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Some Generalizations of Integral Inequalities of Hermite-Hadamard Type for n-Time Differentiable Functions

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Abstract In the paper, by establishing two integral identities and Hölder integral inequality, the authors generalize some integral inequalities of Hermite-Hadamard type for n-time differentiable functions on a closed interval.

Keywords: *generalization, Hermite-Hadamard integral inequality, differentiable function, Hölder integral inequality*

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1. Introduction

Let f(x) be a convex function on [a; b], the famous Hermite-Hadamard integral inequality may be expressed as

$$0 \le \int_a^b f(t)dt - (b-a) f\left(\frac{a+b}{2}\right) \le (b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(t)dt. \tag{1.1}$$

It is well known that Hermite-Hadamard integral inequality is an important cornerstone in mathematical analysis and optimization. There has been a growing literature considering its refinements and interpolations. For more information, please refer to the monographs [3,4], the newly published papers [1,7], and plenty of references therein.

The following theorems are some refinements and generalizations of inequalities in (1.1).

Theorem 1.1 *([2] and [[5], Theorem A]). Let $f:[a,b]\in\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping and suppose that $\gamma \le f''(t) \le \Gamma$ for all $t\in(a,b)$. Then we have*

$$\frac{\gamma(b-a)^2}{24} \le \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(b-a)^2}{24} \tag{1.2}$$

and

$$\frac{\gamma(b-a)^2}{12} \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \le \frac{\Gamma(b-a)^2}{12}. \tag{1.3}$$

This theorem was generalized as follows.

Theorem 1.2 *([6] and [[5], Theorem B]). Let $f:[a,b]\in\mathbb{R}\to\mathbb{R}$ be a twice differentiable mapping and suppose that $\gamma \le f''(t) \le \Gamma$ for all $t\in(a,b)$, then*

$$\frac{3S_2-2\Gamma}{24}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \le \frac{3S_2-2\gamma}{24}(b-a)^2 \tag{1.4}$$

and

$$\frac{3S_2-2\Gamma}{24}(b-a)^2 \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \le \frac{3S_2-2\gamma}{24}(b-a)^2, \tag{1.5}$$

where

$$S_n = \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}, n\in\mathbb{N}. \tag{1.6}$$

The above two theorems were further generalized by the following theorems.

Theorem 1.3 ([[5], Theorem 1]). Let $f(t)$ be n-time differentiable on the closed interval $[a,b]$ such that $\gamma \le f^{(n)}(t) \le \Gamma$ for $t\in[a,b]$ and $n\in\mathbb{N}$. Further, let $u\in[a,b]$ be a parameter. Then

$$(b-a)S_n \max\left\{\frac{(u-a)^n}{n!},\frac{(b-u)^n}{n!}\right\}$$
$$+\left[\begin{array}{l}\frac{(u-a)^{n+1}-(u-b)^{n+1}}{(n+1)!}\\ -(b-a)\max\left\{\frac{(u-a)^n}{n!},\frac{(b-u)^n}{n!}\right\}\end{array}\right]\Gamma$$
$$\le(-1)^n\int_a^b f(t)dt$$
$$+\sum_{i=0}^{n-1}\frac{(u-a)^{n+i}-(u-b)^{n+i}}{(n-i)!}(-1)^i f^{(n-i-1)}(u)$$
$$\le(b-a)S_n\max\left\{\frac{(u-a)^n}{n!},\frac{(b-u)^n}{n!}\right\}$$
$$+\left[\begin{array}{l}\frac{(u-a)^{n+1}-(u-b)^{n+1}}{(n+1)!}\\ -(b-a)\max\left\{\frac{(u-a)^n}{n!},\frac{(b-u)^n}{n!}\right\}\end{array}\right]\lambda, \quad (1.7)$$

where S_n is defined by (1.6).

Theorem 1.4 (*[[5], Theorem 3]). Let $u\in\mathbb{R}$ and f(t) be n-time differentiable on the closed interval $[a,b]$ such that $\gamma\le f^{(n)}(t)\le\Gamma$ for $t\in[a,b]$ and $n\in\mathbb{N}$. Then*

$$\left[\begin{array}{l}(b-a)\max\left\{\frac{(u-a)^n}{n!},\frac{(b-u)^n}{n!}\right\}\\ +\frac{(b-u)^{n+1}-(a-u)^{n+1}}{(n+1)!}\end{array}\right]\gamma$$
$$-(b-a)S_n\max\left\{\frac{|u-a|^n}{n!},\frac{|b-u|^n}{n!}\right\}$$
$$\le(-1)^n\int_a^b f(t)dt$$
$$+\sum_{i=0}^{n-1}(-1)^i\frac{\left[\begin{array}{l}(b-u)^{n-i}f^{(n-i-1)}(b)\\ -(a-u)^{n-i}f^{(n-i-1)}(a)\end{array}\right]}{(n-i)!}$$
$$\le\left[\begin{array}{l}(b-a)\max\left\{\frac{|u-a|^n}{n!},\frac{|b-u|^n}{n!}\right\}\\ +\frac{(b-u)^{n+1}-(a-u)^{n+1}}{(n+1)!}\end{array}\right]\Gamma$$
$$-(b-a)S_n\max\left\{\frac{|u-a|^n}{n!},\frac{|b-u|^n}{n!}\right\}, \quad (1.8)$$

where S_n is defined by (1.6).

Theorem 1.5 ([[5], Theorem 5]). Let $\{P_i(t,x)\}_{i=0}^{\infty}$ be a harmonic sequence of polynomials, that is,

$$P_i'(t):=\frac{\partial P_i(t,x)}{\partial t}=P_{i-1}(t,x):=P_{i-1}(t) \quad (1.9)$$

and $P_0(t,x)=1$ for all defined (t,x) and $i\in N$. Further let $f(t)$ be n -time differentiable on $[a,b]$ such that $\gamma\le f^{(n)}(t)\le\Gamma$ for $t\in[a,b]$ and $n\in N$. Then, for any constant $\alpha\in\mathbb{R}$, we have

$$\left[\alpha+\max_{t\in[a,b]}|p_n(t)+\alpha|\right]S_n$$
$$-\left(\max_{t\in[a,b]}|p_n(t)+\alpha|+\frac{p_{n+1}(b)-p_{n+1}(a)}{b-a}+\alpha\right)\Gamma$$
$$\le(-1)^{n+1}\left[\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right.$$
$$\left.+\sum_{i=1}^{n}(-1)^i\frac{p_i(b)f^{(i-1)}(b)-p_i(a)f^{(i-1)}(a)}{b-a}\right] \quad (1.10)$$
$$\le\left[\alpha-\max_{t\in[a,b]}|p_n(t)+\alpha|\right]S_n$$
$$+\left(\max_{t\in[a,b]}|p_n(t)+\alpha|-\frac{p_{n+1}(b)-p_{n+1}(a)}{b-a}-\alpha\right)\Gamma$$

and

$$\left[\alpha-\max_{t\in[a,b]}|p_n(t)+\alpha|\right]S_n$$
$$+\left(\max_{t\in[a,b]}|p_n(t)+\alpha|-\frac{p_{n+1}(b)-p_{n+1}(a)}{b-a}-\alpha\right)\gamma$$
$$\le(-1)^{n+1}\left[\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right.$$
$$\left.+\sum_{i=1}^{n}(-1)^i\frac{p_i(b)f^{(i-1)}(b)-p_i(a)f^{(i-1)}(a)}{b-a}\right] \quad (1.11)$$
$$\le\left[\alpha+\max_{t\in[a,b]}|p_n(t)+\alpha|\right]S_n$$
$$-\left(\max_{t\in[a,b]}|p_n(t)+\alpha|+\frac{p_{n+1}(b)-p_{n+1}(a)}{b-a}+\alpha\right)\gamma.$$

where S_n is defined by (1.6).

Theorem 1.6 ([[5], **Theorem** 7]). Let $\{P_i(t)\}_{i=0}^{\infty}$ and $\{Q_i(t)\}_{i=0}^{\infty}$ be two harmonic sequences of polynomials, α and β be two real constants, and $u\in[a,b]$. Further let $f(t)$ be n -time differentiable on $[a,b]$ such that $\gamma\le f^{(n)}(t)\le\Gamma$ for $t\in[a,b]$ and $n\in\mathrm{N}$. Then

$$\left[\frac{Q_{n+1}(b)-P_{n+1}(a)}{b-a}+\frac{P_{n+1}(u)-Q_{n+1}(u)}{b-a}\right.$$
$$\left.+\frac{(\alpha-\beta)u+(b\beta-a\alpha)}{b-a}+C(u)\right]\gamma-C(u)S_n$$
$$\le\frac{(-1)^n}{b-a}\int_a^b f(t)\mathrm{d}t \quad (1.12)$$
$$+\sum_{i=0}^{n-1}(-1)^{n+i}\frac{Q_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a}$$

$$+\sum_{i=0}^{n-1}(-1)^{n+i}\frac{P_i(u)-Q_i(u)}{b-a}f^{(i-1)}(u)$$
$$+\frac{\beta f^{(n-1)}(b)-\alpha f^{n}(a)}{b-a}+\frac{(\alpha-\beta)f^{(n-1)}(u)}{b-a} \quad (1.12)$$
$$\le\left[\frac{Q_{n+1}(b)-P_{n+1}(a)}{b-a}+\frac{P_{n+1}(u)-Q_{n+1}(u)}{b-a}\right.$$
$$\left.+\frac{(\alpha-\beta)u+(b\beta-a\alpha)}{b-a}-C(u)\right]\gamma+C(u)S_n$$

and

$$\left[\frac{Q_{n+1}(b)-P_{n+1}(a)}{b-a}+\frac{P_{n+1}(u)-Q_{n+1}(u)}{b-a}\right.$$
$$\left.+\frac{(\alpha-\beta)u+(b\beta-a\alpha)}{b-a}-C(u)\right]\Gamma+C(u)S_n$$
$$\le\frac{(-1)^n}{b-a}\int_a^b f(t)\mathrm{d}t$$
$$+\sum_{i=0}^{n-1}(-1)^{n+i}\frac{Q_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a}$$
$$+\sum_{i=0}^{n-1}(-1)^{n+i}\frac{P_i(u)-Q_i(u)}{b-a}f^{(i-1)}(u) \quad (1.13)$$
$$+\frac{\beta f^{(n-1)}(b)-\alpha f^{n}(a)}{b-a}+\frac{(\alpha-\beta)f^{(n-1)}(u)}{b-a}$$
$$\le\left[\frac{Q_{n+1}(b)-P_{n+1}(a)}{b-a}+\frac{P_{n+1}(u)-Q_{n+1}(u)}{b-a}\right.$$
$$\left.+\frac{(\alpha-\beta)u+(b\beta-a\alpha)}{b-a}+C(u)\right]\Gamma-C(u)S_n,$$

Where S_n is defined by (1.6) and

$$C(u)=\max\left\{\max_{t\in[a,u]}\left|P_n(t)+\alpha\right|,\ \max_{t\in[u,b]}\left|Q_n(t)+\beta\right|\right\}.$$

The aim of this paper is to, by establishing two integral identities and Hölder integral inequality, generalize the above six theorems recited from [5] to more general cases.

2. Lemmas

For generalizing the above six theorems recited from [5] to more general cases, we need the following integral identities.

Lemma 2.1. For $n\in\mathrm{N}$, let $f:[a,b]\to\mathbb{R}$ be a n-time differentiable function on $[a,b]$, and let $g_1:[a,x]\to\mathbb{R}$ and $g_2:[x,b]\to\mathbb{R}$ be n-time differentiable functions for some $x\in[a,b]$, than

$$\int_a^b g(t)f^{(n)}(t)\mathrm{d}t$$
$$=\sum_{i=0}^{n-1}(-1)^i\left[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(n-i-1)}(x)\right. \quad (2.1)$$
$$\left.+(g_2^{(i)}(b)f^{(n-i-1)}(b)-g_1^{(i)}(a)f^{(n-i-1)}(a))\right]$$
$$+(-1)^n\int_a^b g^{(n)}(t)f(t)\mathrm{d}t,$$

where

$$g(t)=\begin{cases}g_1(t),\ t\in[a,x],\\ g_2(t),\ t\in(x,b].\end{cases}$$
$$g^{(i)}(t)=\begin{cases}g_1^{(i)}(t),\ t\in[a,x),\\ g_2^{(i)}(t),\ t\in(x,b],\end{cases} \quad (2.2)$$

and $g^{(i)}(x^-)=g_1^{(i)}(x),g^{(i)}(x^+)=g_2^{(i)}(x)$ for $1\le i\le n$.

Proof. When $n=1$, it is not difficult to obtain that

$$\int_a^b g(t)f'(t)\mathrm{d}t=(g_1(x)-g_2(x))f(x)$$
$$+(g_2(b)f(b)-g_1(a)f(a))-\int_a^b g'(t)f(t)\mathrm{d}t.$$

Suppose that the inequality (2.1) holds for $n=k\ge 2$. For $n=k+1$, by integration by parts, we obtain

$$\int_a^b g(t)f^{(k+1)}(t)\mathrm{d}t=\int_a^b g(t)\left[f'(t)\right]^{(k)}\mathrm{d}t$$
$$=\sum_{i=0}^{k-1}(-1)^i\left[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(k-i)}(x)\right.$$
$$\left.+(g_2^{(i)}(b)f^{(k-i)}(b)-g_1^{(i)}(a)f^{(k-i)}(a))\right]$$
$$+(-1)^k\int_a^b g^{(k)}(t)f'(t)\mathrm{d}t$$
$$=\sum_{i=0}^{k}(-1)^i\left[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(k-i)}(x)\right.$$
$$\left.+(g_2^{(i)}(b)f^{(k-i)}(b)-g_1^{(i)}(a)f^{(k-i)}(a))\right]$$
$$+(-1)^{k+1}\int_a^b g^{(k+1)}(t)f(t)\mathrm{d}t.$$

By induction, the proof of inequality (2.1) is complete.

Lemma 2.2 For $n\in\mathrm{N}$, let $f:[a,b]\to\mathbb{R}$ be a n-time differentiable function on $[a,b]$ and, for $x\in[a,b]$ let $g_1:[a,x]\to\mathbb{R}$ and $g_2:[x,b]\to\mathbb{R}$ be n-time differentiable functions, then

$$\int_a^b g(t)f^{(n)}(t)\mathrm{d}t$$
$$=\sum_{i=0}^{n-1}(-1)^i\left[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(n-i-1)}(x)\right.$$
$$\left.+(g_2^{(i)}(b)f^{(n-i-1)}(b)-g_1^{(i)}(a)f^{(n-i-1)}(a))\right] \quad (2.3)$$
$$+(\alpha-\beta)f^{(n-1)}(x)+(\beta f^{(n-1)}(b)-\alpha f^{(n-1)}(a))$$
$$+(-1)^n\int_a^b g^{(n)}(t)f(t)\mathrm{d}t.$$

where

$$g(t)=\begin{cases}g_1(t)+\alpha,\ t\in[a,x],\\ g_2(t)+\beta,\ t\in(x,b]\end{cases} \quad (2.4)$$

and $g^{(i)}(t)$ for $1\le i\le n$ are same with (2.2).

3. Main results

Now we are in a position to generalize the above six theorems recited from [5] to more general cases.

Theorem 3.1. For $n \in \mathrm{N}$, let $f:[a,b]\subset \mathbb{R}\to\mathbb{R}$ be n-time differentiable such that $\gamma \le f^{(n)}(t) \le \Gamma$ for $t\in[a,b]$. for $x\in[a,b]$ let $g_1:[a,x]\to\mathbb{R}$ $g_2:[x,b]\to\mathbb{R}$ are n-time differentiable functions. Then

$$\begin{aligned}
&(b-a)S_n \max\left\{\max_{t\in[a,x]}|g_1(t)|, \max_{t\in[x,b]}|g_2(t)|\right\} \\
&+\left(G(a,b;g)-(b-a)\max\left\{\begin{matrix}\max_{t\in[a,x]}|g_1(t)|,\\ \max_{t\in[x,b]}|g_2(t)|\end{matrix}\right\}\right)\Gamma \\
&\le \sum_{i=0}^{n-1}(-1)^i\Big[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(n-i-1)}(x) \\
&+(g_2^{(i)}(b)f^{(n-i-1)}(b)-g_1^{(i)}(a)f^{(n-i-1)}(a))\Big] \qquad (3.1)\\
&+(-1)^n\int_a^b g^{(n)}(t)f(t)\mathrm{d}t \\
&\le (b-a)S_n \max\left\{\max_{t\in[a,x]}|g_1(t)|, \max_{t\in[x,b]}|g_2(t)|\right\} \\
&+\left(G(a,b;g)-(b-a)\max\left\{\begin{matrix}\max_{t\in[a,x]}|g_1(t)|,\\ \max_{t\in[x,b]}|g_2(t)|\end{matrix}\right\}\right)\gamma,
\end{aligned}$$

where S_n is defined by (1.6), $g(t)$ and $g^{(i)}(t)$ are defined as in (2.2) and

$$G(\mathrm{a,b;g}) = \frac{1}{b-a}\int_a^b g(t)\mathrm{d}t. \qquad (3.2)$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}
&\int_a^b g(t)[f^{(n)}(t)-\gamma]\mathrm{d}t \\
&=(-1)^n\int_a^b g^{(n)}(t)f(t)\mathrm{d}t-\gamma(b-a)G(a,b;g) \\
&+\sum_{i=0}^{n-1}(-1)^i\Big[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(n-i-1)}(x) \qquad (3.3)\\
&+(g_2^{(i)}(b)f^{(n-i-1)}(b)-g_1^{(i)}(a)f^{(n-i-1)}(a))\Big]
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b g(t)[\Gamma-f^{(n)}(t)]\mathrm{d}t \\
&=\Gamma(b-a)G(a,b;g)-(-1)^n\int_a^b g^{(n)}(t)f(t)\mathrm{d}t \\
&-\sum_{i=0}^{n-1}(-1)^i\Big[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(n-i-1)}(x) \qquad (3.4)\\
&+(g_2^{(i)}(b)f^{(n-i-1)}(b)-g_1^{(i)}(a)f^{(n-i-1)}(a))\Big].
\end{aligned}$$

On the other hand, by the Hölder inequality,

$$\begin{aligned}
&\int_a^b g(t)[f^{(n)}(t)-\gamma]\mathrm{d}t \le \int_a^b |g(t)|\left|f^{(n)}(t)-\gamma\right|\mathrm{d}t \\
&\le \max_{t\in[a,b]}|g(t)|\left|\int_a^b [f^{(n)}(t)-\gamma]\mathrm{d}t\right| \qquad (3.5)\\
&=(b-a)(S_n-\gamma)\max\left\{\max_{t\in[a,x]}|g_1(t)|, \max_{t\in[x,b]}|g_2(t)|\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b g(t)[\Gamma-f^{(n)}(t)]\mathrm{d}t \le \int_a^b |g(t)|\left|\Gamma-f^{(n)}(t)\right|\mathrm{d}t \\
&\le \max_{t\in[a,b]}|g(t)|\left|\int_a^b [\Gamma-f^{(n)}(t)]\mathrm{d}t\right| \qquad (3.6)\\
&=(b-a)(\Gamma-S_n)\max\left\{\max_{t\in[a,x]}|g_1(t)|, \max_{t\in[x,b]}|g_2(t)|\right\}.
\end{aligned}$$

Combining (3.3) to (3.6) yields (3.1). Theorem 3.1 is thus proved.

Remark 1. From taking

$$g(t)=\begin{cases}\dfrac{(t-a)^n}{n!}, & t\in[a,x],\\[2mm] \dfrac{(t-b)^n}{n!}, & t\in(x,b]\end{cases}$$

in (3.1), the double inequality (1.7) followes.

If taking $x=b, g_2(t)=0$ in Theorem 3.1, we can derive the following corollary.

Corollary 3.1.1. For $n\in\mathrm{N}$, let $f:[a,b]\to\mathbb{R}$ be n-time differentiable such that $\gamma\le f^{(n)}(t)\le\Gamma$ for $t\in[a,b]$ and let $g:[a,b]\to\mathbb{R}$ be n-time differentiable. Then

$$\begin{aligned}
&(b-a)S_n\max_{t\in[a,b]}|g(t)| \\
&+(b-a)[G(a,b;g)-\max_{t\in[a,b]}|g(t)|]\Gamma \\
&\le(-1)^n\int_a^b g^{(n)}(t)f(t)\mathrm{d}t \\
&+\sum_{i=0}^{n-1}(-1)^i\Big[g^{(i)}(b)f^{(n-i-1)}(b)-g^{(i)}(a)f^{(n-i-1)}(a)\Big] \qquad (3.7)\\
&\le(b-a)S_n\max_{t\in[a,b]}|g(t)| \\
&+(b-a)[G(a,b;g)-\max_{t\in[a,b]}|g(t)|]\gamma.
\end{aligned}$$

Proof. This follow from putting $x=b$, $g(t)=g_1(t)$, and $g_2(t)=0$ in Theorem 3.1.

Remark 2. If letting $g(t)=\dfrac{(t-u)^n}{n!}$ for $u\in\mathbb{R}$ in (3.7), the double inequality (1.8) may be derived.

Corollary 3.1.2. Under the conditions of Theorem 3.1, if $x=\dfrac{a+b}{2}$, then

$$\begin{aligned}
&(b-a)\left[S_n\max\left\{\max_{t\in[a,x]}|g_1(t)|, \max_{t\in[x,b]}|g_2(t)|\right\}\right. \\
&\left.+\left(G(a,b;g)-\max\left\{\max_{t\in[a,x]}|g_1(t)|, \max_{t\in[x,b]}|g_2(t)|\right\}\right)\Gamma\right] \\
&\le\sum_{i=0}^{n-1}(-1)^i\Big[(g_1^{(i)}(\tfrac{a+b}{2})-g_2^{(i)}(\tfrac{a+b}{2}))f^{(n-i-1)}(\tfrac{a+b}{2}) \\
&+(g_2^{(i)}(b)f^{(n-i-1)}(b)-g_1^{(i)}(a)f^{(n-i-1)}(a))\Big]
\end{aligned}$$

$$+(-1)^n\int_a^b g^{(n)}(t)f(t)\mathrm{d}t$$
$$\le (b-a)\left[S_n \max\left\{\max_{t\in[a,x]}|g_1(t)|, \max_{t\in[x,b]}|g_2(t)|\right\}\right. \quad (3.8)$$
$$\left.+\left(G(a,b;g)-\max\left\{\max_{t\in[a,x]}|g_1(t)|, \max_{t\in[x,b]}|g_2(t)|\right\}\right)\gamma\right].$$

Corollary 3.1.3. Under the conditions of Theorem 3.1, if $n=2$, then

$$(b-a)\left[S_2 \max\left\{\max_{t\in[a,x]}|g_1(t)|, \max_{t\in[x,b]}|g_2(t)|\right\}\right.$$
$$\left.+\left(G(a,b;g)-\max\left\{\max_{t\in[a,x]}|g_1(t)|, \max_{t\in[x,b]}|g_2(t)|\right\}\right)\Gamma\right]$$
$$\le \sum_{i=0}^{1}(-1)^i\left[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(1-i)}(x)\right.$$
$$\left.+g_2^{(i)}(b)f^{(1-i)}(b)-g_1^{(i)}(a)f^{(1-i)}(a)\right] \quad (3.9)$$
$$+\int_a^b g''(t)f(t)\mathrm{d}t$$
$$\le (b-a)\left[S_2 \max\left\{\max_{t\in[a,x]}|g_1(t)|, \max_{t\in[x,b]}|g_2(t)|\right\}\right.$$
$$\left.+\left(G(a,b;g)-\max\left\{\max_{t\in[a,x]}|g_1(t)|, \max_{t\in[x,b]}|g_2(t)|\right\}\right)\gamma\right]$$

Theorem 3.2. For $n\in\mathrm{N}$, let $f:[a,b]\to\mathbb{R}$ be a n-time differentiable function on $[a,b]$ and, for $x\in[a,b]$ let $g_1:[a,x]\to\mathbb{R}$ and $g_2:[x,b]\to\mathbb{R}$ be n-time differentiable functions. Then, for α,β being real constants,

$$[G(a,b;g)+H(x)]\gamma - H(x)S_n$$
$$\le \frac{(-1)^n}{b-a}\int_a^b g^{(n)}(t)f(t)\mathrm{d}t$$
$$+\frac{(\alpha-\beta)f^{(n-1)}(x)+\beta f^{(n-1)}(b)-\alpha f^{(n-1)}(a)}{b-a} \quad (3.10)$$
$$+\sum_{i=0}^{n-1}\frac{(-1)^i}{b-a}\left[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(n-i-1)}(x)\right.$$
$$\left.+g_2^{(i)}(b)f^{(n-i-1)}(b)-g_1^{(i)}(a)f^{(n-i-1)}(a)\right]$$
$$\le [G(a,b;g)-H(x)]\gamma + H(x)S_n$$

and

$$[G(a,b;g)-H(x)]\Gamma + H(x)S_n$$
$$\le \frac{(-1)^n}{b-a}\int_a^b g^{(n)}(t)f(t)\mathrm{d}t$$
$$+\frac{(\alpha-\beta)f^{(n-1)}(x)+\beta f^{(n-1)}(b)-\alpha f^{(n-1)}(a)}{b-a} \quad (3.11)$$
$$+\sum_{i=0}^{n-1}\frac{(-1)^i}{b-a}\left[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(n-i-1)}(x)\right.$$
$$\left.+g_2^{(i)}(b)f^{(n-i-1)}(b)-g_1^{(i)}(a)f^{(n-i-1)}(a)\right]$$
$$\le [G(a,b;g)+H(x)]\Gamma - H(x)S_n,$$

where $S_n, G(a,b;g)$, $g(t)$ and $G^{(i)}(t)$ are defined respectively by (1.6), (3.2), (2.4), (2.2) and

$$H(x)=\max\left\{\max_{t\in[a,x]}|g_1(t)+\alpha|, \max_{t\in[x,b]}|g_2(t)+\beta|\right\}. \quad (3.12)$$

Proof. Applying Lemma 2.2 results in

$$\int_a^b g(t)[f^{(n)}(t)-\gamma]\mathrm{d}t$$
$$=(-1)^n\int_a^b g^{(n)}(t)f(t)\mathrm{d}t-(b-a)\gamma G(a,b;g)$$
$$+(\alpha-\beta)f^{(n-1)}(x)+(\beta f^{(n-1)}(b)-\alpha f^{(n-1)}(a))$$
$$+\sum_{i=0}^{n-1}(-1)^i\left[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(n-i-1)}(x)\right.$$
$$\left.+(g_2^{(i)}(b)f^{(n-i-1)}(b)-g_1^{(i)}(a)f^{(n-i-1)}(a))\right]$$

and

$$\int_a^b g(t)[\Gamma-f^{(n)}(t)]\mathrm{d}t$$
$$=(-1)^{n+1}\int_a^b g^{(n)}(t)f(t)\mathrm{d}t+(b-a)\Gamma G(a,b;g)$$
$$-(\alpha-\beta)f^{(n-1)}(x)-(\beta f^{(n-1)}(b)-\alpha f^{(n-1)}(a))$$
$$-\sum_{i=0}^{n-1}(-1)^i\left[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(n-i-1)}(x)\right.$$
$$\left.+(g_2^{(i)}(b)f^{(n-i-1)}(b)-g_1^{(i)}(a)f^{(n-i-1)}(a))\right].$$

It is easy to show, by the Hölder inequality, that

$$\left|\int_a^b g(t)[f^{(n)}(t)-\gamma]\mathrm{d}t\right|$$
$$\le \int_a^b |g(t)|\left|f^{(n)}(t)-\gamma\right|\mathrm{d}t$$
$$\le \max_{t\in[a,b]}|g(t)|\int_a^b[f^{(n)}(t)-\gamma]\mathrm{d}t$$
$$=(b-a)(S_n-\gamma)H(x)$$

and

$$\left|\int_a^b g(t)[\Gamma-f^{(n)}(t)]\mathrm{d}t\right|$$
$$\le \int_a^b |g(t)|\left|\Gamma-f^{(n)}(t)\right|\mathrm{d}t$$
$$\le \max_{t\in[a,b]}|g(t)|\int_a^b[f^{(n)}(t)-\gamma]\mathrm{d}t$$
$$=(b-a)(\Gamma-S_n)H(x).$$

Combining the above identitie and inequalities yields Theorem 3.2.

Remark 3.3. For $\alpha,\beta\in\mathbb{R}$, setting

$$g(t)=\begin{cases}P_n(t)+\alpha, & t\in[a,x],\\ Q_n(t)+\beta, & t\in(x,b]\end{cases}$$

in Theorem 3.2, where $\{P_i(t)\}_{i=0}^{\infty}$ and $\{Q_i(t)\}_{i=0}^{\infty}$ are two harmonic sequences of polynomials, reveals the double inequalities (1.12) and (1.13).

Corollary 3.1.1. Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be n-time differentiable on the closed interval $[a,b]$ such that

$\gamma \le f^{(n)}(t) \le \Gamma$ for $t \in [a,b]$, $x \in [a,b]$ and $g:[a,b] \to \mathbb{R}$ be n-time differentiable function for $n \in N$, let α, β be a real constant. Then

$$\begin{aligned}
&[G(a,b;g)+H(x)]\gamma - H(x)S_n \\
&\le \frac{(-1)^n}{b-a}\int_a^b g^{(n)}(t)f(t)\mathrm{d}t + \frac{\alpha(f^{(n-1)}(b)-f^{(n-1)}(a))}{b-a} \\
&+\sum_{i=0}^{n-1}(-1)^i \frac{g^{(i)}(b)f^{(n-i-1)}(b)-g^{(i)}(a)f^{(n-i-1)}(a))}{b-a} \\
&\le [G(a,b;g)-H(x)]\gamma + H(x)S_n
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&[G(a,b;g)-H(x)]\Gamma + H(x)S_n \\
&\le \frac{(-1)^n}{b-a}\int_a^b g^{(n)}(t)f(t)\mathrm{d}t + \frac{\alpha(f^{(n-1)}(b)-f^{(n-1)}(a))}{b-a} \\
&+\sum_{i=0}^{n-1}(-1)^i \frac{g(i)(b)f^{(n-i-1)}(b)-g^{(i)}(a)f^{(n-i-1)}(a))}{b-a} \\
&\le [G(a,b;g)+H(x)]\Gamma - H(x)S_n.
\end{aligned} \tag{3.14}$$

Proof. This follows from taking $x = b$, $g(t) = g_1(t)$, $g_2(t) = 0$ and $\alpha = \beta$ in Theorem 3.2.

Remark 3.4. Taking $g(t) = P_n(t) + \alpha$ in (3.13) and (3.14), $\{P_i(t)\}_{i=0}^{\infty}$ be a harmonic of polynomials may derive the double inequalities (1.10) and (1.11).

Corollary 3.2.2. Under the conditions of Theorem 3.2, we have

$$\begin{aligned}
&[G(a,b;g)+H(\tfrac{a+b}{2})]\gamma - H(\tfrac{a+b}{2})S_n \\
&\le \frac{(-1)^n}{b-a}\int_a^b g^{(n)}(t)f(t)\mathrm{d}t + \frac{(\alpha-\beta)f^{(n-1)}(\tfrac{a+b}{2})}{b-a} \\
&+\frac{(\beta f^{(n-1)}(b)-\alpha f^{(n-1)}(a))}{b-a} \\
&+\sum_{i=0}^{n-1}\frac{(-1)^i}{b-a}\Big[(g_1^{(i)}(\tfrac{a+b}{2})-g_2^{(i)}(\tfrac{a+b}{2}))f^{(n-i-1)}(x) \\
&+g_2^{(i)}(b)f^{(n-i-1)}(b)-g_1^{(i)}(a)f^{(n-i-1)}(a)\Big] \\
&\le [G(a,b;g)-H(\tfrac{a+b}{2})]\gamma + H(\tfrac{a+b}{2})S_n
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&[G(a,b;g)-H(\tfrac{a+b}{2})]\Gamma + H(\tfrac{a+b}{2})S_n \\
&\le \frac{(-1)^n}{b-a}\int_a^b g^{(n)}(t)f(t)\mathrm{d}t + \frac{(\alpha-\beta)f^{(n-1)}(\tfrac{a+b}{2})}{b-a} \\
&+\frac{\beta f^{(n-1)}(b)-\alpha f^{(n-1)}(a)}{b-a} \\
&+\sum_{i=0}^{n-1}\frac{(-1)^i}{b-a}\Big[(g_1^{(i)}(\tfrac{a+b}{2})-g_2^{(i)}(\tfrac{a+b}{2}))f^{(n-i-1)}(\tfrac{a+b}{2}) \\
&+(g_2^{(i)}(b)f^{(n-i-1)}(b)-g_1^{(i)}(a)f^{(n-i-1)}(a))\Big] \\
&\le [G(a,b;g)+H(\tfrac{a+b}{2})]\Gamma - H(\tfrac{a+b}{2})S_n.
\end{aligned} \tag{3.16}$$

Proof. This follows from putting $x = \frac{a+b}{2}$ in Theorem 3.2.

Corollary 3.2.3. Under the conditions of Theorem 3.2, if $n = 2$, then

$$\begin{aligned}
&[G(a,b;g)+H(x)]\gamma - H(x)S_2 \\
&\le \frac{1}{b-a}\int_a^b g''(t)f(t)\mathrm{d}t \\
&+\frac{(\alpha-\beta)f'(x)}{b-a} + \frac{\beta f'(b)-\alpha f'(a)}{b-a} \\
&+\sum_{i=0}^{1}\frac{(-1)^i}{b-a}\Big[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(1-i)}(x) \\
&+g_2^{(i)}(b)f^{(1-i)}(b)-g_1^{(i)}(a)f^{(1-i)}(a)\Big] \\
&\le [G(a,b;g)-H(x)]\gamma + H(x)S_2
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&[G(a,b;g)-H(x)]\Gamma + H(x)S_2 \\
&\le \frac{1}{b-a}\int_a^b g''(t)f(t)\mathrm{d}t \\
&+\frac{(\alpha-\beta)f'(x)}{b-a} + \frac{\beta f'(b)-\alpha f'(a)}{b-a} \\
&+\sum_{i=0}^{1}\frac{(-1)^i}{b-a}\Big[(g_1^{(i)}(x)-g_2^{(i)}(x))f^{(1-i)}(x) \\
&+g_2^{(i)}(b)f^{(1-i)}(b)-g_1^{(i)}(a)f^{(1-i)}(a)\Big] \\
&\le [G(a,b;g)+H(x)]\Gamma - H(x)S_2.
\end{aligned} \tag{3.18}$$

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