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BRIDGING K-12 AND UNIVERSITY MATHEMATICS: BUILDING THE STAIRCASE FROM THE TOP

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Abstract

This article is written to promote a didactic idea of connecting computer-enabled experiential approach to K-12 mathematics with the applied, project-based teaching of undergraduate university mathematics as a way of encouraging students to participate in the STEM (science, technology, engineering, mathematics) workforce of the future. It reviews current body of research on how to bring engineering and science into the K-12 mathematics curriculum. The notion of recursion is used as an illustration of how one can bridge K-12 and university mathematics in the context of STEM education. The ideas presented in this article, though based on a North American experience, can be used within a broader international context.

ZDM Subject Classification (2010): R20

1. Introduction

Nowadays the colleges and universities need to provide more and more graduates knowledgeable in science, technology, engineering, and mathematics (STEM) to meet the challenges posed by the global community [37]. Although a degree in a STEM area can lead to a successful career, in the United States fewer and fewer students choose to enter these areas [28]. According to the report of the National Center for Education Statistics [18] about 4% of the U.S. high school graduates (regardless of gender or ethnicity) will obtain an undergraduate degree in mathematics or physical sciences. The situation with other STEM fields seems to be just a little better. This, obviously worrisome, lack of students' interest in the STEM subjects originates at the pre-college level because their learning experiences have usually been associated with little or no connections to real-life problems and methods of solving them. Without practical examples that motivate problem-solving strategies, most pre-college students of all ages struggle to retain the theoretical lessons they have learned. Furthermore, "curricular materials do not portray engineering [and sciences] in ways that seem likely to excite the interest of students from a variety of ethnic and cultural backgrounds" [31, p. 10]. A reform of the K-12 education with a STEM focus is of crucial importance for educational systems in the U.S. and elsewhere. Schoolchildren begin to form the attitudes towards STEM subjects when they first encounter them at the primary level. The importance of the initial impact on their career goals cannot be overestimated. It is imperative that the primary grades become the starting point in improving the situation with STEM education [37]. Consequently, an intervention at the primary level will facilitate STEM education at the secondary level.

The aim of this article is to illustrate a pedagogical strategy originally introduced elsewhere [8, 9] of linking the application-oriented, computer-enabled experiential approach to K-12 mathematics with the applied, project-based approach to the teaching of university mathematics at the undergraduate level. This strategy is based on the notion that many engineering/science problems that are to motivate the study of undergraduate mathematics and its methods can also be rendered to be adopted at the pre-college level to serve as a motivation for 'big ideas' in mathematics through applications. In what follows, the idea of bridging K-12 and university mathematics education systems will be illustrated by focusing on the concept of recursion. The concept turns out to be enormously fruitful for the study of engineering, computer science, biology, economics, physics, and other disciplines that use mathematical methods. The applications of recursive models to engineering and science require different levels of mathematical competence and technological support.

2. STEM and mathematics teacher education

There is a growing body of research on how to bring engineering and science in the school curriculum [17, 21, 31, 34, 44, 47, 50]. In particular, this research shows the importance of incorporating engineering education into teacher education programs. Indeed, as mentioned in [17], the lack of pre-service teacher preparation in the "E" component of STEM could hamper any efforts of introducing schoolchildren to the ideas that develop the foundation of engineering profession. Therefore, an important task of a mathematics teacher education program is to develop cadre of teachers capable to use the available technology tools in teaching mathematics with focus on engineering/sciences methods. To this end, an appropriate ad-hoc redesign of mathematics teacher education courses can help teacher candidates learn, understand and use more complex concepts.

State University of New York (SUNY) at Potsdam, the first author's workplace, has accumulated a wealth of experience in using technology and context (applications) in teaching mathematics schoolchildren and teacher candidates [1, 3]. At the primary level, over the last 10 years a number of projects aimed at the implicit introduction of higher mathematical concepts through the use of technology and real-life applications to 2nd and 3rd grade students have been carried out in two schools by teacher candidates [4, 6, 7, 13]. Likewise, this approach was used at the secondary level [2, 5, 10, 12] as a capstone project experience for teacher candidates in using spreadsheets and computer algebra systems as a means of reaching the depth of the mathematics curriculum not accessible otherwise. The successful implementation of these projects enabled the efficient redesign of elementary and secondary mathematics teacher education courses at SUNY Potsdam in order to address many ideas of contemporary mathematics pedagogy [19, 20, 36] focusing on applications of technology and problem solving. Still, connecting school mathematics and true science and engineering ideas within a teacher education program remains a challenge.

As mentioned in [24], for teachers to be able to significantly improve teaching practices on a larger scale, a comprehensive enhancement and support system based on the notion of "teachers-mentoring-teachers" should be provided. This system allows for the emergence of alternative beliefs and perspectives, it guides each participating teacher in formulating the new classroom methods, and encourages self-evaluation and reflection. Towards this end, the teacher candidates during their field experience can collaborate with sponsor teachers in the context of professional development school using the model introduced by Holmes Group [27]—a consortium of about 100 major U.S. universities, who put forward a new vision of teacher preparation grounded in comprehensive collaboration of university and public school faculty in teaching and research. If teacher candidates are to become comfortable with STEM-oriented teaching strategies, they need the courses informed by the results of a disciplined inquiry into the pedagogy of STEM subject matters.

3. STEM and teaching undergraduate mathematics

The University of South Florida (USF), the second author's workplace, is continually seeking ties to the community, establishing contacts with industry, and integrating the technology provided by these contacts into the educational experiences of the students. The Mathematics Umbrella Group (MUG) program at USF founded by the second author in 1999 is aimed at bringing active experimental learning into the curricula of some basic mathematics courses (such as engineering calculus and life sciences calculus) for non-mathematics majors. It creates service-learning opportunities for students and organizes popular partnerships between mathematics professors and the non-mathematical community along educational lines. The MUG activities include individual instruction, advising, and double supervision (a mathematician and a non-mathematician) of the mathematics application projects, which involve problems from numerous subject areas, businesses, and organizations. The main goal of these activities is to increase the number of qualified STEM graduates [33].

With a choice between taking a traditional final exam and a real life-based project tailored to students' needs and interests, many students choose the latter. It is important to distinguish between many project-based courses currently available elsewhere in which the projects are pre-selected by the instructor from the projects which are customized to each student with significant student input. From an extensive literature review into conceptually similar teaching approaches, Milligan [35] concluded that project options were unique in their employment at USF. Upon comparing project and non-project (final exam) groups, mathematical proficiency was found to be on a similar level for both groups before starting the project work or final exam preparation; then, project students reported higher levels of course satisfaction and an improved positive perception of mathematics. On average, project students spent about twice as much time preparing their projects than non-project students did in preparing for their final exam. In addition to the project-based instruction in the calculus sections at USF, there are a tutoring center, help sessions (sponsored by graduate students), tutors and peer leaders (high performing undergraduate students). The results of the integrated use of projects and help sessions/tutoring have exceeded all expectations. The USF approach to teaching undergraduate mathematics through applications [25] resulted in over one thousand mathematics application projects completed by undergraduate students. The summaries of those projects can be found at the MUG website [http://ciim.usf.edu/mug].

4. Collaboration as a pedagogical idea

The authors' collaboration is based on the idea of juxtaposing the two university approaches. The USF undergraduate projects provide an untapped source of exciting state-of-the-art examples to stimulate the engineering and scientific curiosity of schoolchildren. For instance, successful businesses rely on the accurate inventories before ordering the additional supplies and explore various population characteristics before opening the additional stores. Therefore, such basic mathematical activities as sorting and counting can be considered the rudiments of business analysis. Distance calculations, extensively used in robotic navigation and collision avoidance, are based on vector subtraction. Furthermore, robotic navigation uses geometry and trigonometry—areas of mathematics studies at the secondary level. Finding the area of a circle and the volume of a cylinder can be considered adaptations of mathematical calculations used in environmental engineering and the restaurant business such as finding area of a restored wetland and the volume of a sniffer glasses to justify the importance of using jiggers, respectively [8]. Recursively counting the population growth of mice (Alzheimer's research), cats (feral cat problem in Florida) and dogs (dog vaccination problem) builds the pattern recognition skills and engages the undergraduates into the study of difference equations—mathematical descriptions of discrete dynamical systems. It should be noted that difference equations are associated with another set of real-life problems arising in radio engineering, communication, and computer architecture research [32]. Therefore, counting techniques based on the idea of recursion should be a part of teacher candidates' (and schoolchildren alike) mathematical experience.

In reality, K-12 teachers, when teaching mathematics, a subject matter increasingly seen as a core to academic success across the curriculum [48], often underutilize connections to real life. This stunts schoolchildren's mathematics learning by ignoring the fact that mathematical concepts historically emerged in response to the need to solve practical problems. Many existing K-12 curricula materials superficially address the traditional concern of students that mathematics they are learning would be useful in their lives and do not provide students with a learning environment that encourages and promotes career paths in the STEM area. A logical place to intervene is at the pre-college level when students' career goals are still evolving. Such an intervention, proceeding from the SUNY Potsdam experiential, computer-enabled approach to teaching mathematics to teacher candidates can be enriched and broadened by simplifying engineering and science problems associated with higher mathematical concepts. This leads to comprehensive, application-oriented teaching of K-12 mathematics that is consistent with the view, "when a school subject is taught for which there is a professional counterpart, there should be a conceptual connection to post-secondary studies and to the practice of that subject in the real world" [31, p. 4].

5. A research-based rationale for the collaboration

According to [31], a very few projects exist which describe how students use mathematical models in designing the solutions to problems. This is unfortunate as mathematics has the capability to minimize trial and error methods through modeling techniques. In theory, if students are taught mathematical concepts in the context of solving problems with an engineering/science focus, they will understand mathematics more easily. Current standards for technological literacy [29] describe engineering design as a purposeful (often collaborative) activity with an explicit goal, shaped by specifications and constraints, leading to multiple solutions through systematic reasoning and an iterative process. This is exactly what can be fostered in schoolchildren through grade-appropriate computer-enhanced mathematical activities with an engineering/science focus. By modeling mathematical concepts in a computer environment, teachers and their students can develop the powerful learning experiences in the STEM disciplines that are very different from the traditional educational practices with little or no connection to real life and without using technology as a conceptual tool.

As mentioned in [45], mathematics can be defined as a process of thinking that involves developing and applying abstract, logically interconnected ideas that often arise from the need to solve problems in science, technology, and everyday life—the problems ranging from how to model certain aspects of a complex scientific problem to how to balance a checkbook. One of the main principles that underpin the authors' collaboration is that mathematical abilities can be identified and developed earlier than the other abilities in the STEM area [41, 42]. Research suggests that, in a broad sense, engineering skills represent a combination of mathematical abilities and interest in technical problems [51, 52]. Suppose, for example, a child is interested in the very design of a computer. If a child has mathematical abilities—she/he has a great potential to become an engineer or scientist. Therefore, mathematical abilities can be utilized in the STEM education as a basis for the development of engineering/science skills.

One of the benchmarks for literacy in the STEM area conceptualizes a mathematical model as a tool that enables the demonstration and observation of different phenomena that go beyond one's intuition [15]. Perhaps this is one of the reasons why tertiary mathematics courses taught with an emphasis on engineering applications are considered as useful experiments [46]. As a current research on teaching engineering indicates, it is important to demonstrate the learners that good problems, being grounded in context, have many possible solutions found through a number of iterations [31] and that experiments appearing to be simple may not be simple at all [51]. Such demonstration can first be presented through the teaching of mathematics (the problems of which inherently have more than one solution) by extending seemingly simple mathematical explorations to include more and more challenging situations. That way, schoolchildren can be introduced to and even have an experience with intuitive discoveries in the STEM area. This can be accomplished through the use of curriculum materials emphasizing the role of mathematical model in testing the potential designs.

As mentioned in [46] in the context of preparing engineers, the use of technology as a pedagogical tool fosters concept learning. Consider an electronic spreadsheet. Already in the 1990s, a spreadsheet was singled out as a teaching tool for all engineering/science disciplines [23, 52]. Nowadays, facility at creating spreadsheets is required in many entry-level positions in industry for high-school graduates [20]. Moreover, the *Principles and Standards for School Mathematics* [36] include a recommendation to use a spreadsheet in open-ended problem-solving situations. In addition, many mathematics education researchers made a similar recommendation advocating a spreadsheet as a scaffolding device for the learning of mathematics beginning from the primary grades [40]. Indeed, evidence collected by the first author with collaborators in the past decade [6, 7, 13] strongly suggests that young children have a great potential to acquire quickly the spreadsheet-based skills. In the context of the authors' collaboration, the use of a spreadsheet is particularly advantageous as it includes many features conducive for exploring the engineering/science concepts, which are inherently iterative, interactive, and complex.

Another technological tool that all teacher candidates should learn to use is *The Geometer's Sketchpad* – a dynamic geometry program commonly available at schools across North America. The appropriate use of this computer application allows for the development of the basic geometric skills that can be transformed into the advanced mathematical skills important for engineering applications through the repeated application in purposeful contexts. Like spreadsheets, dynamic geometry software includes many features that support posing the problems and the interactive design of solutions as the center of professional activity of an engineer [46]. Alternatively, one could use the recently developed (and free) software *GeoGebra*. At the secondary level, in addition to the above-mentioned tools, teacher candidates can learn to use computer algebra systems such as *Maple*, *Wolfram Alpha*, and *The Graphing Calculator* [16] to name just most commonly available computer applications.

6. Connecting concepts across the K-16 mathematics curriculum

Through the authors' collaboration two different routes connecting higher and lower level mathematical concepts as a pedagogical method in promoting STEM education have emerged. One route, that can be referred to as conceptual ascend, is to move gradually from early to upper grades along the K-16 staircase by extending the concepts at each step and motivating such extension through a concrete activity. Through this process, a student accumulates new mathematical knowledge and STEM-related skills that can be applied later to the study of problems in science and engineering. In other words, "What students can learn at any particular grade level depends upon what they have learned before" [19, p. 5].

Another route, that can be referred to as conceptual descend, is to start with a non-trivial concept/problem encountered in the practice of science or engineering and gradually split it into special cases each of which brings about a grade-appropriate mathematical task that can be explored with or without a reference to the original problem. The method of conceptual descend motivates students to think about learning more mathematics in order to solve the problems by eliminating the simplified conditions. Jointly, conceptual ascend and descend form a bidirectional concept map of the whole K-16 mathematics teaching (Figure 1).

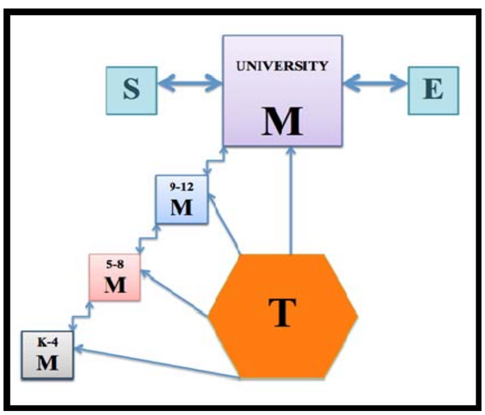


Figure 1. STEM education concept map.

The map intends to demonstrate that the right steps at the base of the staircase are informed by what one needs at the top of the staircase where mathematics knowledge develops through a true interaction with science and engineering. Conversely, it is due to the right steps that one makes at the bottom of the staircase, a success at its top is possible. Only very few individuals can reach successfully the top of the staircase without the appropriate educational support. By analyzing and conceptualizing the success of these *few*, technology-enhanced mathematics education system oriented towards science and engineering that enables such success for *all* can be developed. Put another way, when a child first learns to say "thank-you" he or she does not know how helpful in the adult life this pair of words could (and would) be. Yet, the realization of the usefulness of this simple communicative tool becomes more and more apparent within more and more contexts as the child grows physically and develops cognitively.

7. Illustration: Developing recursive reasoning through conceptual ascend

As was mentioned above, many authentic problems arising in science and engineering are described by difference equations—recursive formulations of discrete concepts. In this section, it will be demonstrated that discrete concepts can be encountered throughout the whole pre-college mathematics curricula. When taught appropriately, that is, through making bidirectional connections across the K-12 mathematics conceptual map, they can provide students with knowledge necessary for dealing with real-life problems involving difference equations. One such problem, offered to an undergraduate student at USF in the context of project-based learning of mathematics, is discussed at the end of this section. Prior to this discussion, it will be shown how knowing what is needed at the top of the staircase can influence activities along the whole staircase beginning from its first steps.

7.1. Reduction to a simpler problem. The ability to reason recursively can be fostered at the pre-operational level by emphasizing a reduction to a simpler problem as an effective counting method. Pólya [43] called this problem-solving strategy ‘try a simpler problem.’ Unfortunately, some schoolteachers discourage their students to use this strategy by insisting on solving the given problem and seeing one’s reduction efforts as an unnecessary distraction.

Already at the kindergarten level, when building different towers out of three different color linking cubes [39, p. 21], a child can be helped to recognize that there are three pairs of towers that differ by the color of the top/middle/bottom cube (Figure 2). Each such pair differs by the arrangement of two cubes only. That way, a systematic reasoning of reducing a problem with three cubes to three problems with two cubes can be seen as a rudiment of recursive reasoning.

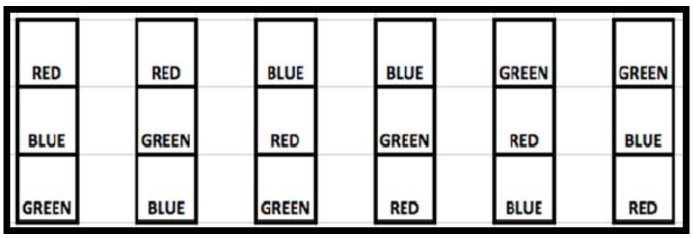


Figure 2. Reduction to a simpler problem.

7.2. Generating counting numbers within a spreadsheet. Noting that each counting (natural) number is one greater than the previous number and the first number is equal to one, a second-grader can use a spreadsheet (Figure 3) to conceptualize the recursive nature of counting numbers by using the so-called *ostensive* definition. This definition is based on the pointing at the first term (entered into cell **A1**: **=1**) and defining the second term in cell **B1** as the previous term plus one (**B1: =A1+1)**. Replicating the formula to the right across the columns to cell **J1** yields a spreadsheet representation of the first ten counting numbers (Figure 3). In the higher grades, the definition of counting numbers in the form of the first-order recurrence $x_{n+1} = x_n + 1, \quad x_1 = 1$, can be introduced with recourse to a spreadsheet. Similarly, other arithmetic sequences (e.g., even and odd numbers) can be generated through a recursive definition by using a spreadsheet. This prepares students for the appropriate use of a spreadsheet in modeling other, more complicated recursive relations. Moreover, a spreadsheet, when used in the context of generating recursively defined sequences, creates background over which mathematics and computer science intersect.

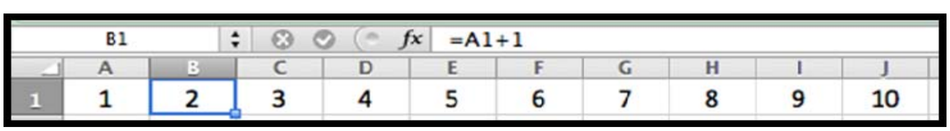


Figure 3. Generating counting numbers through ostensive definition.

7.3. Recognizing recursion in the multiplication table. The skill of interpreting and manipulating data presented in a tabular form is important for engineering/science applications. The addition and multiplication tables are the first experiences of that kind for young children. In particular, the concept of recursion can be emphasized through the study of the multiplication table (that begins in grade three) in which each product can be developed by iterating one of its factors [3]. To clarify, note that by setting $P(x,y)=xy$ each entry of the multiplication table can be expressed in the form of the following recursive definitions

$$P(x,y)=P(x-1,y)+y,\ P(1,y)=y, \tag{1}$$

$$P(x,y)=P(x,y-1)+x,\ P(x,1)=x\,. \tag{2}$$

Indeed, the boundary condition $P(1,y)=y$ means that one group of *y* objects contains *y* objects; the boundary condition $P(x,1)=x$ means that *x* groups of single objects contain *x* objects. Repeatedly adding a group of *y* objects to have *x* such groups, in other words, by iterating *y* until the product $P(x,y)$ defined by formula (1) is reached, the multiplication table can be developed. Likewise, repeatedly adding a group of *x* objects to have *y* such groups can be interpreted as the iteration of *x* until the product $P(x,y)$ defined by formula (2) is reached. In other words, counting by *y*'s is an iteration by *y* and counting by *x*'s is an iteration by *x*. Figure 4 shows the case of $16=4\times 4=4\times(4-1)+4$

y\x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	**16**	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Figure 4. Iterative structure of the multiplication table.

Also, the numbers 1, 4, 9, 16, 25, 36 in the main diagonal of the multiplication table (Figure 4) develop recursively. Seeing the table as a checkerboard, this phenomenon can be given a geometric interpretation: a transition from a square of the side length *n* (the product $n\times n$) to that of *n* + 1 (the product $(n+1)\times(n+1)$) requires augmentation by two rectangles that correspond to the products $n\times 1$ and $(n+1)\times 1$. For example, the transition from 9 to 16 can be interpreted through the chain of equalities

$$4\times 4=3\times 3+3\times 1+4\times 1=3\times 3+2(3\times 1)+1.$$

In other words, the relation $(n+1)^2=n^2+2n+1$—a basic identity studied in algebra—can be interpreted as the recursive relation $s_{n+1}=s_n+2n+1,\ s_1=1$, through which the numbers in the main diagonal of the multiplication table develop. A pictorial representation of this recurrence is shown in Figure 6 (right).

7.4. Counting matchsticks through recursion. Counting activities are among the most fundamental ones in mathematics and, as mentioned above, can be considered as rudiments of business analysis. Recursive reasoning can be effectively applied to the counting context as well. For example, counting matchsticks used to construct a square (or rectangular) grid [26] is another problem that can be used in developing recursive reasoning already at the upper elementary level. For example, if M_n is the number of matchsticks used to construct the $n\times n$ grid, then

$$M_n = M_{n-1} + 4n,\ M_1 = 4 . \qquad (3)$$

In Figure 5 (which prompts a connection to the tiling problems arising in the construction industry), *n* = 3. A useful mathematical exercise is to develop a transition from recursive formula (3) to a formula that expresses M_n as a function of *n*. Such formula has the form

$$M_n = 2n(n+1) \qquad (4)$$

and it can be proved by the method of mathematical induction, another valuable tool of discrete mathematics. Towards this end note that when $n=1$ formula (4) yields $M_1 = 4$. Assuming that formula (4) holds true for *n* = *k*, proving that it is true for *n* = *k* + 1, that is, $M_{k+1} = 2(k+1)(k+2)$, would imply that formula (4) is true for any natural number *n*. Indeed, it follows from formula (3) and the inductive assumption that

$$M_{k+1} = M_k + 4(k+1) = 2k(k+1) + 4(k+1) = 2(k+1)(k+2) .$$

This completes mathematical induction proof of formula (4).

Note that the number of shared edges can be interpreted as the number of matchsticks if the tiles are linked like the linking cubes in the first example (section 7.1).

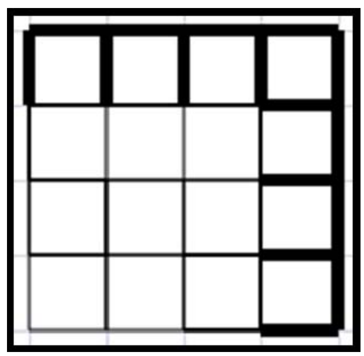


Figure 5. Counting matchsticks through recursion.

7.5. Counting activities on a geoboard. A grid made of matchsticks can then be turned into a geoboard—the basic learning environment for exploring geometry across the grades. As was mentioned above, among many applications of geometry are problems arising in robotic navigation. Utilizing a geoboard, one can count other entities associated with the geometric objects constructed. For example, using either physical or computational geoboard, one can construct evolving isosceles triangles and squares and count pegs on the border and inside each of the shapes (Figure 6). Both the smallest isosceles triangle and square have four pegs on their border and no internal pegs. As the triangles and squares grow, so do the number of border and the number of interior pegs. Both geometric shapes and the number of pegs associated with them develop recursively. The sequences of border pegs, B_n, and internal pegs, I_n, are 4, 8, 12, 16, 20, 24, ... and 0, 1, 4, 9, 16, 25, ... , respectively. One can recognize a recursive nature of these sequences and thus the corresponding definitions can be developed: $B_{n+1} = B_n + 4,\ B_1 = 4$, and $I_{n+1} = I_n + 2n - 1,\ I_1 = 0$. From here, closed formulas for B_n and I_n can also be derived: $B_n = 4n$ and $I_n = (n-1)^2$.

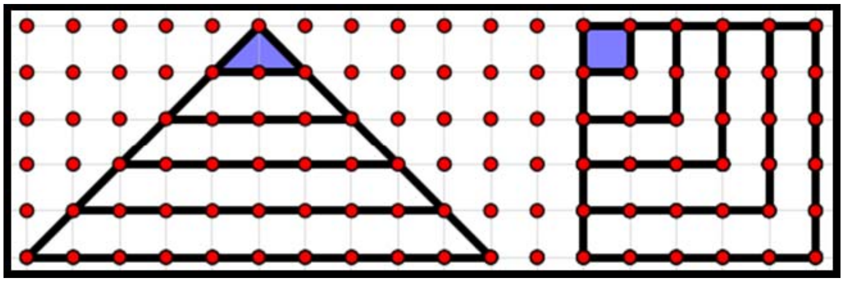


Figure 6. Evolving isosceles triangles and squares on a geoboard.

Equilateral triangles, like squares, develop recursively as a triangle of side length *n* is augmented by an isosceles trapezoid with the bases *n* and *n* + 1 and height $\frac{\sqrt{3}}{2}$. These explorations can be carried out by the joint use of a spreadsheet and *The Geometer's Sketchpad*. It should be noted that a grid can be used in calculating areas of irregular shapes through an approximation—a problem frequently arising in engineering contexts.

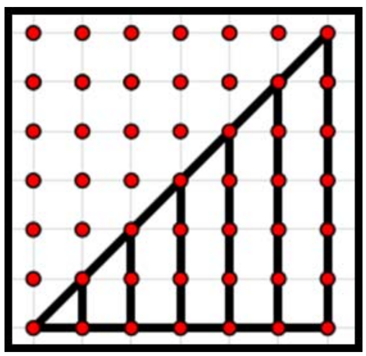


Figure 7. Evolving right-angled triangles on a geoboard.

7.6. Recursion and polygonal numbers. A different number sequence can be developed if one considers evolving right-angled triangles on a geoboard and counts all pegs associated with a triangle at each step without distinguishing between border and internal pegs. The sequence 1, 3, 6, 10, 15, 21, ... , known already at the elementary level [38, p. 14] as triangular numbers, also develops recursively through the formula $t_n = t_{n-1} + n, t_1 = 1$. Another related sequence is 1, 4, 9, 16, 25, ... , called square numbers, has the following recursive representation $s_n = s_{n-1} + 2n + 1, s_1 = 1$. Just like counting numbers, both sequences can be generated within a spreadsheet through the ostensive definition by pointing at the appropriate cells. Their closed formulas can be found by using computer algebra system *Maple* as shown in Figure 8. It is important to emphasize that triangular numbers t_n are the sums of the first *n* counting numbers (as shown in Figure 7 for *n* = 7)

$$t_n = \sum_{k=1}^{n} k . \tag{5}$$

Likewise, square numbers s_n are the sums of the first *n* odd numbers (as shown in Figure 6 for *n* = 6)

$$s_n = \sum_{k=1}^{n} (2k-1) . \tag{6}$$

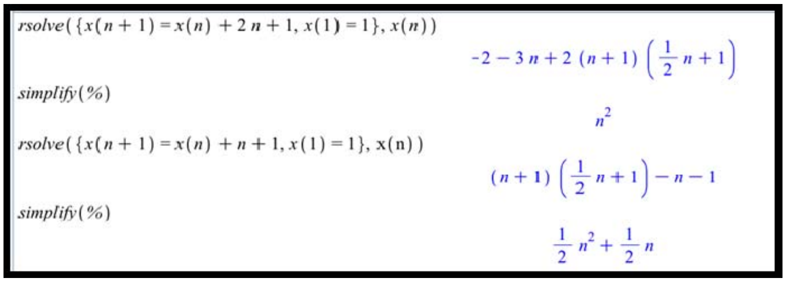


Figure 8. Computing closed formulas using *Maple*.

Interestingly, both sequences can be described through the same third-order recurrence

$$x_n = 3(x_{n-1} - x_{n-2}) + x_{n-3} \tag{7}$$

with initial conditions $x_1 = 1, x_2 = 3, x_3 = 6$ for t_n and $x_1 = 1, x_2 = 4, x_3 = 9$ for s_n. A geometric representation of relation (7) in the case of the fourth triangular number t_4 is shown in Figure 9.

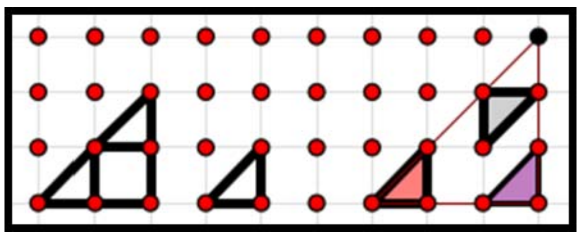
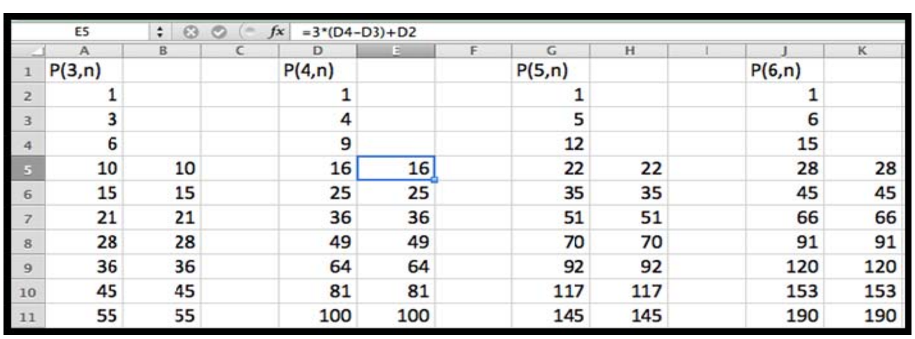


Figure 9. Recursive relation (7) for the fourth triangular number.

In fact, recurrence (7) holds true for all polygonal numbers $P(m, n)$ with the initial conditions $x_1 = P(m,1), x_2 = P(m,3), x_3 = P(m,3)$. At the same time, a closed formula for a polygonal number of side m and rank n is

$$P(m,n) = \frac{n(n-1)}{2}(m-2) + n .$$

The spreadsheet pictured in Figure 10 provides numerical evidence of the equivalence of different symbolic representations of polygonal numbers for m = 3, 4, 5, 6. This example, in particular, demonstrates the importance of initial conditions in defining recursive relations.



E5 =3*(D4-D3)+D2

	A	B	C	D	E	F	G	H	I	J	K
1	P(3,n)			P(4,n)			P(5,n)			P(6,n)	
2	1			1			1			1	
3	3			4			5			6	
4	6			9			12			15	
5	10	10		16	16		22	22		28	28
6	15	15		25	25		35	35		45	45
7	21	21		36	36		51	51		66	66
8	28	28		49	49		70	70		91	91
9	36	36		64	64		92	92		120	120
10	45	45		81	81		117	117		153	153
11	55	55		100	100		145	145		190	190

Figure 10. Relation (7) for triangular, square, pentagonal and hexagonal numbers.

7.7. Pick's formula as a double recursion. Let A_n be the area of an isosceles triangle/square with B_n and I_n pegs. Then, according to Pick's formula [22],

$$A_{n+1} = \frac{B_n}{2} + I_n + 2n,\ B_1 = 4,\ I_1 = 0,$$

where $B_{n+1} = B_n + 4,\ B_1 = 4$, and $I_{n+1} = I_n + 2n - 1,\ I_1 = 0$. One can see how a recursive formula may include components which satisfy recursive definitions as well (compare to section 9.11.1 below).

7.8. Fibonacci numbers. One of the most common uses of a computer in high school mathematics for demonstrating the recursive nature of calculations is to generate Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, 21, ... within a spreadsheet. This remarkable number sequence and its recursive structure can be introduced earlier than it is being done now (i.e., earlier than in high school) through the following hands-on task [3, p. 114] (that incorporates manupulative materials familiar to young learners):

Find the number of different arrangements of one, two, three, four, and so on two-sided (red-yellow) counters in which no two red counters appear in a row.

Indeed, the number of such arrangements of, say, four counters is the sum of those with two and three counters, respectively (Figure 11). Here the idea of reduction to a simpler problem, already familiar from the kindeegarten level, comes into play. Just like all the towers constructed out of three cubes of different color were put in three groups depending on the color of the top cube, all the arrangements of four counters can be put in two groups depending on the colour of the far-right counter. When the far-right counter is red, its immediate neighbour has to be yellow; thus only two counters have to be arranged according to the rules of the task; this can be done in three ways. When the far-right counter is yellow, its immediate neighbour may be of either colour. This implies that the number of the arrangements of counters in the second group equals to that with three counters. In a numeric form, the diagram of Figure 11 can be expressed as $8 = 3 + 5$. Obvisously, there are two ways to arrange one counter; therefore, beginning from the third number, Fibonacci numbers provide the solution to the above hands-on task.

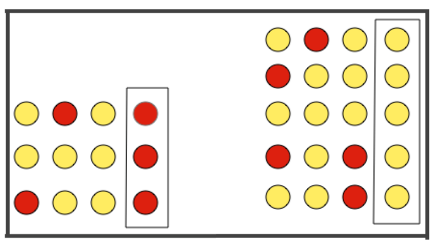


Figure 11. Reduction to a simpler problem: 3 + 5 = 8.

The recursive structure of Fibonacci numbers can be described in the form of the following second-order difference equation

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2 \text{ and } F_0 = F_1 = 1. \qquad (8)$$

Relation (8) can be shown to describe the growth of the population of rabits breeding in ideal circumstances, something that connects recurrences to a classic real-life problem (studied by Fibonacci in the 13th century) though not necessarily of an authentic nature. However, the realization of such a connection prepares studens for work on authentic research problems dealing with the calculation of growth of different speices like the one described below at the conculusion of the conceptual ascend. Furthermore, in the context of Fibonacci numbers one can learn about the qualitative difference between recursive and closed formulas for discrete concepts—while the recursive form may look rather simple, the closed form may be very complicated or even unknown. In particular, a closed formula for Fibonacci numbers (the derivation of which is included in section 7.10) has the form

$$F_n = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1}],\ n = 0, 1, 2, \ldots \qquad (9)$$

Formula (9) is commonly referred to as Binet's formula named after Jacques Binet—a French mathematician of the 19th century. One can be surprised to see that the computerization (e.g., using a spreadsheet) of formula (9) produces whole numbers only.

7.9. Recursion in Pascal's triangle. The right triangle pictured in Figure 12 can be turned into Pascal's triangle if one counts the number of ways to reach an internal peg starting from the far-left vertex when travelling either North or East only. For example, in order to reach any *internal* peg in the second row of the pegs (to which pegs A and B belong) requires one to make two decisions: where to go North and where to go East. In particular, there are $C_5^2 = 10$ ways to reach peg A (column 5 and row 2) and $C_4^2 = 6$ ways to reach peg B (column 4 and row 2). Similarly, one has to make three choices to reach a peg in the third (internal) row. In particular, there are $C_4^3 = 4$ ways to reach point D. In general, the peg (*n*, *k*), $n \geq k$, located at the intersection of the *n*-th internal column and the *k*-th internal row of the evolving triangle shown in Figure 12 can be reached in C_n^k ways. The recursive development of the numbers C_n^k known as the binomial coefficients can be revealed through the understanding that one can reach the peg (*n*, *k*) in the following two ways—either through the peg (*n* – 1, *k*) or through the peg (*n* – 1, *k* – 1). That is, the relation

$$C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$$

betweeen the entries of Pascal's triangle holds true.

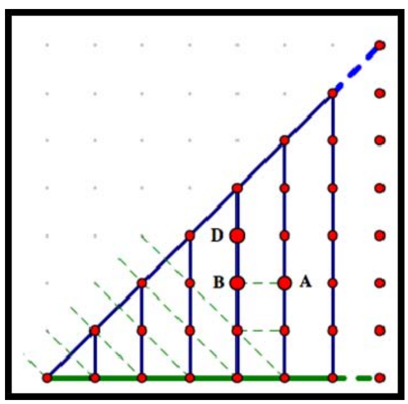


Figure 12. Pascal's triangle on a geoboard.

Finally, by travelling North-West only strating from a bottom border peg of the triangle and adding the binomial coefficients located at the pegs which are being passed, one can arrive at Fibonacci numbers F_n and through this process the identity

$$C_n^0 + C_{n-1}^1 + C_{n-2}^2 + ... + C_{n-r}^r = F_n, \; n = 0, 1, 2, ... , \tag{10}$$

where $r = [\frac{n}{2}]$, can be developed.[1]

[1] Hereafter, [*x*] denotes the largest integer smaller than or equal to *x*.

7.10. Fibonacci numbers revisited. Consider the function $F(x)=\sum_{n=0}^{\infty}F_n x^n$, where F_n are Fibonacci numbers satisfying recurrence relation (8). Noting that $\sum_{n=2}^{\infty}F_{n-1}x^{n-1}=\sum_{n=1}^{\infty}F_n x^n=F(x)-1$ and $\sum_{n=2}^{\infty}F_{n-2}x^{n-2}=\sum_{n=0}^{\infty}F_n x^n=F(x)$, one can write

$$F(x)=1+x+\sum_{n=2}^{\infty}(F_{n-1}+F_{n-2})x^n=1+x+x\sum_{n=2}^{\infty}F_{n-1}x^{n-1}+x^2\sum_{n=2}^{\infty}F_{n-2}x^{n-2}$$
$$=1+x+x[F(x)-1]+x^2F(x)=1+xF(x)+x^2F(x).$$

Hence, using the rule of summation of geometric series, the binomial theorem, and the substitution $n = k + r$ yields the so-called generating function for Fibonacci numbers

$$F(x)=\frac{1}{1-(x+x^2)}=\sum_{k=0}^{\infty}x^k(1+x)^k=\sum_{k=0}^{\infty}x^k\sum_{r=0}^{k}C_k^r x^r=\sum_{n=0}^{\infty}x^n\sum_{r=0}^{[n/2]}C_{n-r}^r\,. \quad (11)$$

Thus, due to the uniqueness of the power series expansion about $x = 0$,

$$F_n=\sum_{r=0}^{[n/2]}C_{n-r}^r\,. \quad (12)$$

Formula (12) is the exact replica of formula (10) already discovered within Pascal's triangle.

In addition, using the function $F(x)$ defined in (11) makes it possible to derive formula (9) without much difficulty. To this end, note that the characteristic equation of difference equation (8) has the form $x^2-x-1=0$ the roots of which are $x_1=\frac{1+\sqrt{5}}{2}$ and $x_2=\frac{1-\sqrt{5}}{2}$. Due to the following factorization

$$1-x-x^2=x^2[(x^{-1})^2-(x^{-1})-1]=x^2(x^{-1}-x_1)(x^{-1}-x_2)=(1-x_1x)(1-x_2x)$$

and the equality $x_1-x_2=\sqrt{5}$, the function $F(x)$ can be decomposed in partial fractions as follows

$$F(x)=\frac{1}{1-x-x^2}=\frac{1}{(1-x_1x)(1-x_2x)}=\frac{1}{\sqrt{5}}\left(\frac{x_1}{1-x_1x}-\frac{x_2}{1-x_2x}\right).$$

Using, once again, the rule of summation of geometric series yields

$$F(x)=\frac{x_1}{\sqrt{5}}\sum_{n=0}^{\infty}(x_1x)^n-\frac{x_2}{\sqrt{5}}\sum_{n=0}^{\infty}(x_2x)^n=\sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]x^n.$$

As $F(x)=\sum_{n=0}^{\infty}F_n x^n$, the uniqueness of the power series expansion about $x = 0$ implies formula (9). One can also establish formula (9) through the direct use of the characteristic equation rather than using the method of generating functions. Namely, formula (9) can be found as the unique linear combination of x_1^n and x_2^n that satisfies the initial conditions $F_0=F_1=1$.

Furthermore, any Fibonacci number can be represented through a sum of consecutive Fibonacci numbers. Using mathematical induction, one can prove that

$$F_{n+2}=\sum_{l=0}^{n}F_l+1. \quad (13)$$

Setting n = 0 in (13) yields $F_2 = F_0 + 1$—a true identity. Assuming (13) to be true for $n = k$, one can show that it remains true for $n = k + 1$. Indeed,

$$F_{k+3} = F_{k+2} + F_{k+1} = \sum_{l=0}^{k} F_l + 1 + F_{k+1} = \sum_{l=0}^{k+1} F_l + 1.$$

In other words, any Fibonacci number is a linear combination of a sum of consecutive Fibonacci numbers starting from one.

7.11. Recurrence relations in authentic science research. Finally, we present a modified version of the mouse population project associated with Alzheimer's research. The original project (see [25] for details) deals with the study of transgenic mice population and focuses, given a budget limitation, on a financial feasibility of purchasing two parent mice (male and female) and raising a population of mice of a specified size. To this end, one should come up with a recursive algorithm defining the population size as a function of time, determine the amount of time required for the population of mice to reach a given quantity, and see if one can afford the resulting population of offspring for this amount of time. Laboratory data lead to several assumptions that underlie an idealized mathematical model, namely: the average length of gestation period (20 days), its relation to the period of sexual maturation (60 days), and the litter size of offspring (3 mails and 3 females). Under these assumptions, one can easily find an approximate "power" solution to the mice population problem. However, an effective approach to this problem, which leads to the closed formula for the mice population, involves the theory of recurrence relations.

7.11.1. Developing a recursive model. To begin, note that the population of mice depends on one's interpretation of the initial purchase of the parent couple. This interpretation would be reflected in the initial conditions of a recursive model to be constructed. Namely, one can buy either two separate adult mice (male and female) or a family couple. In the former case, six babies will appear in 20 days according to the considered model, in the second case, it can happen earlier (actually, it can be any number of days not greater than 20). The solution that follows is based on the case of two separate adult mice purchase[2].

Let m_n and f_n be, respectively, the number of mice and reproducing females at the time t_n. The time unit is 20 days, thus $t_n = 20n$. Because each of the f_{n-1} mature (reproducing) female gives birth to six mice at the time t_n, the equation

$$m_n = m_{n-1} + 6 f_{n-1}, \quad m_0 = 2,\ f_0 = 1 \tag{14}$$

is a recurrence relation for the number of mice at that point of time. From (14), it follows that

$$m_n = 2 + 6\sum_{k=0}^{n-1} f_k,\ \ n = 1, 2, \ldots . \tag{15}$$

Indeed,

$$m_n = m_{n-1} + 6 f_{n-1} = m_{n-2} + 6 f_{n-2} + 6 f_{n-1} = m_{n-3} + 6 f_{n-3} + 6 f_{n-2} + 6 f_{n-1}$$
$$= m_0 + 6(f_0 + f_1 + ... + f_{n-1}) = 2 + 6\sum_{k=0}^{n-1} f_k \ .$$

[2] Some "average" initial conditions generated by the case of the parent couple purchase are used in [25]. Obviously, the complexity of a recurrence model increases if additional restrictions are to be taken into consideration. For example, one can consider the case when reproducing females die by the end of the sixth reproduction period.

Thus, the problem of determining the population size m_n at the time t_n is reduced to the need to find the sequence f_n. Because it takes three time periods for a female to mature, the equation

$$f_n = f_{n-1} + 3f_{n-3}, \quad f_1 = f_2 = f_3 = 1, \quad n = 4,5,... \ . \tag{16}$$

is a recurrence relation for the number of mature females at the time t_n. Indeed, because f_{n-3} represents those females who were mature at the time t_{n-3}, only half of their offspring (that is, three females) are able to reproduce at the time t_n. Also, the reproducing females (except the original purchased mouse) cannot immediately become impregnated for they need 60 days to mature.

One can use relations (14) and (16) to prove by induction that

$$m_n = 6 + 2f_{n+2}\ , \quad n = 1, 2, \dots \ . \tag{17}$$

Indeed, setting n = 1 in (17) implies $m_1 = 8$ because, according to (16), $f_3 = 1$. It follows from (14) that $m_1 = 8$, thus formula (17) holds for n = 1. If (17) holds for $n = k \geq 1$, i.e., assuming $m_k = 6 + 2f_{k+2}$ for $k \geq 1$ and then setting $n = k+1$ in (14) and $n = k+3$ in (16) yields

$$m_{k+1} = m_k + 6f_k = 6 + 2f_{k+2} + 6f_k = 6 + 2(f_{k+2} + 3f_k) = 6 + 2f_{k+3}.$$

This completes inductive proof of formula (17).

Furthermore, it follows from (15) and (17) that

$$f_{n+2} = 3\sum_{k=0}^{n-1} f_k - 2, \quad n = 1,2,3,... \ . \tag{18}$$

Formula (18) may be considered as an analogue of formula (13) established for Fibonacci numbers. One can explore if the representation of recursively defined sequences as a linear combination of a sum of its consecutive terms starting from the first one holds true in other cases. For example, given positive integer *n*, one can develop an algorithm of finding the smallest triangular number (the sum of consecutive integers starting from one) greater than or equal to *n*. As discussed elsewhere [1], finding such a triangular number may be motivated by a real-life context.

7.11.2. Spreadsheet modeling approach. Note that recurrence relations (14) and (16) (as well as (16) and (17)) can be modeled numerically within a spreadsheet directly without using any additional mathematical tools. Here are some values of f_n and m_n, respectively, for n = 0, 1., …, 10: 1, 1, 1, 1, 4, 7, 10, 22, 43, 73, 139; and 2, 8, 14, 20, 26, 50, 92, 152, 284, 542, 980. These numbers will be used below to verify theoretical results.

7.11.3. Generating function approach. The method of generating functions, already employed in section 7.10, can be used again to express f_n and m_n algebraically through the binomial coefficients. Let $f(x) = \sum_{n=0}^{\infty} f_n x^n$. Towards this end, using recurrence relation and initial conditions defined by (16) we have

$$\begin{aligned} f(x) &= 1 + x + x^2 + x^3 + \sum_{n=4}^{\infty}(f_{n-1} + 3f_{n-3})x^n \\ &= 1 + x + x^2 + x^3 + x\sum_{n=3}^{\infty} f_n x^n + 3x^3\sum_{n=1}^{\infty} f_n x^n = 1 - 3x^3 + (x + 3x^3)\, f(x)\,. \end{aligned} \tag{19}$$

From (19), using the corresponding geometric series, binomial theorem, and the substitution $n = k + 2r$, we obtain the generating function for the sequence f_n as follows

$$f(x)=\frac{1-3x^3}{1-(x+3x^3)}=(1-3x^3)\sum_{k=0}^{\infty}x^k(1+3x^2)x^k$$
$$=(1-3x^3)\sum_{k=0}^{\infty}x^k\sum_{r=0}^{k}C_k^r 3^r x^{2r}=(1-3x^3)\sum_{n=0}^{\infty}x^n\sum_{r=0}^{[n/3]}C_{n-2r}^r 3^r \; . \tag{20}$$

That is,

$$f(x)=(1-3x^3)\sum_{n=0}^{\infty}x^n\sum_{r=0}^{[n/3]}C_{n-2r}^r 3^r \, . \tag{21}$$

Formula (21) implies the following representation of f_n through binomial coefficients

$$f_n=\sum_{r=0}^{[n/3]}C_{n-2r}^r 3^r-3\sum_{r=0}^{[n/3]-1}C_{n-3-2r}^r 3^r$$
$$=C_{n-2[n/3]}^{[n/3]}\cdot 3^{[n/3]}+\sum_{r=0}^{[n/3]-1}(C_{n-2r}^r-3C_{n-1-2r}^r)3^r, \tag{22}$$
$$n=0,1,2,... \, .$$

In turn, relations (17) and (22) imply the binomial presentation for m_n

$$m_n=6+2C_{n+2-2[(n+2)/3]}^{[(n+2)/3]}\cdot 3^{[(n+2)/3]}+2\sum_{r=0}^{[(n+2)/3]-1}(C_{n+2-2r}^r-3C_{n-1-2r}^r)\cdot 3^r, \tag{23}$$
$$n=0,1,2,.... \, .$$

For instance, plugging n = 5 into (23) yields

$$m_5=6+2C_3^2\cdot 3^2+2[(C_7^0-3C_4^0)\cdot 3^0+(C_5^1-3C_2^1)\cdot 3]=6+54-10=50\, .$$

As was shown above, spreadsheet modeling of relations (14) and (16) gives the same result. Furthermore, approaches to the relation (23) can be modeled within a spreadsheet to confirm numerically two different calculation of the mice population size. Note that a sum is considered to be equal zero when its upper bound is less than its lower bound thus making formulas (22) and (23) work for n = 0.

7.11.4. Characteristic equation approach. To obtain formulas for m_n and f_n without the sums of binomial coefficients, the characteristic equation method can be used. Just like in the case of Fibonacci numbers, this standard approach to solving recurrence relations allows one to represent f_n in (16) as a linear combination of the n^{th} powers of the roots of the auxiliary (characteristic) equation

$$x^3=x^2+3\, . \tag{24}$$

Equation (24) has three roots: one real root a and two complex conjugates b and $\bar{b}$ (see, e.g., [30] and also [25]). Solving the cubic equation yields the real root

$$a=\frac{y^{1/3}+1+y^{-1/3}}{3}, \text{ where } y=41.5+4.5\sqrt{85}, \tag{25}$$

and then the complex root b as a function of a

$$b=\frac{a-a^2+i\sqrt{9a+3}}{2a} \qquad . \tag{26}$$

Note that (26) is easily implied by the factorization $x^3-x^2-3=(x-a)[x^2+(a-1)x+\frac{3}{a}]$ and the identity $a^3=a^2+3$.

Representing f_n as a linear combination of the n^{th} powers of *a*, *b*, and $\overline{b}$ yields

$$f_n=ua^n+vb^n+w\overline{b}^{\,n}.$$

As f_n is real for all *n*, it follows that

$$f_n=\Re f_n=\Re x_1\cdot a^n+\Re(x_2b^n+x_3\overline{b}^{\,n})=\Re x_1\cdot a^n+\Re[(x_2+\overline{x}_3)]b^n].$$

Setting $\alpha=\Re x_1$ and $\beta=x_2+\overline{x}_3$ yields

$$f_n=\alpha a^n+\Re(\beta b^n), \qquad n\ge 1 , \tag{27}$$

$$\alpha a+\Re(\beta b)=\alpha a^2+\Re(\beta b^2)=\alpha a^3+\Re(\beta b^3)=1 . \tag{28}$$

The exact formulas for f_n and m_n (n = 1, 2, …) implied by relations (25)-(28) and (17) in terms of the constant a are

$$f_n=\frac{1}{a^2+9}\left\{a^{n+2}+6\Re\left[\left(1-\frac{ai}{\sqrt{9a+3}}\right)\left(\frac{1-a+i\sqrt{9a+3}/a}{2}\right)^{n-1}\right]\right\}, \tag{29}$$

$$m_n=6+\frac{2}{a^2+9}\left\{a^{n+4}+6\Re\left[\left(1-\frac{ai}{\sqrt{9a+3}}\right)\left(\frac{1-a+i\sqrt{9a+3}/a}{2}\right)^{n+1}\right]\right\} . \tag{30}$$

Plugging n = 10 into (29) and (30) and using a spreadsheet as a computing tool yield exactly $f_{10}=139$ and $m_{10}=980$—the values already listed in section 9.11.2.

Note that formulas (29) and (30) do not include the values f_0 and m_0. This is due to the fact that the initial conditions $f_k=1$ $(k=0,1,2,3)$ cannot be satisfied by any combination of the available parameters $\alpha, \Re\beta$, and $\Im\beta$ in (27). This observation becomes more obvious if one notes that the generating function in (20) is not a proper rational function. Such problem has not occurred in the case of Fibonacci numbers for the generating function in (11) is a proper rational function. As

$$f(x)=\frac{1-3x^3}{1-x-3x^3}=1+\frac{x}{1-x-3x^3},$$

the constant term $f_0=1$ does not depend on the roots of characteristic equation (24). Hence,

$$\sum_{n=1}^{\infty}f_nx^n= f(x)-1=\frac{x}{(1-ax)(1-bx)(1-\overline{b}x)}=\frac{A}{1-ax}+\frac{B}{1-bx}+\frac{C}{1-\overline{b}x}.$$

As

$$A(1-bx)(1-\overline{b}x)+B(1-ax)(1-\overline{b}x)+C(1-ax)(1-bx)=x,$$

it follows

$$A = \frac{a}{|a-b|^2},\ B = \frac{\bar{b}}{[2(b-a)\Im b]},\ C = \bar{B},$$

and $f_n = Aa^n + 2\Re[Bb^n]$ for n = 1, 2, 3, ... , which is equivalent to (29).

8. Concluding remarks

One may argue that computing technology commonly available nowadays in a variety of educational settings has reduced the need for teaching formal mathematical methods like those used in developing rather sophisticated representations of recursive sequences f_n and m_n. Indeed, as mentioned in section 7.11.2, the use of a spreadsheet allows one to get a numerical solution of the mice population size problem without much difficulty. Consequently, one may doubt the importance of preparing students to study mathematics conceptually at the lower levels of the map shown in Figure 1. The concluding remarks that follow are designed to counter such perspectives on mathematics education and justify the need for teaching and learning mathematical theory which is supported rather than replaced by powerful computational tools.

To begin, note that from an educational perspective, binomial representations of f_n and m_n demonstrate one of the most profound notions of mathematics epistemology—the recurring structure of mathematical concepts on different levels of abstraction that preserves the unity of mathematics despite its enormous growth [14]. At the early grades, some rudiments of binomial summation formulas can be developed using pictorial representations of the sums of numbers as shown in Figures 6 and 7. Likewise, formulas (5) and (6) can be considered as precursors to formulas (12), (22), and (23). Furthermore, formulas (22) and (23), developed through the method of generating functions, imply (and thus confirm) formula (17) found through other means and then proved by induction. As the National Council of Teachers of Mathematics put it, “When students can see the connections across content areas, they develop a view of mathematics as an integrated whole” [36, p. 355]. On a practical level, formula (23) enables one to find the smallest n for which the mice population size m_n is greater or equal to the specified amount.

The educational aspect of formulas (29) and (30) can also be revealed. The fact that f_n and m_n defined through complex numbers turn out to be natural numbers when calculated within a spreadsheet (or through other computational tools) for each n greatly fascinates students. These calculations demonstrate a robust interplay between mathematics and technology—two major pillars of STEM education—and show how one can use computing to verify theoretical results. Furthermore, it follows from (29), (30), and (25) that $\lim_{n\to\infty}[m_n / f_n] = 2a^2 = 6.9468...$. Here is a simple, yet powerful application of this limit. Assume that m_n is not known while f_n is known for sufficiently large n. Instead of counting all the mice, one can use the approximation $m_n \approx 7f_n$. Indeed, spreadsheet modeling (section 9.11.2) yields $m_{10} = 980$ and $f_{10} = 139$, hence $\frac{980}{139} = 7.0504...$. Some approximate formulas in the trigonometric form are good for applications and education also. Equations (29) and (30) allow one to have approximations for f_n and m_n (without using complex numbers) to any required degree of accuracy. For example,

$$f_n \approx 0.5189765(1.863707)^{n-1} - 0.5215636(1.268738)^{n-1}\cos[157.2605^\circ + 109.9001^\circ(n-1)]$$

and

$$m_n \approx 6 + 2\cdot\{0.5189765(1.863707)^{n-1} - 0.5215636(1.268738)^{n-1}\cos[157.2605^\circ + 109.9001^\circ(n-1)]\}.$$

Once again, computing yields $f_{10} \approx 139.0003$ and $m_{10} \approx 980.002$—pretty good approximations to the exact values found in section 7.11.2.

Although the material of section 7 presented an example of conceptual ascend, it could also serve to illustrate conceptual descend. Indeed, when n = 1 formula (29) yields $f_1 = \frac{a^3+6}{a^2+9}$ and this may prompt a simple, yet quite an educative computational task to show that $f_1 = 1$. An educative value of such a task is to show how a complex arithmetical combination of irrational numbers is in fact a whole number. At a more basic level, one may recall or can be asked to show that division of fractions might lead to a whole number. Another task descending from formula (29) is to prove that if $\frac{a^3+6}{a^2+9} = 1$ then a is a root of equation (24). Likewise, as $f_2 = f_3 = 1$, one can be asked to prove that setting n = 2 (or n = 3) in formula (29) implies that a, once again, satisfies equation (24). Similarly, formula (29) can be explored with other values of n.

A simple task of that kind may bring about many similar tasks when a real root of an algebraic equation of degree higher than two satisfies other algebraic equations. This leads to a problem posing activity—"the creation of a new problem from a situation or experience" [49, p. 20]. Such a situation and/or experience originates at the top of the conceptual map of Figure 1 and it is the notion of conceptual descend that results in the creation of a new problem. Thereby, one can say that the notion of conceptual descend can be used as a springboard into mathematical problem posing—another important tool of STEM education. The discussion of the use of technology in problem posing is, however, beyond the scope of this article.

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