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New Extensions of Some Known Special Polynomials under the Theory of Multiple q-Calculus

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Abstract In the year 2011, the idea of multiple q-calculus was formulated and introduced in the Ph.D. dissertation of Nalci [9] in which this idea is simple but elegant method in order to derive new generating functions of some special polynomials that are generalizations of known q-polynomials. In this paper, we will use Nalci's method in order to find a systematic study of new types of the Bernoulli polynomials, Euler polynomials and Genocchi polynomials. Also we will obtain recursive formulas for these polynomials.

Keywords: *Quantum calculus, Multiple quantum calculus, q-Bernoulli polynomials, q-Euler polynomials, q-Genocchi polynomials, Generating function.*

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1. Introduction

1.1. q-**Calculus.** The usual quantum calculus (or recalled q-calculus) has been extensively studied for a long time by many mathematicians, physicists and engineers. The development of q-calculus stems from the applications in many fields such as engineering, economics, math-ematics, and so on. One of the important branches of q-calculus is q-special polynomials. For example, Kim [18] constructed q-generalized Euler polynomials based on q-exponential function. Moreover, Srivastava *et al* investigated Apostol q-Bernoulli, Apostol q-Euler polynomials and Apostol q-Genocchi polynomials. This is why q-calculus is thought as one of the useful tools to study with special numbers and polynomias. For more information related these issues, see, e.g. [1,2,3,5,6,8-13,16-21].

Before starting at multiple q-calculus, we first give some basic notations about q-calculus which can be found in [3].

For a real number (or complex number) x, q-number (quantum number) is known as

$$[x]_q := \begin{cases} \dfrac{1-q^x}{1-q}, & if\ q \neq 1, \\ x, & if\ q = 1 \end{cases} \quad (1.1)$$

which is also called non-symmetrical q-number. The followings can be easily derived using (1.1):

$$[x+y]_q = [x]_q + q^x [y]_q \quad (q\text{-addition formula}) \quad (1.2)$$

$$[x-y]_q = -q^{x-y}[y]_q + [x]_q \quad (q\text{-substraction formula}) \quad (1.3)$$

$$[xy]_q = [x]_q [y]_{q^x} \quad (q\text{-product rule}) \quad (1.4)$$

$$\left[\frac{x}{y}\right]_q = \frac{[x]_q}{[y]_{q^{\frac{x}{y}}}} = \frac{[x]_{q^{\frac{1}{y}}}}{[y]_{q^{\frac{1}{y}}}} \quad (q\text{-division rule}) \quad (1.5)$$

where x, y are real or complex numbers.

The q-binomial coefficients are defined for positive integer n, k as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \quad (1.6)$$

where $[n]_q! = [n]_q [n-1]_q [n-2]_q \dots [1]_q$, $\quad n = 1, 2, \dots;$ $[0]_q! = 1.$

The q-derivative $D_q f(x)$ of a function f is given as

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, q \neq 1, x \neq 0, D_q f(0) = f'(0),$$

provided $f'(0)$ exists.

For any $z \in \mathbb{C}$ with $|z| < 1$,

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} \ and\ E_q(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}.$$

For the q-commuting variables x and y such as $yx = qxy$, we know that

$$e_q(x+y) = e_q(x) e_q(y).$$

The q-integral was defined by Jackson as follows:

$$\int_0^x f(y)d_q y = (1-q)x\sum_{n=0}^{\infty} f\left(q^n x\right)q^n$$

provided that the series on right hand side converges absolutely.

1.2. Multiple q-calculus. All notations and all corollaries written in this part have been taken from the Ph.D. dissertation of Nalci [9].

Consider basis vector $\vec{q}$ with coordinates $q_1, q_2, ..., q_N$ so that the multiple q-number can be de.ned as

$$[n]_{q_i,q_j} := \frac{q_i^n - q_j^n}{q_i - q_j} = [n]_{q_j,q_i}, \tag{1.7}$$

which is symmetric. Hence, we can write $N\times N$ matrix with q-numbers elements in the following form:

$$\left([n]_{q_i,q_j}\right) = \begin{pmatrix} [n]_{q_1,q_1} & [n]_{q_1,q_2} & \cdots & [n]_{q_1,q_N} \\ [n]_{q_2,q_1} & [n]_{q_2,q_2} & & [n]_{q_2,q_N} \\ \cdots & \cdots & \cdots & \cdots \\ [n]_{q_N,q_1} & [n]_{q_N,q_2} & \cdots & [n]_{q_N,q_N} \end{pmatrix}.$$

Diagonal terms of this matrix are defined in the limit $q_j \to q_i$ as

$$\lim_{q_j\to q_i}[n]_{q_i,q_j} = \lim_{q_j\to q_i}\frac{q_i^n - q_j^n}{q_i - q_j} = nq_i^{n-1}. \tag{1.8}$$

So, by (1.8), we see that this symmetric matrix can be shown as

$$\left([n]_{q_i,q_j}\right) = \begin{pmatrix} nq_i^{n-1} & [n]_{q_1,q_2} & \cdots & [n]_{q_1,q_N} \\ [n]_{q_2,q_1} & nq_2^{n-1} & & [n]_{q_2,q_n} \\ \cdots & \cdots & \cdots & \cdots \\ [n]_{q_N,q_1} & [n]_{q_N,q_2} & \cdots & nq_N^{n-1} \end{pmatrix}.$$

The followings can be easily derived using (1.7):

$$[n+m]_{q_i,q_j} = q_i^n [m]_{q_i,q_j} + q_j^m [n]_{q_i,q_j}$$

(q-multiple addition formula)

$$[n-m]_{q_i,q_j} = -q_j^{-m}\left([n]_{q_i,q_j} - q_i^{n-m}[m]_{q_i,q_j}\right)$$

(q-multiple substraction formula)

$$[nm]_{q_i,q_j} = [m]_{q_i,q_j}[n]_{q_i^m,q_j^m}$$

(q-multiple product rule)

$$\left[\frac{n}{m}\right]_{q_i,q_j} = \frac{[n]_{q_i,q_j}}{[m]_{q_i^{\frac{n}{m}},q_j^{\frac{n}{m}}}} = \frac{[n]_{q_i^{\frac{1}{m}},q_j^{\frac{1}{m}}}}{[m]_{q_i^{\frac{1}{m}},q_j^{\frac{1}{m}}}}$$

(q-multiple division rule)

where n, m are real or complex numbers.

In multiple q-calculus, multiple q-derivative with base q_i, q_j is given by

$$\mathrm{D}_{q_i,q_j} f(x) = \frac{f(q_i x) - f(q_j x)}{(q_i - q_j)x}$$

representing $N\times N$ matrix of multiple q-derivative operators $D := \left(D_{q_i,q_j}\right)$ which is sym-metric: $D_{q_i,q_j} = D_{q_j,q_i}$ where i and $j = 1, 2, ..., N$.

$$D = \left(D_{q_i,q_j}\right) = \begin{pmatrix} D_{q_1,q_1} & D_{q_1,q_2} & \cdots & D_{q_1,q_N} \\ D_{q_2,q_1} & D_{q_2,q_2} & \cdots & D_{q_2,q_N} \\ \cdots & \cdots & \cdots & \cdots \\ D_{q_N,q_1} & D_{q_N,q_2} & \cdots & D_{q_N,q_N} \end{pmatrix}.$$

Corollary 1. *For N = 1 case and* $q_1 = q_2 \equiv q$, *we have*

$$[n]_{q,q} = nq^{n-1} \ \ and \ \ D_{q,q} = M_q \frac{d}{dx}$$

where $M_q = q^{x\frac{d}{dx}}$. Also, in the case $q = 1$, we have the standard number $[n]_{1,1} = n$ and the usual derivative $D_{1,1} = \frac{d}{dx}$.

Corollary 2. *For N* = 2 *case, we have*

$$[n]_{q_1,q_1} = nq_1^{n-1}, [n]_{q_1,q_2} = [n]_{q_2,q_1} = \frac{q_1^n - q_2^x}{q_1 - q_2},$$

$$[n]_{q_2,q_2} = nq_2^{n-1}$$

$$D_{q_1,q_1} = M_{q_1}\frac{d}{dx}, D_{q_1,q_2} = D_{q_2,q_1} = \frac{M_{q_1} - M_{q_2}}{(q_1 - q_2)x},$$

$$D_{q_2,q_2} = M_{q_2}\frac{d}{dx}$$

Corollary 3. *Choosing* q_1 = 1 *and* q_2 = q *gives non-symmetrical case as*

$$[n]_{1,1} = n, \ [n]_{1,q} = [n]_{q,1} = [n]_q, [n]_{q,q} = nq^{n-1},$$

$$D_{1,1} = \frac{d}{dx}, D_{1,q} = D_{q,1} = \frac{1-M_q}{(1-q)^x}, D_{q,q} = M_q\frac{d}{dx}.$$

Corollary 4. Taking $q_1 = q$ and $q_2 = \frac{1}{q}$ gives symmetrical case as

$$[n]_{q,q} = nq^{n-1}, \ [n]_{1,\frac{1}{q}} = [n]_{\tilde{q}}, [n]_{\frac{1}{q},\frac{1}{q}} = n\left(\frac{1}{q}\right)^{n-1},$$

$$D_{q,q} = M_q\frac{d}{dx}, \ D_{\frac{1}{q},q} = D_{q,\frac{1}{q}} = \frac{M_{\frac{1}{q}} - M_q}{\left(\frac{1}{q} - q\right)^x},$$

$$D_{\frac{1}{q},\frac{1}{q}} = M_{\frac{1}{q}}\frac{d}{dx}.$$

The multiple q-analogue of $(x-a)^n$ is the polynomial

$$(x+a)^n_{q_i,q_j} := \begin{cases} \left[\begin{matrix}\left(x+q_i^{n-1}a\right)\left(x+q_i^{n-2}q_ja\right)\ldots \\ \left(x+q_iq_j^{n-2}a\right)\left(x+q_j^{n-2}a\right)\end{matrix}\right], & if\ n\geq 1 \\ 1, & if\ n=1 \end{cases}$$

or equivalently

$$(x+a)^n_{q_i,q_j} = \sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_{q_i,q_j}(q_iq_j)^{\frac{k(k-1)}{2}}x^{n-k}a^k$$

where x and a is commutative, $xa = ax$. q-multiple Binomial coefficients and multiple q-factorial are defined by

$$[n]_{q_i,q_j}! = \begin{cases}[n]_{q_i,q_j}[n-1]_{q_i,q_j}\ldots[2]_{q_i,q_j}[1]_{q_i,q_j} & if\ \ n\geq 1 \\ 1 & if\ \ n\geq 1\end{cases} (n\in\mathbb{N}).$$

Two types of multiple q-exponential functions are de.ned by

$$e_{q_i,q_j}(x)=\sum_{n=0}^{\infty}\frac{x^n}{[n]_{q_i,q_j}!}$$

$$E_{q_i,q_j}(x)=\sum_{n=0}^{\infty}(q_iq_j)^{\frac{n(n-1)}{2}}\frac{x^n}{[n]_{q_i,q_j}!}$$

which satisfy the following condition for commutative x and y, $xy = yx$

$$e_{q_i,q_j}(x+y)_{q_iq_j} = e_{q_i,q_j}(x)E_{q_i,q_j}(y).$$

The generalization of Jackson's integral (called multiple q-integral) is given by

$$\int f\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=\left(q_i-q_j\right)\sum_{k=0}^{\infty}\frac{q_j^kx}{q_i^{k+1}}f\left(\frac{q_j^k}{q_i^{k+1}}x\right).$$

Let $f(x)=\sum_{k=0}^{\infty}a_kx^k$ be formal power series. Applying multiple q-integral to the both sides of $f(x)$ gives

$$\int f\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=\sum_{k=0}^{\infty}q_i^{k+1}a_k\frac{x^{k+1}}{[k+1]_{q_i,q_j}}+C$$

where C is constant.

In the next section, we will use Nalci's method in order to find a systematic study of new types of the Bernoulli polynomials, Euler polynomials and Genocchi polynomials. Also we will obtain recursive formulas for these polynomials.

2. Main Results

Recently, analogues of Bernoulli, Euler and Genocchi polynomials were studied by many mathematicians [1,2,5,6,11,12,13,17,18,19,20,21]. We are now ready to give the definition of generating functions, corresponding to multiple q-calculus, of Bernoulli type, Euler type and Genocchi type polynomials.

Definition 1. *Let n be positive integer, we define*

$$\mathcal{S}(x,z:q_i,q_j)=\sum_{n=0}^{\infty}\mathcal{B}_n(x:q_i,q_j)\frac{z^n}{[n]_{q_i,q_j}!}$$

$$=\frac{z}{e_{q_i,q_j}(z)-1}e_{q_i,q_j}(xz)\quad(|z|<2\pi)$$

$$\mathcal{U}(x,z:q_i,q_j)=\sum_{n=0}^{\infty}\mathcal{E}_n(x:q_i,q_j)\frac{z^n}{[n]_{q_i,q_j}!}$$

$$=\frac{[2]_{q_i,q_j}}{e_{q_i,q_j}(z)+1}e_{q_i,q_j}(xz)\quad(|z|<\pi)$$

$$\mathcal{M}(x,z:q_i,q_j)=\sum_{n=0}^{\infty}\mathcal{G}_n(x:q_i,q_j)\frac{z^n}{[n]_{q_i,q_j}!}$$

$$=\frac{[2]_{q_i,q_j}z}{e_{q_i,q_j}(z)+1}e_{q_i,q_j}(xz)\quad(|z|<\pi)$$

where $\mathcal{B}_n(x:q_i,q_j)$, $\mathcal{E}_n(x:q_i,q_j)$ *and* $\mathcal{G}_n(x:q_i,q_j)$ *are called, respectively, Bernoulli-type, Euler-type and Genocchi-type polynomials.*

Corollary 5. *Taking* $q_i = q_j = 1$ *for indexes i and j in the case N = 1 in Definition 1, we have*

$$\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!}=\frac{z}{e^z-1}e^{xz}\quad(|z|<2\pi)$$

$$\sum_{n=0}^{\infty}E_n(x)\frac{z^n}{n!}=\frac{2}{e^z+1}e^{xz}\quad(|z|<\pi)$$

$$\sum_{n=0}^{\infty}G_n(x)\frac{z^n}{n!}=\frac{2z}{e^z+1}e^{xz}\quad(|z|<\pi)$$

where $B_n(x)$, $E_n(x)$ *and* $G_n(x)$ *are called Bernoulli polynomials, Euler polynomials and Genocchi polynomials, respectively (see* [4,7,14,15,19]).

Corollary 6. *Substituting* $q_i = 1$ *and* $q_j = q$ *for indexes i and j in the case N =* 1 *in Definition* 1, *we have*

$$\sum_{n=0}^{\infty}B_n(x\,|\,q)\frac{z^n}{[n]_q!}=\frac{z}{e_q(z)-1}e_q(xz)\quad(|z|<2\pi)$$

$$\sum_{n=0}^{\infty}E_n(x\,|\,q)\frac{z^n}{[n]_q!}=\frac{[2]_q}{e_q(z)+1}e_q(xz)\quad(|z|<\pi)$$

$$\sum_{n=0}^{\infty}G_n(x\,|\,q)\frac{z^n}{[n]_q!}=\frac{[2]_qz}{e_q(z)+1}e_q(xz)\quad(|z|<\pi)$$

where $B_n(x\,|\,q)$, $E_n(x\,|\,q)$ *and* $G_n(x\,|\,q)$ *are called q-Bernoulli polynomials, q-Euler polynomials and q-Genocchi polynomials, respectively (see* [18,20,21]).

Taking $x = 0$ in the above definition, we have

$$\mathcal{B}_n(0:q_i,q_j):=\mathcal{B}_n(q_i,q_j)\quad\text{(Bernoulli-type number)}$$

$$\mathcal{E}_n\left(0:q_i,q_j\right):=\mathcal{E}_n\left(q_i,q_j\right) \quad \text{(Euler-type number)}$$

$$\mathcal{G}_n\left(0:q_i,q_j\right):=\mathcal{G}_n\left(q_i,q_j\right) \quad \text{(Genocchi-type number)}$$

and from the above, we write

$$\begin{aligned}\mathcal{S}\left(0,z:q_i,q_j\right)&:=\mathcal{S}\left(z:q_i,q_j\right),\\ \mathcal{U}\left(0,z:q_i,q_j\right)&:=\mathcal{U}\left(z:q_i,q_j\right),\\ \mathcal{M}\left(0,z:q_i,q_j\right)&:=\mathcal{M}\left(z:q_i,q_j\right).\end{aligned} \tag{2.1}$$

From Definition 1 and (2.1), we get the following corollary.

Corollary 7. *The following functional equations hold true:*

$$\begin{aligned}\mathcal{S}\left(x,z:q_i,q_j\right)&:=\mathcal{S}\left(z:q_i,q_j\right)e_{q_i,q_j}(xz),\\ \mathcal{U}\left(x,z:q_i,q_j\right)&:=\mathcal{U}\left(z:q_i,q_j\right)e_{q_i,q_j}(xz),\\ \mathcal{M}\left(x,z:q_i,q_j\right)&:=\mathcal{M}\left(z:q_i,q_j\right)e_{q_i,q_j}(xz).\end{aligned}$$

By using Definition 1 and Corollary 7, it becomes

$$\sum_{n=0}^{\infty}\mathcal{B}_n\left(x:q_i,q_j\right)\frac{z^n}{[n]_{q_i,q_j}!}=\left(\sum_{n=0}^{\infty}\mathcal{B}_n\left(q_i,q_j\right)\frac{z^n}{[n]_{q_i,q_j}!}\right)\left(\sum_{n=0}^{\infty}x^n\frac{z^n}{[n]_{q_i,q_j}!}\right).$$

From the rule of Cauchy product, we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathcal{B}_n\left(x:q_i,q_j\right)\frac{z^n}{[n]_{q_i,q_j}!}\\ &=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{B}_k\left(q_i,q_j\right)x^{n-k}\right)\frac{z^n}{[n]_{q_i,q_j}!}.\end{aligned} \tag{2.2}$$

Comparing the coefficients of $\frac{z^n}{[n]_{q_i,q_j}!}$ in (2.2), we have

$$\mathcal{B}_n\left(x:q_i,q_j\right)=\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{B}_k\left(q_i,q_j\right)x^{n-k}. \tag{2.3}$$

From this, we can get similar identities for Euler-type and Genocchi-type polynomials. Therefore, we state the following theorem.

Theorem 1. *The following identities hold true:*

$$\begin{aligned}\mathcal{B}_n\left(x:q_i,q_j\right)&=\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{B}_k\left(q_i,q_j\right)x^{n-k},\\ \mathcal{E}_n\left(x:q_i,q_j\right)&=\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{E}_k\left(q_i,q_j\right)x^{n-k},\\ \mathcal{G}_n\left(x:q_i,q_j\right)&=\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{G}_k\left(q_i,q_j\right)x^{n-k}.\end{aligned}$$

Now we are in a position to investigate some properties of Bernoulli-type numbers and polynomials, Euler-type numbers and polynomials and Genocchi-type numbers and polynomials as follows.

From Definition 1 and by using Cauchy product, we get

$$\frac{z}{e_{q_i,q_j}(z)-1}e_{q_i,q_j}(z)=z+\frac{z}{e_{q_i,q_j}(z)-1}$$

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}\mathcal{B}_n\left(q_i,q_j\right)\frac{z^n}{[n]_{q_i,q_j}!}\right)\left(\sum_{n=0}^{\infty}\frac{z^n}{[n]_{q_i,q_j}!}\right)\\ &=z+\sum_{n=0}^{\infty}\mathcal{B}_n\left(q_i,q_j\right)\frac{z^n}{[n]_{q_i,q_j}!}\end{aligned}$$

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{B}_n\left(q_i,q_j\right)\right)\frac{z^n}{[n]_{q_i,q_j}!}\\ &=z+\sum_{n=0}^{\infty}\mathcal{B}_n\left(q_i,q_j\right)\frac{z^n}{[n]_{q_i,q_j}!}.\end{aligned}$$

If we compute both of side and then compare coefficent of $\frac{z^n}{[n]_{q_i,q_j}!}$, then for $n > 1$, we acquire

$$\sum_{k=0}^{\infty}\frac{\mathcal{B}_k\left(q_i,q_j\right)}{[k]_{q_i,q_j}![n-k]_{q_i,q_j}!}-\frac{\mathcal{B}_n\left(q_i,q_j\right)}{[n]_{q_i,q_j}!}=0$$

$$\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{B}_k\left(q_i,q_j\right)=\mathcal{B}_n\left(q_i,q_j\right). \tag{2.4}$$

From this, we can get similar identities for Euler-type numbers and Genocchi-type numbers. The following theorem is an immediate consequence of Eq. (2.4).

Theorem 2. *(Recurrence Formula) For $n > 1$, we have*

$$\sum_{k=0}^{n-1}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{B}_k\left(q_i,q_j\right)=\begin{cases}1, & n=1\\ 0, & n>1\end{cases}$$

$$\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{E}_k\left(q_i,q_j\right)+\mathcal{E}_n\left(q_i,q_j\right)=\begin{cases}[2]_{q_i,q_j}, & n=0\\ 0, & n\geq 1\end{cases}$$

$$\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{G}_k\left(q_i,q_j\right)+\mathcal{G}_n\left(q_i,q_j\right)=\begin{cases}[2]_{q_i,q_j}, & n=1\\ 0, & n>1\end{cases}.$$

It is not diffucult to show the following equality:

$$\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\begin{bmatrix}n-k\\m\end{bmatrix}_{q_i,q_j}=\begin{bmatrix}n\\m+k\end{bmatrix}_{q_i,q_j}\begin{bmatrix}m+k\\m\end{bmatrix}_{q_i,q_j}. \tag{2.5}$$

By (2.5), we get readily the following theorem.

Theorem 3. *For $n\in\mathbb{N}$, the followings hold true*

$$\begin{aligned}\mathcal{B}_n\left(x+y:q_i,q_j\right)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{B}_k\left(x:q_i,q_j\right)y^{n-k},\\ \mathcal{E}_n\left(x+y:q_i,q_j\right)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{E}_k\left(x:q_i,q_j\right)y^{n-k},\\ \mathcal{G}_n\left(x+y:q_i,q_j\right)&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{G}_k\left(x:q_i,q_j\right)y^{n-k}.\end{aligned}$$

Proof. If we change x by $x + y$ in $\mathcal{S}_n\left(x,z:q_i,q_j\right)$, then we acquire

$$\sum_{n=0}^{\infty}\mathcal{B}_n\left(x+y:q_i,q_j\right)\frac{z^n}{[n]_{q_i,q_j}!}$$

$$=\frac{z}{e_{q_i,q_j}(z)-1}e_{q_i,q_j}\left((x+y)z\right)$$

$$=\sum_{n=0}^{\infty}\mathcal{B}_n\left(q_i,q_j\right)\frac{z^n}{[n]_{q_i,q_j}!}\sum_{n=0}^{\infty}\frac{z^n}{[n]_{q_i,q_j}!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{B}_n\left(x:q_i,q_j\right)y^{n-k}\right)\frac{z^n}{[n]_{q_i,q_j}!}.$$

By computing the coefficient $\frac{z^n}{[n]_{q_i,q_j}!}$ of both of side, then we have

$$\mathcal{B}_n\left(x+y:q_i,q_j\right)=\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{B}_n\left(x:q_i,q_j\right)y^{n-k}.$$

The others can be proved in a like manner.

Now we consider the special cases of Theorem 3 as Corollary 8 and Corollary 9.

Corollary 8. *Letting y = 1 in the Theorem* 3, *we then get*

$$\mathcal{B}_n\left(x+1:q_i,q_j\right)=\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{B}_k\left(x:q_i,q_j\right),$$

$$\mathcal{E}_n\left(x+1:q_i,q_j\right)=\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{E}_k\left(x:q_i,q_j\right),$$

$$\mathcal{G}_n\left(x+1:q_i,q_j\right)=\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{G}_k\left(x:q_i,q_j\right).$$

Corollary 9. *Letting x* = 0 *in the Theorem 3, we then get*

$$\mathcal{B}_n\left(x:q_i,q_j\right)=\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{B}_k\left(q_i,q_j\right)x^{n-k},$$

$$\mathcal{E}_n\left(x:q_i,q_j\right)=\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{E}_k\left(q_i,q_j\right)x^{n-k},$$

$$\mathcal{G}_n\left(x:q_i,q_j\right)=\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\mathcal{G}_k\left(q_i,q_j\right)x^{n-k}.$$

Theorem 4. *The following expressions hold true for* $n\in\mathbb{N}$

$$\mathcal{B}_n\left(x+1:q_i,q_j\right)-\mathcal{B}_n\left(x:q_i,q_j\right)=[n]_{q_i,q_j}x^{n-1}$$

$$\mathcal{E}_n\left(x+1:q_i,q_j\right)+\mathcal{E}_n\left(x:q_i,q_j\right)+=[2]_{q_i,q_j}x^{n}$$

$$\mathcal{G}_n\left(x+1:q_i,q_j\right)+\mathcal{G}_n\left(x:q_i,q_j\right)=[2]_{q_i,q_j}[n]_{q_i,q_j}x^{n-1}.$$

Proof. By using definitions of these polynomials and numbers, one can easily obtain these relations.

Theorem 5. *(Identity of Symmetry) The followings hold true for* $n\in\mathbb{N}$*:*

$$\mathcal{B}_n\left(1-x:q_i,q_j\right)=(-1)^n\mathcal{B}_n\left(x:q_i,q_j\right),$$

$$\mathcal{E}_n\left(1-x:q_i,q_j\right)=(-1)^n\mathcal{E}_n\left(x:q_i,q_j\right),$$

$$\mathcal{G}_n\left(1-x:q_i,q_j\right)=(-1)^{n+1}\mathcal{G}_n\left(x:q_i,q_j\right).$$

Proof. Setting $1-x$ instead of x in $\mathcal{S}_n\left(x,z:q_i,q_j\right)$, we then get

$$\frac{z}{e_{q_i,q_j}(z)-1}e_{q_i,q_j}\left((1-x)z\right)$$

$$=\sum_{n=0}^{\infty}\mathcal{B}_n\left(1-x:q_i,q_j\right)\frac{z^n}{[n]_{q_i,q_j}!}$$

$$=\frac{(-z)}{e_{q_i,q_j}(-z)-1}e_{q_i,q_j}\left(x(-z)\right)$$

$$=\sum_{n=0}^{\infty}(-1)^n\mathcal{B}_n\left(x:q_i,q_j\right)\frac{z^n}{[n]_{q_i,q_j}!}.$$

Compairing coefficients $\frac{z^n}{[n]_{q_i,q_j}!}$ both of side in above equality, we have desired the result. Similar to that of this proof, it can be proved for Euler-type polynomials and Genocchi-type polynomials. So we completed this proof.

Theorem 6. *(Raabe's Formula) For* $n\in\mathbb{N}$ *, the followings hold true*

$$\mathcal{B}_n\left(dx:q_i,q_j\right)=d^{n-1}\sum_{k=0}^{d-1}\mathcal{B}_n\left(x+\frac{k}{d}:q_i,q_j\right)\quad\left(d\in\mathbb{Z}^{+}\right)$$

$$\mathcal{E}_n\left(dx:q_i,q_j\right)$$

$$=d^{n}\sum_{k=0}^{d-1}(-1)^k\mathcal{E}_n\left(x+\frac{k}{d}:q_i,q_j\right)\qquad(\mathrm{d}\equiv 1 \pmod 2)$$

$$\mathcal{G}_n\left(dx:q_i,q_j\right)$$

$$=d^{n-1}\sum_{k=0}^{d-1}(-1)^k\mathcal{G}_n\left(x+\frac{k}{d}:q_i,q_j\right)\qquad(\mathrm{d}\equiv 1 \pmod 2)).$$

Proof. By using Definition 1, then we have

$$\sum_{n=0}^{\infty}\left(d^{n-1}\sum_{k=0}^{d-1}\mathcal{B}_n\left(x+\frac{k}{d}:q_i,q_j\right)\right)\frac{z^n}{[n]_{q_i,q_j}!}$$

$$=\frac{1}{d}\sum_{k=0}^{d-1}\sum_{n=0}^{\infty}\mathcal{B}_n\left(x+\frac{k}{d}:q_i,q_j\right)\frac{(dz)^n}{[n]_{q_i,q_j}!}$$

$$=\frac{1}{d}\sum_{k=0}^{d-1}\frac{dz}{e_{q_i,q_j}(dz)-1}e_{q_i,q_j}\left(\left(x+\frac{k}{d}\right)dz\right)$$

$$=\frac{z}{e_{q_i,q_j}(dz)-1}e_{q_i,q_j}(xdz)\sum_{k=0}^{d-1}e_{q_i,q_j}(kz)$$

$$=\sum_{n=0}^{\infty}\mathcal{B}_n\left(dx:q_i,q_j\right)\frac{z^n}{[n]_{q_i,q_j}!}.$$

Similarly, we can prove this theorem for Euler-type numbers and Genocchi-type polynomials. So we omit them. Hence, we complete the proof of this theorem.

Theorem 7. *The three relations between Euler-type numbers and polynomials and Genocchi-type numbers and polynomials are given by*

$$\mathcal{E}_n\left(q_i,q_j\right)=\frac{\mathcal{G}_{n+1}\left(q_i,q_j\right)}{n+1},$$

$$\mathcal{E}_n\left(x:q_i,q_j\right)=\frac{\mathcal{G}_{n+1}\left(x:q_i,q_j\right)}{n+1},$$

$$\mathcal{E}_n\left(x:q_i,q_j\right)=\sum_{k=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_{q_i,q_j}\frac{\mathcal{G}_{k+1}\left(q_i,q_j\right)}{k+1}x^{n-k}.$$

Proof. By Definition 1, we can easily obtain these relations. So we omit the proof.

Let us now apply the multiple q-derivative D_{q_i,q_j}, with respect to x, on the both sides of Definition 1,

$$D_{q_i,q_j}\left(\sum_{n=0}^{\infty}\mathcal{B}_n\left(x:q_i,q_j\right)\frac{z^n}{[n]_{q_i,q_j}!}\right)$$

$$=\sum_{n=0}^{\infty}D_{q_i,q_j}\mathcal{B}_n\left(x:q_i,q_j\right)\frac{z^n}{[n]_{q_i,q_j}!}$$

$$=\frac{z}{e_{q_i,q_j}(z)-1}D_{q_i,q_j}e_{q_i,q_j}(xz)$$

$$=\frac{z^2}{e_{q_i,q_j}(z)-1}e_{q_i,q_j}(xz)$$

$$=\sum_{n=0}^{\infty}\mathcal{B}_n\left(x:q_i,q_j\right)\frac{z^{n+1}}{[n]_{q_i,q_j}!}.$$

Matching the coefficients of $\frac{z^n}{[n]_{q_i,q_j}!}$ gives us

$$D_{q_i,q_j}\mathcal{B}_n\left(x:q_i,q_j\right)=[n]_{q_i,q_j}\mathcal{B}_{n-1}\left(x:q_i,q_j\right).$$

Thus we procure the following theorem.

Theorem 8. *The following identities hold true:*

$$D_{q_i,q_j}\mathcal{B}_n\left(x:q_i,q_j\right)=[n]_{q_i,q_j}\mathcal{B}_{n-1}\left(x:q_i,q_j\right),$$

$$D_{q_i,q_j}\mathcal{E}_n\left(x:q_i,q_j\right)=[n]_{q_i,q_j}\mathcal{E}_{n-1}\left(x:q_i,q_j\right)$$

and

$$D_{q_i,q_j}\mathcal{G}_n\left(x:q_i,q_j\right)=[n]_{q_i,q_j}\mathcal{G}_{n-1}\left(x:q_i,q_j\right).$$

Applying k-times the operator D_{q_i,q_j} denoted by $D^k_{q_i,q_j}$ and the limit $t\to 0$, respectively, to the Definition 1, we derive that

$$\mathcal{B}_n\left(x:q_i,q_j\right)=\lim_{t\to 0}D^k_{q_i,q_j}\frac{ze_{q_i,q_j}(xz)}{e_{q_i,q_j}(z)-1}.$$

So we conclude the following theorem.

Theorem 9. *For* $k\geq 0$ *and* $n\geq 0$, *we have*

$$\mathcal{B}_n\left(x:q_i,q_j\right)=\lim_{t\to 0}D^k_{q_i,q_j}\frac{ze_{q_i,q_j}(xz)}{e_{q_i,q_j}(z)-1},$$

$$\mathcal{E}_n\left(x:q_i,q_j\right)=\lim_{t\to 0}D^k_{q_i,q_j}\frac{[2]_{q_i,q_j}}{e_{q_i,q_j}(z)+1}e_{q_i,q_j}(xz)$$

and

$$\mathcal{G}_n\left(x:q_i,q_j\right)=\lim_{t\to 0}D^k_{q_i,q_j}\frac{[2]_{q_i,q_j}z}{e_{q_i,q_j}(z)+1}e_{q_i,q_j}(xz).$$

Definition 2. *Let* $0<a<b$. *The definite multiple q-integral has the following representation:*

$$\int_a^b f\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=\left(q_i-q_j\right)b\sum_{k=0}^{\infty}\frac{q_j^k}{q_i^{k+1}}f\left(\frac{q_j^k}{q_i^{k+1}}b\right)$$

and

$$\int_a^b f\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x=\int_0^b f\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x-\int_0^a f\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x.$$

Theorem 10. *The following holds true:*

$$\int_0^b f\left(\frac{x}{q_i}\right)D_{q_i,q_j}g\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x$$

$$=\sum_{n=0}^{\infty}f\left(\frac{q_j^k}{q_i^{k+1}}b\right)\left(g\left(\frac{q_j^k b}{q_i^k}\right)-g\frac{q_j^{k+1}b}{q_i^{k+1}}\right).$$

Proof. From Definition 2, we write that

$$\begin{aligned}&\int_0^b f\left(\frac{x}{q_i}\right)D_{q_i,q_j}g\left(\frac{x}{q_i}\right)d_{\frac{q_j}{q_i}}x\\&=\left(q_i-q_j\right)b\sum_{k=0}^{\infty}\frac{q_j^k}{q_i^{k+1}}f\left(\frac{q_j^k}{q_i^{k+1}}b\right)D_{q_i,q_j}g\left(\frac{q_j^k}{q_i^{k+1}}b\right),\end{aligned}\tag{2.6}$$

where $D_{q_i,q_j}g\left(\frac{q_j^k}{q_i^{k+1}}b\right)$ *equals to*

$$\frac{g\left(\frac{q_j^k}{q_i^k}b\right)-g\left(\frac{q_j^{k+1}}{q_i^{k+1}}b\right)}{\left(q_i-q_j\right)b}.\tag{2.7}$$

Combining the Eq. (2.6) with the Eq. (2.7) gives us the proof of the theorem.

Theorem 11. $\left|\frac{q_j}{q_i}\right|<1$ *and* $i,j\in\{1,2,...,N\}$. *Then we have*

$$\int_0^1\mathcal{B}_n\left(\frac{x}{q_i}:q_i,q_j\right)d_{\frac{q_j}{q_i}}x=\frac{\mathcal{B}_{n+1}\left(1:q_i,q_j\right)-\mathcal{B}_{n+1}\left(q_i,q_j\right)}{[n+1]_{q_i,q_j}}$$

$$\int_0^1\mathcal{E}_n\left(\frac{x}{q_i}:q_i,q_j\right)d_{\frac{q_j}{q_i}}x=\frac{\mathcal{E}_{n+1}\left(1:q_i,q_j\right)-\mathcal{E}_{n+1}\left(q_i,q_j\right)}{[n+1]_{q_i,q_j}}$$

$$\int_0^1 \mathcal{G}_n\left(\frac{x}{q_i}:q_i,q_j\right)d_{\frac{q_j}{q_i}}x=\frac{\mathcal{G}_{n+1}\left(1:q_i,q_j\right)-\mathcal{G}_{n+1}\left(q_i,q_j\right)}{[n+1]_{q_i,q_j}}.$$

Proof. By using Theorem 1, Definition 2 and for $\left|\frac{q_j}{q_i}\right|<1$, we have

$$\int_0^1 \mathcal{B}_n\left(\frac{x}{q_i}:q_i,q_j\right)d_{\frac{q_j}{q_i}}x$$

$$=\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{q_i,q_j}\mathcal{B}_{n-l}\left(q_i,q_j\right)\left(\left(q_i-q_j\right)\sum_{k=0}^{\infty}\frac{q_j^k}{q_i^{k+1}}\left(\frac{q_j^k}{q_i^{k+1}}\right)^l\right)$$

$$=\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{q_i,q_j}\mathcal{B}_{n-l}\left(q_i,q_j\right)\frac{1}{[l+1]_{q_i,q_j}}$$

$$=\frac{\mathcal{B}_{n+1}\left(1:q_i,q_j\right)-\mathcal{B}_{n+1}\left(q_i,q_j\right)}{[n+1]_{q_i,q_j}}.$$

Similarly, the identities of Euler-type polynomials and Genocchi-type polynomials can be shown. Therefore, we complete the proof of theorem.

3. Further Remarks

Here we list a few values of Bernoulli-type, Euler-type and Genocchi-type numbers as follows:

Bernoulli-type number:

Table 1.

$\mathcal{B}_0\left(q_i,q_j\right)=1$
$\mathcal{B}_1\left(q_i,q_j\right)=-\frac{1}{[2]_{q_i,q_j}}$
$\mathcal{B}_2\left(q_i,q_j\right)=-\frac{1}{[3]_{q_i,q_j}}+\frac{1}{[2]_{q_i,q_j}}$
$\mathcal{B}_3\left(q_i,q_j\right)=-\frac{1}{[4]_{q_i,q_j}}+\frac{2}{[2]_{q_i,q_j}}-\frac{[3]_{q_i,q_j}}{[2]^2_{q_i,q_j}}$
$\mathcal{B}_4\left(q_i,q_j\right)=-\frac{1}{[5]_{q_i,q_j}}+\frac{1}{[2]_{q_i,q_j}}+\frac{[4]_{q_i,q_j}}{[2]_{q_i,q_j}}\left(\frac{1}{[3]_{q_i,q_j}}-\frac{1}{[2]_{q_i,q_j}}\right)$
$+\frac{[4]_{q_i,q_j}}{[2]_{q_i,q_j}}\left(\frac{1}{[4]_{q_i,q_j}}-\frac{2}{[2]_{q_i,q_j}}+\frac{[3]_{q_i,q_j}}{[2]^2_{q_i,q_j}}\right)$

Substituting q_i = 1 and q_j = q for indexes i and j in the case N = 1 in the Table 1, then we get

$\mathscr{B}_0(q)=1$
$\mathscr{B}_1(q)=-\frac{1}{[2]_q}$
$\mathscr{B}_2(q)=-\frac{1}{[3]_q}+\frac{1}{[2]_q}$
$\mathscr{B}_3(q)=-\frac{1}{[4]_q}+\frac{2}{[2]_q}-\frac{[3]_q}{[2]_q^2}$
$\mathscr{B}_4(q)=-\frac{1}{[5]_q}+\frac{1}{[2]_q}+\frac{[4]_q}{[2]_q}\left(\frac{1}{[3]_q}-\frac{1}{[2]_q}\right)$
$+\frac{[4]_q}{[2]_q}\left(\frac{1}{[4]_q}-\frac{2}{[2]_q}+\frac{[3]_q}{[2]_q^2}\right)$

Substituting q_i = 1 and q_j = q for indexes i and j in the case N = 1 in the Table 1, then we get

$B_0=1$
$B_1=-\frac{1}{2}$
$B_2=\frac{1}{6}$
$B_3=0$
$B_4=-\frac{1}{30}$

Moreover the first few Bernoulli-type numbers can be shown $N\times N$ matrix with multiple q-numbers elements in the following form

$$\left(\mathcal{B}_0\left(q_i,q_j\right)\right)=\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

$$\left(\mathcal{B}_1\left(q_i,q_j\right)\right)=\begin{pmatrix} -\frac{1}{2q_1} & -\frac{1}{[2]_{q_1,q_2}} & \dots & -\frac{1}{[2]_{q_1,q_N}} \\ -\frac{1}{[2]_{q_2,q_1}} & -\frac{1}{2q_2} & \dots & -\frac{1}{[2]_{q_2,q_N}} \\ \dots & \dots & \dots & \dots \\ -\frac{1}{[2]_{q_N,q_1}} & -\frac{1}{[2]_{q_N,q_1}} & \dots & -\frac{1}{[2]q_N} \end{pmatrix}$$

$$\left(\mathcal{B}_2\left(q_i,q_j\right)\right)=\begin{pmatrix} \left(-\frac{1}{3q_1^2}+\frac{1}{2q_1}\right) & \left(-\frac{1}{[3]_{q_1,q_2}}+\frac{1}{[2]_{q_1,q_2}}\right) & \dots & \left(-\frac{1}{[3]_{q_1,q_N}}+\frac{1}{[2]_{q_1,q_N}}\right) \\ \left(-\frac{1}{[3]_{q_2,q_1}}+\frac{1}{[2]_{q_2,q_1}}\right) & \left(-\frac{1}{3q_2^2}+\frac{1}{2q_2}\right) & \dots & \left(-\frac{1}{[3]_{q_2,q_N}}+\frac{1}{[2]_{q_2,q_N}}\right) \\ \dots & \dots & \dots & \dots \\ \left(-\frac{1}{[3]_{q_N,q_1}}+\frac{1}{[2]_{q_N,q_1}}\right) & \left(-\frac{1}{[3]_{q_N,q_2}}+\frac{1}{[2]_{q_N,q_2}}\right) & \dots & \left(-\frac{1}{3q_N^2}+\frac{1}{2q_N}\right) \end{pmatrix}$$

From Definition 1 and the Table 1, we easily acquire the first few Bernoulli-type poly-nomials

Bernoulli-type polynomials

$$\mathcal{B}_0\left(x:q_i,q_j\right)=1$$

$$\mathcal{B}_1\left(x:q_i,q_j\right)=x-\frac{1}{[2]_{q_i,q_j}}$$

$$\mathcal{B}_2\left(x:q_i,q_j\right)=x^2-x-\frac{1}{[3]_{q_i,q_j}}+\frac{1}{[2]_{q_i,q_j}}$$

Usual Bernoulli polynomials

$$B_0=1$$

$$B_1=x-\frac{1}{2}$$

$$B_2=x^2-x-\frac{1}{6}$$

Moreover, the first few Bernoulli-type polynomials can be shown $N\times N$ matrix with q-numbers elements in the following form

$$\left(\mathcal{B}_0\left(x:q_i,q_j\right)\right)=\begin{pmatrix}1&1&\dots&1\\1&1&\dots&1\\\dots&\dots&\dots&\dots\\1&1&\dots&1\end{pmatrix}=\left(\mathcal{B}_0\left(q_i,q_j\right)\right)$$

$$\left(\mathcal{B}_1\left(x:q_i,q_j\right)\right)=\begin{pmatrix}x-\frac{1}{2q_1}&x-\frac{1}{[2]_{q_1,q_2}}&\dots&x-\frac{1}{[2]_{q_1,q_N}}\\x-\frac{1}{[2]_{q_2,q_1}}&x-\frac{1}{2q_2}&\dots&x-\frac{1}{[2]_{q_2,q_N}}\\\dots&\dots&\dots&\dots\\x-\frac{1}{[2]_{q_N,q_1}}&x-\frac{1}{[2]_{q_N,q_1}}&\dots&x-\frac{1}{[2]q_N}\end{pmatrix}$$

$$\left(\mathcal{B}_2\left(x:q_i,q_j\right)\right)=\begin{pmatrix}x^2-x-\frac{1}{3q_1^2}+\frac{1}{2q_1}&x^2-x-\frac{1}{[3]_{q_1,q_2}}+\frac{1}{[2]_{q_1,q_2}}&\dots&x^2-x-\frac{1}{[3]_{q_1,q_N}}+\frac{1}{[2]_{q_1,q_N}}\\x^2-x-\frac{1}{[3]_{q_2,q_1}}+\frac{1}{[2]_{q_2,q_1}}&x^2-x-\frac{1}{3q_2^2}+\frac{1}{2q_2}&\dots&x^2-x-\frac{1}{[3]_{q_2,q_N}}+\frac{1}{[2]_{q_2,q_N}}\\\dots&\dots&\dots&\dots\\x^2-x-\frac{1}{[3]_{q_N,q_1}}+\frac{1}{[2]_{q_N,q_1}}&x^2-x-\frac{1}{[3]_{q_N,q_2}}+\frac{1}{[2]_{q_N,q_2}}&\dots&x^2-x-\frac{1}{3q_N^2}+\frac{1}{2q_N}\end{pmatrix}$$

Euler-type Numbers and Polynomials:

We begin to compute the first few value of $\mathcal{E}_n\left(q_i,q_j\right)$ as follows:

Table 2.

$\mathcal{E}_0\left(q_i,q_j\right)=\frac{[2]_{q_i,q_j}}{2}$
$\mathcal{E}_1\left(q_i,q_j\right)=-\frac{[2]_{q_i,q_j}}{4}$
$\mathcal{E}_2\left(q_i,q_j\right)=-\frac{[2]_{q_i,q_j}}{4}+\frac{\left([2]_{q_i,q_j}\right)^2}{8}$
$\mathcal{E}_3\left(q_i,q_j\right)=-\frac{[2]_{q_i,q_j}}{4}+\frac{[3]_{q_i,q_j}[2]_{q_i,q_j}}{4}\left(1-\frac{[2]_{q_i,q_j}}{4}\right)$
$\mathcal{E}_4\left(q_i,q_j\right)=-\frac{[2]_{q_i,q_j}}{4}+\frac{[4]_{q_i,q_j}[2]_{q_i,q_j}}{4}+\frac{[4]_{q_i,q_j}[3]_{q_i,q_j}[2]_{q_i,q_j}}{8}\left(\frac{1}{[2]_{q_i,q_j}}-\frac{3}{2}+\frac{[2]_{q_i,q_j}}{4}\right)$
$\mathcal{E}_5\left(q_i,q_j\right)=-\frac{[2]_{q_i,q_j}}{4}+\frac{[5]_{q_i,q_j}[2]_{q_i,q_j}}{4}+\frac{[5]_{q_i,q_j}[4]_{q_i,q_j}[2]_{q_i,q_j}}{4}\left(-\frac{3}{4}-\frac{1}{[2]_{q_i,q_j}}\right)$
$+\frac{[5]_{q_i,q_j}[4]_{q_i,q_j}[3]_{q_i,q_j}[2]_{q_i,q_j}}{16}\left(\frac{3}{[2]_{q_i,q_j}}-2+\frac{[2]_{q_i,q_j}}{4}\right)$

Substituting $q_i = 1$ and $q_j = q$ for indexes i and j in the case $N = 1$ in the Table 2, we have

$E_0(q)=\frac{[2]_q}{2}$
$E_1(q)=-\frac{[2]_q}{4}$
$E_2(q)=-\frac{[2]_q}{4}+\frac{\left([2]_q\right)^2}{8}$
$E_3(q)=-\frac{[2]_q}{4}+\frac{[3]_q[2]_q}{4}\left(1-\frac{[2]_q}{4}\right)$
$E_4(q)=-\frac{[2]_q}{4}+\frac{[4]_q[2]_q}{4}+\frac{[4]_q[3]_q[2]_q}{8}\left(\frac{1}{[2]_q}-\frac{3}{2}+\frac{[2]_q}{4}\right)$
$E_5(q)=-\frac{[2]_q}{4}+\frac{[5]_q[2]_q}{4}+\frac{[5]_q[4]_q[2]_q}{4}\left(-\frac{3}{4}-\frac{1}{[2]_q}\right)$
$+\frac{[5]_q[4]_q[3]_q[2]_q}{16}\left(\frac{3}{[2]_q}-2+\frac{[2]_q}{4}\right)$

Substituting $q_i = 1$ and $q_j = q$ for indexes i and j in the case $N = 1$ in the Table 2, we have

$E_0(q)=1$
$E_1(q)=-\frac{1}{2}$
$E_2(q)=0$
$E_3(q)=\frac{1}{4}$
$E_4(q)=0$
$E_5(q)=-\frac{1}{2}$

Moreover the first few Euler-type numbers can be shown $N\times N$ matrix with q-numbers elements in the following form

$$\left(\mathcal{E}_0\left(q_i,q_j\right)\right)=\begin{pmatrix} q_1 & \frac{[2]_{q_1,q_2}}{2} & \cdots & \frac{[2]_{q_1,q_N}}{2} \\ \frac{[2]_{q_2,q_1}}{2} & q_2 & \cdots & \frac{[2]_{q_2,q_N}}{2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{[2]_{q_N,q_1}}{2} & \frac{[2]_{q_N,q_2}}{2} & \cdots & q_N \end{pmatrix}$$

$$\left(\mathcal{E}_1\left(q_i,q_j\right)\right)=\begin{pmatrix} -\frac{q_1}{2} & -\frac{[2]_{q_1,q_2}}{4} & \cdots & -\frac{[2]_{q_1,q_N}}{4} \\ -\frac{[2]_{q_2,q_1}}{4} & -\frac{q_2}{2} & \cdots & -\frac{[2]_{q_2,q_N}}{4} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{[2]_{q_N,q_1}}{4} & -\frac{[2]_{q_N,q_2}}{4} & \cdots & -\frac{q_N}{2} \end{pmatrix}$$

$$\left(\mathcal{E}_2\left(x:q_i,q_j\right)\right)$$

$$=\begin{pmatrix} -\frac{q_1}{2}+\frac{q_1^2}{2} & \left(-\frac{[2]_{q_1,q_2}}{4}+\frac{\left([2]_{q_1,q_2}\right)^2}{8}\right) & \cdots & \left(-\frac{[2]_{q_1,q_N}}{4}+\frac{\left([2]_{q_1,q_N}\right)^2}{8}\right) \\ \left(-\frac{[2]_{q_2,q_1}}{4}+\frac{\left([2]_{q_2,q_1}\right)^2}{8}\right) & -\frac{q_2}{2}+\frac{q_2^2}{2} & \cdots & \left(-\frac{[2]_{q_2,q_N}}{4}+\frac{\left([2]_{q_2,q_N}\right)^2}{8}\right) \\ \cdots & \cdots & \cdots & \cdots \\ \left(-\frac{[2]_{q_N,q_1}}{4}+\frac{\left([2]_{q_N,q_1}\right)^2}{8}\right) & \left(-\frac{[2]_{q_N,q_2}}{4}+\frac{\left([2]_{q_N,q_2}\right)^2}{8}\right) & \cdots & -\frac{q_N}{2}+\frac{q_N^2}{2} \end{pmatrix}$$

From Definition 1 and the Table 2, we easily acquire the first few Euler-type polynomials

Euler-type polynomials

$$\mathcal{E}_0\left(x:q_i,q_j\right)=\frac{[2]_{q_i,q_j}}{2}$$

$$\mathcal{E}_1\left(x:q_i,q_j\right)=\frac{[2]_{q_i,q_j}}{2}x-\frac{[2]_{q_i,q_j}}{4}$$

$$\mathcal{E}_2\left(x:q_i,q_j\right)=\frac{[2]_{q_i,q_j}}{2}x^2-\frac{\left([2]_{q_i,q_j}\right)^2}{4}x$$
$$-\frac{[2]_{q_i,q_j}}{4}+\frac{\left([2]_{q_i,q_j}\right)^2}{8}$$

Usual Euler polynomials

$$E_0=1$$

$$E_1=x-\frac{1}{2}$$

$$E_2=x^2-x$$

Moreover the first few Euler-type polynomials can be shown $N\times N$ matrix with multiple q-numbers elements in the following form

$$\left(\mathcal{E}_0\left(q_i,q_j\right)\right)$$
$$=\begin{pmatrix} q_1 & \frac{[2]_{q_1,q_2}}{2} & \cdots & \frac{[2]_{q_1,q_N}}{2} \\ \frac{[2]_{q_2,q_1}}{2} & q_2 & \cdots & \frac{[2]_{q_2,q_N}}{2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{[2]_{q_N,q_1}}{2} & \frac{[2]_{q_N,q_2}}{2} & \cdots & q_N \end{pmatrix}$$

$$\left(\mathcal{E}_1\left(x:q_i,q_j\right)\right)=\begin{pmatrix} q_1x-\frac{q_1}{2} & \frac{[2]_{q_1,q_2}}{2}x-\frac{[2]_{q_1,q_2}}{4} & \cdots & \frac{[2]_{q_1,q_2}}{2}x-\frac{[2]_{q_1,q_N}}{4} \\ \frac{[2]_{q_1,q_2}}{2}x-\frac{[2]_{q_2,q_1}}{4} & q_2x-\frac{q_2}{2} & \cdots & \frac{[2]_{q_1,q_2}}{2}x-\frac{[2]_{q_2,q_N}}{4} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{[2]_{q_1,q_2}}{2}x-\frac{[2]_{q_N,q_1}}{4} & \frac{[2]_{q_1,q_2}}{2}x-\frac{[2]_{q_N,q_2}}{4} & \cdots & q_Nx-\frac{q_N}{2} \end{pmatrix}$$

$$\left(\mathcal{E}_2\left(x:q_i,q_j\right)\right)=\begin{pmatrix} q_1x^2-q_1^2x-\frac{q_1}{2}+\frac{q_1^2}{2} & \cdots & \frac{[2]_{q_1,q_N}}{2}x^2-\frac{\left([2]_{q_1,q_N}\right)^2}{4}x-\frac{[2]_{q_1,q_N}}{4}+\frac{\left([2]_{q_1,q_N}\right)^2}{8} \\ \frac{[2]_{q_1,q_2}}{2}x^2-\frac{\left([2]_{q_2,q_1}\right)^2}{4}x-\frac{[2]_{q_1,q_2}}{4}+\frac{\left([2]_{q_1,q_2}\right)^2}{8} & \cdots & \frac{[2]_{q_2,q_N}}{2}x^2-\frac{\left([2]_{q_2,q_N}\right)^2}{4}x-\frac{[2]_{q_1,q_N}}{4}+\frac{\left([2]_{q_1,q_N}\right)^2}{8} \\ \cdots & \cdots & \cdots \\ \frac{[2]_{q_N,q_1}}{2}x^2-\frac{\left([2]_{q_N,q_1}\right)^2}{4}x-\frac{[2]_{q_N,q_1}}{4}+\frac{\left([2]_{q_N,q_1}\right)^2}{8} & \cdots & q_Nx^2-q_N^2x-\frac{q_N}{2}+\frac{q_N^2}{2} \end{pmatrix}$$

Genocchi-type Numbers and Polynomials:

We begin to compute the first few value of $\mathcal{G}_1\left(q_i,q_j\right)$ as follows:

Table 3.

$\mathcal{G}_0\left(q_i,q_j\right)=0$
$\mathcal{G}_1\left(q_i,q_j\right)=\frac{[2]_{q_i,q_j}}{2}$
$\mathcal{G}_2\left(q_i,q_j\right)=-\frac{\left([2]_{q_i,q_j}\right)^2}{4}$
$\mathcal{G}_3\left(q_i,q_j\right)=\frac{[3]_{q_i,q_j}}{4}\left(-[2]_{q_i,q_j}+\frac{\left([2]_{q_i,q_j}\right)^2}{2}\right)$
$\mathcal{G}_4\left(q_i,q_j\right)=-\frac{[4]_{q_i,q_j}[2]_{q_i,q_j}}{4}+\frac{[4]_{q_i,q_j}[3]_{q_i,q_j}[2]_{q_i,q_j}}{8}-\frac{[4]_{q_i,q_j}[3]_{q_i,q_j}}{8}\left(-[2]_{q_i,q_j}+\frac{\left([2]_{q_i,q_j}\right)^2}{2}\right)$
$\mathcal{G}_5\left(q_i,q_j\right)=-\frac{[5]_{q_i,q_j}[2]_{q_i,q_j}}{4}+\frac{[5]_{q_i,q_j}[4]_{q_i,q_j}[2]_{q_i,q_j}}{8}--\frac{[5]_{q_i,q_j}[4]_{q_i,q_j}[3]_{q_i,q_j}}{8}\left(-1+\frac{[2]_{q_i,q_j}}{2}\right)$
$-\frac{[5]_{q_i,q_j}}{8}\left(-[4]_{q_i,q_j}[2]_{q_i,q_j}+\frac{[4]_{q_i,q_j}[3]_{q_i,q_j}[2]_{q_i,q_j}}{2}-\frac{[4]_{q_i,q_j}[3]_{q_i,q_j}}{2}\left(-[2]_{q_i,q_j}+\frac{\left([2]_{q_i,q_j}\right)^2}{2}\right)\right)$

Substituting $q_i = 1$ and $q_j = q$ for indexes i and j in the case $N = 1$ in the Table 3, we have

$\mathscr{G}_0(q)=0$
$\mathscr{G}_1(q)=\frac{[2]_q}{2}$

$\mathcal{G}_2(q)=\frac{\left([2]_q\right)^2}{4}$
$\mathcal{G}_3(q)=\frac{[3]_q}{4}\left(-[2]_q+\frac{\left([2]_q\right)^2}{2}\right)$
$\mathcal{G}_4(q)=-\frac{[4]_q[2]_q}{4}+\frac{[4]_q[3]_q[2]_q}{8}-\frac{[4]_q[3]_q}{8}\left(-[2]_q+\frac{\left([2]_q\right)^2}{2}\right)$
$\mathcal{G}_5(q)=-\frac{[5]_q[2]_q}{4}+\frac{[5]_q[4]_q[2]_q}{8}--\frac{[5]_q[4]_q[3]_q}{8}\left(-1+\frac{[2]_q}{2}\right)$
$-\frac{[5]_q}{8}\left(-[4]_q[2]_q+\frac{[4]_q[3]_q[2]_q}{2}-\frac{[4]_q[3]_q}{2}\left(-[2]_q+\frac{\left([2]_q\right)^2}{2}\right)\right)$

Substituting $q_i = 1$ and $q_j = q$ for indexes i and j in the case $N = 1$ in the Table 3, we have

$G_0=0$
$G_1=1$
$G_2=-1$
$G_3=0$
$G_4=1$
$G_5=0$

Moreover the first few Genocchi-type numbers can be shown $N\times N$ matrix with multiple q-numbers elements in the following form

$$\left(\mathcal{G}_0\left(q_i,q_j\right)\right)=\begin{pmatrix}0 & 0 & \ldots & 0\\ 0 & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & 0\end{pmatrix}$$

$$\left(\mathcal{G}_1\left(q_i,q_j\right)\right)=\begin{pmatrix}q_1 & \frac{[2]_{q_1,q_2}}{2} & \ldots & \frac{[2]_{q_1,q_N}}{2}\\ \frac{[2]_{q_2,q_1}}{2} & q_2 & \ldots & \frac{[2]_{q_2,q_N}}{2}\\ \ldots & \ldots & \ldots & \ldots\\ \frac{[2]_{q_N,q_1}}{2} & \frac{[2]_{q_N,q_2}}{2} & \ldots & q_N\end{pmatrix}$$

$$\left(\mathcal{G}_2\left(q_i,q_j\right)\right)=\begin{pmatrix}-\frac{q_1^2}{2} & -\frac{\left([2]_{q_1,q_2}\right)^2}{4} & \ldots & -\frac{\left([2]_{q_1,q_N}\right)^2}{4}\\ -\frac{\left([2]_{q_2,q_1}\right)^2}{4} & -\frac{q_2^2}{2} & \ldots & -\frac{\left([2]_{q_2,q_N}\right)^2}{4}\\ \ldots & \ldots & \ldots & \ldots\\ -\frac{\left([2]_{q_N,q_1}\right)^2}{4} & -\frac{\left([2]_{q_N,q_2}\right)^2}{4} & \ldots & -\frac{q_N^2}{2}\end{pmatrix}$$

From Definition 1 and the Table 3, we easily acquire the first few Genocchi-type poly-nomials

Genocchi-type polynomials

$$\mathcal{G}_0\left(x:q_i,q_j\right)=0$$

$$\mathcal{G}_1\left(x:q_i,q_j\right)=\frac{[2]_{q_i,q_j}}{2}$$

$$\mathcal{G}_2\left(x:q_i,q_j\right)=\frac{\left([2]_{q_i,q_j}\right)^2}{2}x-\frac{\left([2]_{q_i,q_j}\right)^2}{4}$$

Usual Genocchi polynomials

$$G_0=0$$

$$G_1=1$$

$$G_2=2x-1$$

Moreover the first few Genocchi-type polynomials can be shown $N\times N$ matrix with multiple q-numbers elements in the following form

$$\left(\mathcal{G}_0\left(q_i,q_j\right)\right)=\begin{pmatrix}0 & 0 & \ldots & 0\\ 0 & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & 0\end{pmatrix}$$

$$\left(\mathcal{G}_1\left(q_i,q_j\right)\right)=\begin{pmatrix}q_1 & \frac{[2]_{q_1,q_2}}{2} & \ldots & \frac{[2]_{q_1,q_N}}{2}\\ \frac{[2]_{q_2,q_1}}{2} & q_2 & \ldots & \frac{[2]_{q_2,q_N}}{2}\\ \ldots & \ldots & \ldots & \ldots\\ \frac{[2]_{q_N,q_1}}{2} & \frac{[2]_{q_N,q_2}}{2} & \ldots & q_N\end{pmatrix}$$

$$\left(\mathcal{G}_2\left(x:q_i,q_j\right)\right)=\begin{pmatrix} 2q_1^2x-\frac{q_1^2}{2} & \frac{\left([2]_{q_1,q_2}\right)^2}{2}-\frac{\left([2]_{q_1,q_2}\right)^2}{4} & \cdots & \frac{\left([2]_{q_1,q_N}\right)^2}{2}-\frac{\left([2]_{q_1,q_N}\right)^2}{4} \\ \frac{\left([2]_{q_2,q_1}\right)^2}{2}-\frac{\left([2]_{q_2,q_1}\right)^2}{4} & 2q_2^2x-\frac{q_2^2}{2} & \cdots & \frac{\left([2]_{q_2,q_N}\right)^2}{2}-\frac{\left([2]_{q_2,q_N}\right)^2}{4} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\left([2]_{q_N,q_1}\right)^2}{2}-\frac{\left([2]_{q_N,q_1}\right)^2}{4} & \frac{\left([2]_{q_N,q_2}\right)^2}{2}-\frac{\left([2]_{q_N,q_2}\right)^2}{4} & \cdots & 2q_N^2x-\frac{q_N^2}{2} \end{pmatrix}$$

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