

Hermite-Hadamard Type Inequalities for (m, h_1, h_2) -Convex Functions Via Riemann-Liouville Fractional Integrals

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Abstract In the paper, via Riemann-Liouville fractional integration, the authors present some new inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are (m, h_1, h_2) -convex.

Keywords: Riemann-Liouville fractional integral, (m, h_1, h_2) -convex function, integral inequality of Hermite-Hadamard type

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1. Introduction

The following definitions are well known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad (1)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. If the inequality (1) reverses, then f is said to be concave on I .

The well-known Hermite-Hadamard inequality reads that for every convex function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}, \quad \text{where}$$

$a, b \in I$ with $a < b$. If f is concave, the above inequalities reverse.

Definition 1.2. ([2]) For $s \in (0, 1]$, a function $f : I \subseteq \mathbb{R}_0 = [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \quad (2)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. If the above inequality (2) reverses, then f is said to be s -concave on I .

Definition 1.3. ([6]) Let $(0, 1) \subseteq J \subseteq \mathbb{R}$, $I \subseteq \mathbb{R}$ be an interval, and $h : J \rightarrow \mathbb{R}_0$. A function $f : I \rightarrow \mathbb{R}_0$ is said to be h -convex if the inequality

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y) \quad (3)$$

If the above inequality (3) reverses, then f is said to be h -concave on I .

Definition 1.4. ([10]) For $f : [0, b] \rightarrow \mathbb{R}$, $\alpha, m \in (0, 1]$, if

$$f(\lambda x + m(1-\lambda)y) \leq \lambda^\alpha f(x) + m(1-\lambda^\alpha)f(y)$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that $f(x)$ is a (α, m) -convex function on $[0, b]$.

There have been many inequalities of Hermite-Hadamard type for the above convex functions. Some of them may be recited as follows.

Theorem 1.1. ([2]) Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° , where $a, b \in I^\circ$ and $a < b$. If $|f'(x)|$ is a convex function on $[a, b]$, then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 1.2. ([3]) Let f is a differentiable function on $[a, b]$, $|f'|^q$ is convex function on $[a, b]$, where $q > 1$, then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}.$$

Theorem 1.3. ([4]) Let $f : I \rightarrow \mathbb{R}$ is a differentiable function on I° , where $a, b \in I^\circ, a < b, p > 1$, if $|f'(x)|^{p/(p-1)}$ is convex function on $[a, b]$, then

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{16} \left(\frac{4}{p+1} \right)^{1/p} \\ & \times \left\{ \left[|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)} \right]^{1-1/p} \right. \\ & \left. + \left[3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)} \right]^{1-1/p} \right\}. \end{aligned}$$

Theorem 1.4. ([5]) Let $f : \mathbb{R}_0 \rightarrow \mathbb{R}_0$ is s -convex function, where $s \in (0, 1), a, b \in \mathbb{R}_0, a < b$, if $f' \in L_1([a, b])$, then

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{s+1}.$$

Definition 1.6. ([1]) Let $f \in L_1([a, b])$, The Riemann-Liouville integrals $J_{a+}^\alpha f(x)$ and $J_{b-}^\alpha f(x)$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, x < b$$

Respectively, where $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$. and $\Gamma(\alpha)$ is the classical Euler gamma function be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt.$$

Theorem 1.5. ([12]) Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b, f \in L_1([a, b])$, If f is a convex function on $[a, b]$ then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) & \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \\ & \leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

with $\alpha > 0$.

Theorem 1.6. ([12]) Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$, If $|f'|$ is convex on $[a, b]$, then

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right| \\ & \leq \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^\alpha} \right) [f'(a) + f'(b)]. \end{aligned}$$

In this paper, motivated by the above results, we will establish a Riemann-Liouville fractional integral identity involving a differentiable mapping and present some new inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals for (m, h_1, h_2) -convex functions.

2. A Definition and A Lemma

In the most recent paper [11], Maksa and Palés introduced even more general notion of convexity. More precisely, (α, β, a, b) -convex functions are defined as solutions f of the functional inequality

$$f(\alpha(t)x + \beta(t)y) \leq a(t)f(x) + b(t)f(y),$$

where $0 \neq T \subseteq [0, 1]$ and $\alpha, \beta, a, b : T \rightarrow \mathbb{R}$ are given functions.

We first introduce a definition of (m, h_1, h_2) -convex functions.

Definition 2.1. Assume $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}, h_1, h_2 : [0, 1] \rightarrow \mathbb{R}_0$, and $m \in (0, 1]$. Then f is said to be (m, h_1, h_2) -convex if the inequality

$$f(\lambda x + m(1-\lambda)y) \leq h_1(\lambda)f(x) + mh_2(\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$. If the above inequality reverses, then f is said to be (m, h_1, h_2) -concave on I .

Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on (a, b) and $f' \in L_1([a, b])$. Denote $M_\alpha(a, b)$ by

$$\begin{aligned} M_\alpha(a, b) & = \frac{1}{3} \left[f(a) + \frac{1}{2} \left(f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right) \right) \right] \\ & - \frac{\Gamma(\alpha+1)3^{\alpha-1}}{(b-a)^\alpha} \times \left[J_{a+}^\alpha f\left(\frac{2a+b}{3}\right) + J_{\frac{a+2b}{3}+}^\alpha f\left(\frac{a+2b}{3}\right) \right] \\ & + J_{\frac{a+2b}{3}+}^\alpha f(b). \end{aligned}$$

Specially, when $\alpha = 1$, we have

$$\begin{aligned} M_1(a, b) & = \frac{1}{3} \left[f(a) + \frac{1}{2} \left(f\left(\frac{2a+b}{3}\right) \right. \right. \\ & \left. \left. + f\left(\frac{a+2b}{3}\right) \right) \right] + f(b) - \frac{1}{b-a} \int_a^b f(t) dt. \end{aligned}$$

Lemma 2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on (a, b) such that $f' \in L_1([a, b])$. Then

$$\begin{aligned} M_\alpha(a, b) & = \frac{b-a}{9} \\ & \times \left\{ \int_0^1 (0-t^\alpha) f' \left(ta + (1-t) \frac{2a+b}{3} \right) dt \right. \\ & + \int_0^1 \left(\frac{1}{2} - t^\alpha \right) f' \left(t \frac{2a+b}{3} + (1-t) \frac{a+2b}{3} \right) dt \\ & \left. + \int_0^1 (1-t^\alpha) f' \left(t \frac{a+2b}{3} + (1-t)b \right) dt \right\}. \end{aligned}$$

Proof. Letting $u = at + (1-t)\frac{2a+b}{3}$. By integration by parts, we have

$$\begin{aligned} & \frac{b-a}{9} \int_0^1 (-t^\alpha) f' \left(ta + (1-t)\frac{2a+b}{3} \right) dt \\ &= \frac{1}{3} \left[f(a) - \alpha \int_0^1 f \left(ta + (1-t)\frac{2a+b}{3} \right) t^{\alpha-1} dt \right] \\ &= \frac{1}{3} f(a) - \frac{\alpha 3^{\alpha-1}}{(b-a)^\alpha} \int_a^{\frac{2a+b}{3}} \left(\frac{2a+b}{3} - u \right)^{\alpha-1} f(u) du \\ &= \frac{1}{3} f(a) - \frac{\Gamma(\alpha+1)3^{\alpha-1}}{(b-a)^\alpha} J_{a^+}^\alpha f \left(\frac{2a+b}{3} \right). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} & \frac{b-a}{9} \int_0^1 \left(\frac{1-t^\alpha}{2} \right) f' \left(t\frac{2a+b}{3} + (1-t)\frac{a+2b}{3} \right) dt \\ &= \frac{1}{6} \left(f \left(\frac{2a+b}{3} \right) + f \left(\frac{a+2b}{3} \right) \right), \\ & \frac{b-a}{9} \int_0^1 \left(\frac{1-t^\alpha}{2} \right) f' \left(t\frac{2a+b}{3} + (1-t)\frac{a+2b}{3} \right) dt \\ &= \frac{1}{6} \left(f \left(\frac{2a+b}{3} \right) + f \left(\frac{a+2b}{3} \right) \right) \\ & \quad - \frac{\Gamma(\alpha+1)3^{\alpha-1}}{(b-a)^\alpha} J_{\frac{2a+b}{3}^+}^\alpha f \left(\frac{a+2b}{3} \right) \end{aligned}$$

and

$$\begin{aligned} & \frac{b-a}{9} \int_0^1 (1-t^\alpha) f' \left(t\frac{a+2b}{3} + (1-t)b \right) dt \\ &= \frac{1}{3} f(b) - \frac{\Gamma(\alpha+1)3^{\alpha-1}}{(b-a)^\alpha} J_{\frac{a+2b}{3}^+}^\alpha f(b). \end{aligned}$$

The proof of Lemma 2.1 is complete.

3. New Inequalities for (m, h_1, h_2) -Convex Functions

Theorem 3.1. Let $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$ be a differentiable function such that $f' \in L_1([a, b])$ for $0 < a < b$. If $|f'|^q$ is (m, h_1, h_2) -convex on $[a, b]$ for $q > 1$ and $h_1, h_2 \in L_1([a, b])$, then

$$\begin{aligned} |M_\alpha(a, b)| &\leq \frac{b-a}{9} \left[\left(\frac{q-1}{\alpha q + q-1} \right)^{1-1/q} \right. \\ &\times \left(|f'(a)|^q \|h_1\|_1 + m \left| f' \left(\frac{2a+b}{3m} \right) \right|^q \|h_2\|_1 \right)^{1/q} \\ &+ \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \right)^{1-1/q} \\ &\times \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q \|h_1\|_1 + m \left| f' \left(\frac{a+2b}{3m} \right) \right|^q \|h_2\|_1 \right)^{1/q} \end{aligned}$$

$$\begin{aligned} & + \left(\frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right) \right)^{1-1/q} \\ & \times \left(\left| f' \left(\frac{a+2b}{3} \right) \right|^q \|h_1\|_1 + m \left| f' \left(\frac{b}{m} \right) \right|^q \|h_2\|_1 \right)^{1/q} \end{aligned}$$

where $\|h_1\|_p = \left(\int_0^1 h_1^p(t) dt \right)^{1/p}$ for $p \geq 1$ and $B(x, y)$ is the classical Beta function which may be defined by

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x > 0, y > 0.$$

Proof. From Lemma 2.1, Hölder inequality, and the (m, h_1, h_2) -convexity of $|f'|^q$, we obtain

$$\begin{aligned} |M_\alpha(a, b)| &\leq \frac{b-a}{9} \left[\left(\int_0^1 t^{\alpha q/(q-1)} dt \right)^{1-1/q} \right. \\ &\times \left(|f'(a)|^q \|h_1\|_1 + m \left| f' \left(\frac{2a+b}{3m} \right) \right|^q \|h_2\|_1 \right)^{1/q} \\ &+ \left(\frac{1}{2} \right)^{1/q} \left(\int_0^1 \left| \frac{1-t^\alpha}{2} \right| dt \right)^{1-1/q} \\ &\times \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q \|h_1\|_1 + m \left| f' \left(\frac{a+2b}{3m} \right) \right|^q \|h_2\|_1 \right)^{1/q} \\ &+ \left(\int_0^1 (1-t^\alpha)^{q/(q-1)} dt \right)^{1-1/q} \\ &\times \left(\left| f' \left(\frac{a+2b}{3} \right) \right|^q \|h_1\|_1 + m \left| f' \left(\frac{b}{m} \right) \right|^q \|h_2\|_1 \right)^{1/q} \end{aligned}$$

where

$$\begin{aligned} \int_0^1 t^{\alpha q/(q-1)} dt &= \frac{q-1}{\alpha q + q-1}, \\ \int_0^1 (1-t^\alpha)^{q/(q-1)} dt &= \frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right), \end{aligned}$$

and

$$\int_0^1 \left| \frac{1-t^\alpha}{2} \right| dt = \frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{2(\alpha + 1)}.$$

The proof of Theorem 3.1 is complete.

Corollary 3.1.1. Under the conditions of Theorem 3.1, if $h_1(t) = h(t)$, $h_2(t) = h(1-t)$, then

$$\begin{aligned} |M_\alpha(a, b)| &\leq \frac{(b-a) \|h_1\|_1^{1/q}}{9} \left[\left(\frac{q-1}{\alpha q + q-1} \right)^{1-1/q} \right. \\ &\times \left(|f'(a)|^q + m \left| f' \left(\frac{2a+b}{3m} \right) \right|^q \right)^{1/q} \\ &+ \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \right)^{1-1/q} \end{aligned}$$

$$\begin{aligned} & \times \left[\left| f' \left(\frac{2a+b}{3} \right) \right|^q + m \left| f' \left(\frac{a+2b}{3m} \right) \right|^q \right]^{1/q} \\ & + \left(\frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right) \right)^{1-1/q} \\ & \times \left[\left| f' \left(\frac{a+2b}{3} \right) \right|^q + m \left| f' \left(\frac{b}{m} \right) \right|^q \right]^{1/q} \end{aligned}$$

Furthermore, if $m = 1$, then

$$\begin{aligned} |M_\alpha(a, b)| & \leq \frac{(b-a) \|h\|_1^{1/q}}{9} \left[\left(\frac{q-1}{\alpha q + q - 1} \right)^{1-1/q} \right. \\ & \times \left. \left(|f'(a)|^q + \left| f' \left(\frac{2a+b}{3} \right) \right|^q \right)^{1/q} \right. \\ & + \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \right)^{1-1/q} \\ & \times \left. \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q + \left| f' \left(\frac{a+2b}{3} \right) \right|^q \right)^{1/q} \right. \\ & + \left. \left(\frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right) \right)^{1-1/q} \right. \\ & \times \left. \left(\left| f' \left(\frac{a+2b}{3} \right) \right|^q + |f'(b)|^q \right)^{1/q} \right]. \end{aligned}$$

Corollary 3.1.2. Under the conditions of Corollary

3.1.1, if $h_1(t) = h(t) = t^s$, $m = 1$, then

$$\begin{aligned} |M_\alpha(a, b)| & \leq \frac{b-a}{9} \left(\frac{1}{s+1} \right)^{1/q} \\ & \times \left[\left(\frac{q-1}{\alpha q + q - 1} \right)^{1-1/q} \left(|f'(a)|^q + \left| f' \left(\frac{2a+b}{3} \right) \right|^q \right)^{1/q} \right. \\ & + \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \right)^{1-1/q} \times \left. \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q \right. \right. \\ & \left. \left. + \left| f' \left(\frac{a+2b}{3} \right) \right|^q \right)^{1/q} \right. \\ & + \left. \left(\frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right) \right)^{1-1/q} \times \left(\left| f' \left(\frac{a+2b}{3} \right) \right|^q \right. \right. \\ & \left. \left. + |f'(b)|^q \right)^{1/q} \right]; \end{aligned}$$

Specially, if $\alpha = s = m = 1$, then

$$\begin{aligned} |M_1(a, b)| & \leq \frac{b-a}{9} \left(\frac{1}{2} \right)^{1/q} \\ & \times \left[\left(\frac{q-1}{2q-1} \right)^{1-1/q} \left(|f'(a)|^q + \left| f' \left(\frac{2a+b}{3} \right) \right|^q \right)^{1/q} \right. \\ & + \left. \left(\frac{1}{2} \right)^{2-1/q} \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q + \left| f' \left(\frac{a+2b}{3} \right) \right|^q \right)^{1/q} \right. \\ & + \left. \left(\frac{q-1}{2q-1} \right)^{1-1/q} \left(\left| f' \left(\frac{a+2b}{3} \right) \right|^q + |f'(b)|^q \right)^{1/q} \right]. \end{aligned}$$

Corollary 3.1.3. Under the conditions of Theorem 3.1, if $h_1(t) = t^{\alpha_1}$ and $h_2(t) = 1 - t^{\alpha_1}$, then

$$\begin{aligned} |M_\alpha(a, b)| & \leq \frac{b-a}{9} \left(\frac{1}{\alpha_1 + 1} \right)^{1/q} \\ & \times \left[\left(\frac{q-1}{\alpha q + q - 1} \right)^{1-1/q} \times \left(|f'(a)|^q \right. \right. \\ & \left. \left. + m \alpha_1 \left| f' \left(\frac{2a+b}{3m} \right) \right|^q \right)^{1/q} \right. \\ & + \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \right)^{1-1/q} \\ & \times \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q + m \alpha_1 \left| f' \left(\frac{a+2b}{3m} \right) \right|^q \right)^{1/q} \\ & + \left(\frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right) \right)^{1-1/q} \\ & \times \left(\left| f' \left(\frac{a+2b}{3} \right) \right|^q + m \alpha_1 \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{1/q} \right]. \end{aligned}$$

Specially, if $m = 1$, then

$$\begin{aligned} |M_\alpha(a, b)| & \leq \frac{b-a}{9} \left(\frac{1}{\alpha_1 + 1} \right)^{1/q} \\ & \times \left[\left(\frac{q-1}{\alpha q + q - 1} \right)^{1-1/q} \left(|f'(a)|^q \right. \right. \\ & \left. \left. + \alpha_1 \left| f' \left(\frac{2a+b}{3} \right) \right|^q \right)^{1/q} \right. \\ & + \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \right)^{1-1/q} \\ & \times \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q + \alpha_1 \left| f' \left(\frac{a+2b}{3} \right) \right|^q \right)^{1/q} \\ & + \left(\frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right) \right)^{1-1/q} \\ & \times \left(\left| f' \left(\frac{a+2b}{3} \right) \right|^q + \alpha_1 |f'(b)|^q \right)^{1/q} \right]. \end{aligned}$$

Theorem 3.2. Let $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$ be a differentiable function such that $f' \in L_1([a, b])$ and $0 < a < b$. If $|f'|^q$ is (m, h_1, h_2) -convex on $[a, b]$ for $h_1, h_2 \in L_1([a, b])$ and $q \geq 1$, then

$$\begin{aligned} |M_\alpha(a, b)| &\leq \frac{b-a}{9} \left(\frac{1}{\alpha+1}\right)^{1-1/q} \\ &\times \left\{ \left[\frac{1}{2} |f'(a)|^q \left(\frac{1}{2\alpha+1} + \|h_1\|_2^2 \right) \right. \right. \\ &+ \left. \frac{m}{2} \left| f' \left(\frac{2a+b}{3m} \right) \right|^q \left(\frac{1}{2\alpha+1} + \|h_2\|_2^2 \right) \right]^{1/q} \\ &+ \frac{1}{2} \left(\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1 \right)^{1-1/q} \\ &\times \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q \|h_1\|_1 + m \left| f' \left(\frac{a+2b}{3m} \right) \right|^q \|h_2\|_1 \right)^{1/q} \\ &+ \alpha^{1-1/q} \left[\frac{1}{2} \left| f' \left(\frac{a+2b}{3} \right) \right|^q \left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)} \right. \right. \\ &+ \left. \left. \frac{m}{2} \left| f' \left(\frac{b}{m} \right) \right|^q \left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)} + \|h_2\|_2^2 \right) \right] \right\}^{1/q}, \end{aligned}$$

where where $\|h_1\|_p$ is as given in Theorem 3.1.

Proof. From Lemma 2.1, Hölder inequality, and the (m, h_1, h_2) -convexity of $|f'|^q$, we obtain

$$\begin{aligned} |M_\alpha(a, b)| &\leq \frac{b-a}{9} \left[\left(\int_0^1 t^\alpha dt \right)^{1-1/q} \right. \\ &\times \left. \left(\left| f'(a) \right|^q \int_0^1 t^\alpha h_1(t) dt + m \left| f' \left(\frac{2a+b}{3m} \right) \right|^q \int_0^1 t^\alpha h_2(t) dt \right)^{1/q} \right. \\ &+ \left(\int_0^1 \left| \frac{1}{2} - t^\alpha \right| dt \right)^{1-1/q} \left(\frac{1}{2} \right)^{1/q} \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q \int_0^1 h_1(t) dt \right. \\ &+ \left. m \left| f' \left(\frac{a+2b}{3m} \right) \right|^q \int_0^1 h_2(t) dt \right)^{1/q} + \left(\int_0^1 (1-t^\alpha) dt \right)^{1-1/q} \\ &\times \left(\left| f' \left(\frac{a+2b}{3} \right) \right|^q \int_0^1 (1-t^\alpha) h_1(t) dt \right. \\ &+ \left. m \left| f' \left(\frac{b}{m} \right) \right|^q \int_0^1 (1-t^\alpha) h_2(t) dt \right)^{1/q} \\ &\leq \frac{b-a}{9} \left[\left(\int_0^1 t^\alpha dt \right)^{1-1/q} \left(|f'(a)|^q \int_0^1 \frac{t^{2\alpha} + h_1^2(t)}{2} dt \right. \right. \\ &+ \left. \left. \left(\frac{1}{2} \right)^{1/q} \left(\int_0^1 \left| \frac{1}{2} - t^\alpha \right| dt \right)^{1-1/q} \times \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q \int_0^1 h_1(t) dt \right) \right. \right. \end{aligned}$$

$$\begin{aligned} &+ \left. m \left| f' \left(\frac{a+2b}{3m} \right) \right|^q \int_0^1 h_2(t) dt \right)^{1/q} \\ &+ \left(\int_0^1 (1-t^\alpha) dt \right)^{1-1/q} \times \left[\left| f' \left(\frac{a+2b}{3} \right) \right|^q \right. \\ &\left. \int_0^1 \frac{(1-t^\alpha)^2 + h_1^2(t)}{2} dt \right. \\ &+ \left. m \left| f' \left(\frac{b}{m} \right) \right|^q \int_0^1 \frac{(1-t^\alpha)^2 + h_2^2(t)}{2} dt \right]^{1/q} \\ &= \frac{b-a}{9} \left(\frac{1}{\alpha+1} \right)^{1-1/q} \left[\left(\frac{1}{2} |f'(a)|^q \left(\frac{1}{2\alpha+1} \right. \right. \right. \\ &+ \left. \left. \left. \frac{m}{2} \left| f' \left(\frac{2a+b}{3m} \right) \right|^q \left(\frac{1}{2\alpha+1} + \|h_2\|_2^2 \right) \right) \right)^{1/q} \right. \\ &+ \frac{1}{2} \left(\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1 \right)^{1-1/q} \\ &\times \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q \|h_1\|_1 + m \left| f' \left(\frac{a+2b}{3m} \right) \right|^q \|h_2\|_1 \right)^{1/q} \\ &+ \alpha^{1-1/q} \left[\frac{1}{2} \left| f' \left(\frac{a+2b}{3} \right) \right|^q \left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)} \right) \right. \\ &+ \left. \frac{m}{2} \left| f' \left(\frac{b}{m} \right) \right|^q \left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)} + \|h_2\|_2^2 \right) \right]^{1/q}. \end{aligned}$$

The proof of Theorem 3.2 is complete.

Corollary 3.2.1. Under the conditions of Theorem 3.2,

(1) if $h_1(t) = t^{\alpha_1}$ and $h_2(t) = 1 - t^{\alpha_1}$, then

$$\begin{aligned} |M_\alpha(a, b)| &\leq \frac{b-a}{9} \left(\frac{1}{\alpha+1} \right)^{1-1/q} \\ &\times \left[\left(\frac{1}{2} |f'(a)|^q \left(\frac{1}{2\alpha+1} + \frac{1}{2\alpha_1+1} \right) \right. \right. \\ &+ \left. \frac{m}{2} \left| f' \left(\frac{2a+b}{3m} \right) \right|^q \left(\frac{1}{2\alpha+1} \right. \right. \\ &+ \left. \left. \frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)} \right) \right]^{1/q} \\ &+ \frac{1}{2} \left(\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1 \right)^{1-1/q} \\ &\times \left[\left| f' \left(\frac{2a+b}{3} \right) \right|^q \frac{1}{\alpha_1+1} \right. \\ &+ \left. m \left| f' \left(\frac{a+2b}{3m} \right) \right|^q \frac{\alpha_1}{\alpha_1+1} \right]^{1/q} \end{aligned}$$

$$\begin{aligned}
& + \alpha^{1-1/q} \left(\frac{1}{2} \left| f' \left(\frac{a+2b}{3} \right) \right|^q \right. \\
& \times \left[\left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)} + \frac{1}{2\alpha_1+1} \right) + \frac{m}{2} \left| f' \left(\frac{b}{m} \right) \right|^q \right. \\
& \left. \left. \times \left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)} \right) \right]^{1/q} \right. \\
& \left. \times \left[\frac{2\alpha_1^2}{(\alpha_1+1)(2\alpha_1+1)} \right] \right];
\end{aligned}$$

(2) if $h_1(t) = h(t)$, $h_2(t) = h(1-t)$, and $m = 1$, then

$$\begin{aligned}
|M_\alpha(a,b)| & \leq \frac{b-a}{9} \left(\frac{1}{\alpha+1} \right)^{1-1/q} \\
& \times \left[\left(\frac{1}{2} \left(\frac{1}{2\alpha+1} + \|h\|_2^2 \right) \left(|f'(a)|^q + \left| f' \left(\frac{2a+b}{3} \right) \right|^q \right) \right)^{1/q} \right. \\
& + \frac{1}{2} \left(\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1 \right)^{1-1/q} \\
& \times \left(\|h\|_1 \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q + \left| f' \left(\frac{a+2b}{3} \right) \right|^q \right) \right)^{1/q} \\
& + \alpha^{1-1/q} \left(\frac{1}{2} \left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)} + \|h\|_2^2 \right) \right. \\
& \left. \times \left[\left| f' \left(\frac{a+2b}{3} \right) \right|^q + |f'(b)|^q \right] \right]^{1/q};
\end{aligned}$$

(3) if $h_1(t) = h(t) = t^s$, $h_2(t) = h(1-t)$, and $m = 1$, then

$$\begin{aligned}
|M_\alpha(a,b)| & \leq \frac{b-a}{9} \left(\frac{1}{\alpha+1} \right)^{1-1/q} \\
& \times \left[\left(\frac{1}{2} \left(\frac{1}{2\alpha+1} + \frac{1}{2s+1} \right) \left(|f'(a)|^q + \left| f' \left(\frac{2a+b}{3} \right) \right|^q \right) \right)^{1/q} \right. \\
& + \frac{1}{2} \left(\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1 \right)^{1-1/q} \\
& \times \left(\frac{1}{s+1} \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q + \left| f' \left(\frac{a+2b}{3} \right) \right|^q \right) \right)^{1/q} \\
& + \alpha^{1-1/q} \left(\frac{1}{2} \left(\frac{2\alpha^2}{(1+\alpha)(1+2\alpha)} + \frac{1}{2s+1} \right) \right. \\
& \left. \times \left[\left| f' \left(\frac{a+2b}{3} \right) \right|^q + |f'(b)|^q \right] \right]^{1/q};
\end{aligned}$$

(4) if $h_1(t) = h(t) = t^s$, $h_2(t) = h(1-t)$ and $\alpha = s = m = 1$, then

$$\begin{aligned}
|M_\alpha(a,b)| & \leq \frac{b-a}{9} \left(\frac{1}{2} \right)^{1-1/q} \\
& \times \left[\left(\frac{1}{3} \left(|f'(a)|^q + \left| f' \left(\frac{2a+b}{3} \right) \right|^q \right) \right)^{1/q} \right. \\
& + \frac{1}{2} \left(\frac{1}{2} \left(\left| f' \left(\frac{2a+b}{3} \right) \right|^q + \left| f' \left(\frac{a+2b}{3} \right) \right|^q \right) \right)^{1/q} \\
& \left. + \left(\frac{1}{3} \left(\left| f' \left(\frac{a+2b}{3} \right) \right|^q + |f'(b)|^q \right) \right)^{1/q} \right];
\end{aligned}$$

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