rangle = \langle f(x), \langle (g*h)(y), \phi(z,y)\rangle\rangle$$

$$= \langle f(x), [(g*h)*\phi](x)\rangle$$

$$= \langle [f*(g*h)](x), \phi(x)\rangle .$$

This is the conclusion of part (ii).

REMARK 5.3. Convolutions for Legendre transform [11], Chebyshev transform [2], Jacobi transform [5], Laguerre transform [3], Hermite transform [6] and many other discrete transforms can be derived as special cases of (5.1), and hence their properties can be investigated.

ACKNOWLEDGEMENTS

I express our gratitude to Professor R. S. Pathak of Department of Mathematics (Ex.), Banaras Hindu University Varanasi. As our Ph.D. thesis Co-Supervisor he lectured on distribution theory, provided us with extensive literature, and conveyed important insights about the field. I would also like to thanks Prof. O. P. Singh for Ph.D. thesis guidance. Part of this work was done while the author as research scholar (Ph.D. scholar from January 1, 2006 to June 30, 2009) in the Department of Applied Mathematics, Indian Institute of Technology (BHU), Varanasi, India.

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