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On Quasi Multiplicative Function

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Abstract In this paper we introduce two new Arithmetic functions, that is, Quasi-Multiplicative (QM) and omega (ω) functions. The Omega (ω) function is based on Euler's Phi (ϕ) function and is used to find the sum of coprime integers. Euler's Phi (ϕ) function, Dedekind's psi (ψ) function, the sigma (σ) function and τ -function play significant role in this work.

Keywords: *arithmetic function, quasi-multiplicative function, omega function*

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1. Introduction

Recall that an Arithmetic function [2,6] f is Multiplicative if for each pair of coprime integers m and n, it is satisfied $f(mn) = f(m)f(n)$. Missana [5] established some significant results on multiplicative functions. Dehaye [3] constructed some algebraic structures using arithmetic functions. Some classical examples of multiplicative functions that have important meaning in Number theory are Euler's Phi (ϕ) function, Dedekind's psi (ψ) function, the sigma (σ) function and $\tau -$ function. Recently Hoque and Kalita [4] studied perfect numbers and their generalizations using these multiplicative functions. For any Positive integer $n > 1$ having the factorization $n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$, where $p_1, p_2, \dots, p_k$ are distinct prime numbers and $k, r_1, r_2, \dots, r_k \geq 1$ are integers, these functions admit the following multiplicative representations:

$$\phi(n) = n\prod_{i=1}^{k}\left(1 - \frac{1}{p_i}\right)$$

$$\psi(n) = n\prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right)$$

$$\sigma(n) = \prod_{i=1}^{k}\left(\frac{p_i^{k_i+1} - 1}{p_i - 1}\right)$$

$$\tau(n) = \prod_{i=1}^{k}(k_i + 1).$$

In this paper, we introduce the notions of Omega (ω) function and quasi-multiplicative function. The Omega (ω) function is based on Euler's Phi (ϕ) function and is used to find the sum of coprime integers. We establish some important results on these two functions via Euler's Phi (ϕ) function, Dedekind's psi (ψ) function, the sigma (σ) function and τ -function.

2. Main Results

Definition 2.1: An Arithmetic function f is Quasi-Multiplicative (QM) if for each pair of coprime integers m and n there exists a positive integer k such that $f(mn) = kf(m)f(n)$. The positive k is defined as the multiplicative index of f.

It is clear that a quasi-multiplicative function with index one is Multiplicative.

Definition 2.2: For any positive integer n, we define the Omega function, $\omega(n)$ as the sum of the positive integers less than n and relatively prime to n.

By Theorem 7.7 of [2], we see that $\omega(n) = \frac{1}{2}n\phi(n)$.

Proposition 2.1: If f is a multiplicative function and k is a positive integer then $g = \frac{1}{k}f$ is quasi-multiplicative function with index k.

Proof: Let m and n be any two positive and coprime integers. Then

$$g(mn) = \left(\frac{1}{k}f\right)(mn) = \frac{1}{k}f(mn)$$

$$= \frac{1}{k}f(m)f(n) = kg(m)g(n).$$

This proves the result.

Theorem 2.2: The function ω is quasi multiplicative with index 2.

Proof: By definition,

$$\omega(n) = \frac{1}{2}n\phi(n).$$

Since ϕ is multiplicative, the function f defined by $f(n)=n\phi(n)$ is also multiplicative.

Thus by Proposition 2.1, ω is quasi multiplicative with index 2.

Proposition 2.3: For any prime number p,

$$2\omega(p)=(\sigma(p)-1)(\sigma(p)-2).$$

Proof: We have

$$\omega(p)=\frac{1}{2}p\phi(p)=\frac{1}{2}p(p-1) \quad (1)$$

Again

$$\sigma(p)=p+1 \quad (2)$$

From equation (1) and equation (2), the result follows.

Lemma 2.4 [1]: For every natural number $n\geq 2$,

$$\phi(n)\psi(n)\sigma(n)\geq n^3+n^2-n-1$$

Proposition 2.5: For every natural number $n\geq 2$,

$$2\omega(n)\psi(n)\sigma(n)\geq n^4+n^3-n^2-n.$$

Theorem 2.6: There are infinitely many positive integers m,n and t such that

(i) $\phi(\omega(m))>\omega(\phi(m))>m$

(ii) $\omega(\phi(n))<\phi(\omega(n))<n^2$

(iii) $\omega(\phi(t))=2\phi(\omega(t))$

Proof. (i) Let us suppose, $m=3.2^k$ for any positive integer k. Then

$$\phi(m)=2^k \text{ and } \omega(m)=3.2^{2k-1}.$$

Now, $\phi(\omega(m))=\phi\left(3.2^{2k-1}\right)=2^{2k-1}$

Also, $\omega(\phi(m))=\omega\left(2^k\right)=2^{2k-2}=\frac{m2^k}{6}>m$ for $k>3$.

Thus $\phi(\omega(m))>\omega(\phi(m))$ for $k\geq 1$.

Moreover, $\phi(\omega(m))>\omega(\phi(m))>m$ for $k>3$.

(ii) Let $n=2^k5^l$ for any positive integer k and l. Then

$$\phi(n)=2^{k+1}5^{l-1}$$
$$\omega(n)=2^{2k}5^{2l-1}$$

Now, $\phi(\omega(n))=\phi\left(2^{2k}5^{2l-1}\right)=2^{2k+1}5^{2l-2}=\frac{2n^2}{25}<n^2$.

Also $\omega(\phi(n))=\omega\left(2^{k+1}5^{l-1}\right)=2^{2k+2}5^{2l-3}=\frac{4n^2}{125}$.

(iii) Let $t=2^k$ for any positive integer k. Then

$$\phi(\omega(t))=2^{2k-3}=\frac{t^2}{8}$$
$$\omega(\phi(t))=2^{2k-4}=\frac{t^2}{16}.$$

Thus $\omega(\phi(t))=2\phi(\omega(t))$.

Theorem 2.7: There are infinitely many positive integers m,n and t such that

(i) $m^2>\omega(\psi(m))>\psi(\omega(m))>m$

(ii) $n<\omega(\psi(n))<\psi(\omega(n))<n^2$

(iii) $\omega(\psi(t))=\psi(\omega(t))$

Proof. (i) Let $m=2^k3^l$ for any positive integers k and l. Then

$$\psi(m)=2^{k+1}3^l$$
$$\omega(m)=2^{2k-1}3^{2l-1}.$$

Now, $\omega(\psi(m))=\omega\left(2^{k+1}3^l\right)=2^{2k+1}3^{2l-1}=\frac{2m^2}{3}<m^2$.

And, $\psi(\omega(m))=\psi\left(2^{2k-1}3^{2l-1}\right)=2^{2k}3^{2l-1}>m$ for $l\geq 1$.

Thus we have $m^2>\omega(\psi(m))=2\psi(\omega(m))>\psi(\omega(m))>m$.

(ii) Let $n=2^k5^l$ for any positive integer k and $l\geq 2$. Then

$$\psi(n)=3^22^k5^{l-1}$$
$$\omega(n)=2^{2k}5^{2l-1}$$

Now, $\omega(\psi(n))=\omega\left(3^22^k5^{l-1}\right)=3^32^{2k}5^{2l-3}=\frac{27n^2}{125}>n$ for $l\geq 2$

And, $\psi(\omega(n))=\psi\left(2^{2k}5^{2l-1}\right)=3^22^{2k}5^{2l-2}=\frac{9n^2}{25}<n^2$.

Thus we have $n<\omega(\psi(n))=\frac{3}{5}\psi(\omega(n))<\psi(\omega(n))<n^2$.

(iv) Let $t=2^k$ for any positive integer k. Then

$$\psi(t)=2^{k-1}3$$
$$\omega(t)=2^{2k-2}$$

Now, $\omega(\psi(t))=2^{2k-3}3$ and $\psi(\omega(t))=2^{2k-3}3$.

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