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Some Fixed Point Theorems in b-metric Space

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Abstract In this paper we have obtained some fixed point theorems on *b*- metric space which is an extension of a fixed point theorem by Hardy [13] and Reich [20].

Keywords: *b-metric space, fixed point*

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1. Introduction

In the development of non-linear analysis, fixed point theory plays a very important role. Also, it has been widely used in different branches of engineering and sciences.

Metric fixed point theory is an essential part of mathematical analysis because of its applications in different areas like variational and linear inequalities, improvement, and approximation theory. The fixed point theorem in metric spaces plays a significant role to construct methods to solve the problems in mathematics and sciences.

Although metric fixed point theory is vast field of study and is capable of solving many equations. To overcome the problem of measurable functions w.r.t. a measure and their convergence, Czerwik [8] needs an extension of metric space. Using this idea, he presented a generalization of the renowned Banach fixed point theorem in the b-metric spaces (see also [9,10,11]). Many researchers including Aydi [1], Boriceanu [3,4,5], Bota [6], Chug [7], Du [12], Kir [14], Olaru [15], Olatinwo [16], Păcurar [17,18], Rao [19], Roshan [21] and Shi [22] studied the extension of fixed point theorems in b-metric space.

In this paper, our aim is to show the validity of some important fixed point results into b-metric spaces.

2. Preliminaries

We recall some definitions and properties for b-metric spaces given by Czerwik [8].

Definition 2.1. If $M(\neq \phi)$ is a set having $s(>=1) \in \mathbb{R}$ then a self-map ρ on M is called a b-metric if the following conditions are satisfied:

(i) $\rho(x,y)=0$ if and only x = y;

(ii) $\rho(x,y)=\rho(y,x)$;

(iii) $\rho(x,z) \le s.[\rho(x,y)+\rho(y,z)]$ for all $x,y,z \in M$.

The pair (M,ρ) s called a b-metric space.

From the above definition it is evident that the b-metric space extended the metric space. Here, for s = 1 it reduces into standard metric space.

Let us have a look on some example [2] of b-metric space:

Example 2.1. *The space* $l_p,(0<p<1)$,

$$l_p = \{(x_n) \subset R : \sum_{n=1}^{\infty} |x_n|^p < \infty\},$$

together with the function $\rho : l_p \times l_p \to \mathbb{R}$ *where*

$$\rho(x,y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$$

where $x = x_n, y = y_n \in l_p$ *is a b-metric space. By an elementary calculation we obtain that*

$$\rho(x,y) \le 2^{\frac{1}{p}}[\rho(x,y)+\rho(y,z)]$$

Example 2.2. The space $l_p,(0<p<1)$, of all real functions $x(t), t \in [0,1]$ such that

$$\int_0^1 |x(t)|^p dt < \infty,$$

is b-metric space if we take

$$\rho(x,y) = (\int_0^1 \left| |x(t)-y(t)|^p dt \right)^{\frac{1}{p}}$$

for each $x,y \in l_p$.

Now we present the definition of Cauchy sequence, convergent sequence and complete b-metric space.

Definition 2.2. [8] Let (*M*; ρ) be a *b*-metric space then $\{x_n\}$ in *M* is called

(a) A Cauchy sequence iff $\forall\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$, such that for each $n, m \geq n(\epsilon)$ we have $\rho(x_n, x_m) < \epsilon$.

(b) convergent sequence if and only if there exist $x \in M$ such that for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$, such that for every $n \geq n(\epsilon)$ we have $\rho(x_n, x) < \epsilon$.

Definition 2.3. [8] 1. If (M, ρ) is a b-metric space then a subset $L \subset M$ is called

(i) compact iff for every sequence of elements of L there exists a subsequence that converges to an element of L.

(ii) closed iff for each sequence $\{x_n\}$ in L which converges to an element x, we have x □ L.

2. The b-metric space is complete if every Cauchy sequence converges.

To prove the theorem 3.2 and 3.4 we will use the following lemma 2.1 [23].

Lemma 2.1. *Suppose* (M, ρ) *be a b-metric space and* $\{y_n\}$ *be a sequence in M such that*

$$\rho(y_{n+1}, y_{n+2}) \leq \lambda\rho(y_n, y_{n+1}),\ n = 0,1,\ldots \qquad (2.1)$$

where $0 \leq \lambda < 1$ *Then the sequence* $\{y_n\}$ *is a Cauchy sequence in M provided that* $s.\lambda < 1$.

3. Main Result

The following theorem is given by Reich [20]:

Theorem 3.1. *Let M be a complete metric space with metric* ρ *and let* $T : M \to M$ *be a function with the following property*

$$\rho(T(x), T(y)) \leq a\rho(x, T(x)) + b\rho(y, T(y)) + c\rho(x, y)$$

for all $x, y \in M$ *where* a, b, c *are non-negative and satisfy a + b + c < 1. Then T has a unique fixed point.*

We have extended the above theorem 3.1 to the b-metric space.

Theorem 3.2. *Let M be a complete b-metric space with metric* ρ *and let* $T : M \to M$ *be a function with the following*

$$\rho(T(x), T(y)) \leq a\rho(x, T(x)) + b\rho(y, T(y)) + c\rho(x, y) \qquad (3.1)$$

$\forall$x; y $\in$ M, *where a, b, c are non-negative real numbers and satisfy* $a + s(b + c) < 1$ *for* $s \geq 1$ *then T has a unique fixed point.*

Proof. Let $x_0 \in M$ and $\{x_n\}$ be a sequence in *M*, such that

$$x_n = Tx_{n-1} = T^n x_0$$

Now

$$\rho(x_{n+1}, x_n) = \rho(Tx_n, Tx_{n-1})$$
$$\leq a\rho(x_n, T(x_n)) + b\rho(x_{n-1}, T(x_{n-1})) + c\rho(x_n, x_{n-1})$$
$$= a\rho(x_n, x_{n+1}) + b\rho(x_{n-1}, x_n) + c\rho(x_n, x_{n-1})$$
$$\Rightarrow (1-a)\rho(x_{n+1}, x_n) \leq (b+c)\rho(x_n, x_{n-1})$$
$$\Rightarrow \rho(x_{n+1}, x_n) \leq \frac{(b+c)}{(1-a)}\rho(x_n, x_{n-1}) = p\rho(x_n, x_{n-1})$$

continuing this process we can easily say that

$$\rho(x_{n+1}, x_n) \leq p^n \rho(x_0, x_1)$$

This implies that *T* is a contraction mapping.

Now, it is to show that $\{x_n\}$ is a Cauchy sequence in *M*. Let *m*, *n* > 0, with *m* > *n*

$$\rho(x_n, x_m) \leq s[\rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_m)]$$
$$\leq s\rho(x_n, x_{n+1}) + s^2\rho(x_{n+1}, x_{n+2})$$
$$+ s^3\rho(x_{n+2}, x_{n+3}) + \ldots$$
$$\leq sp^n\rho(x_0, x_1) + s^2p^{n+1}\rho(x_0, x_1)$$
$$+ s^3p^{n+2}\rho(x_0, x_1) + \ldots$$
$$= sp^n\rho(x_0, x_1)[1 + sp + (sp)^2 + (sp)^3 + \ldots]$$
$$= \frac{sp^n}{1-sp}\rho(x_0, x_1)$$

Now using lemma 2.1 and taking limit $n \to \infty$ we get

$$\lim_{n\to\infty} \rho(x_n, x_m) = 0$$

$\Rightarrow \{x_n\}$ is a Cauchy sequence in *M*. Since *M* is complete, we consider that $\{x_n\}$ converges to x^*.

Now, we show that x^* is fixed point of *T*.

we have

$$\rho(x^*, T(x^*)) \leq s[\rho(x^*, x_n) + \rho(x_n, T(x^*))]$$
$$\leq s[\rho(x^*, x_n) + \rho(T(x_{n-1}), T(x^*))]$$
$$\leq s[\rho(x^*, x_n) + a\rho(x^*, T(x^*))$$
$$+ b\rho(x_{n-1}, T(x_{n-1})) + c\rho(x_{n-1}, x^*)]$$

$$(1-as)\rho(x^*, T(x^*)) \leq s[\rho(x^*, x_n)$$
$$+ b\rho(x_{n-1}, x_n) + c\rho(x_{n-1}, x^*)]$$

$$\rho(x^*, T(x^*) \leq \frac{s}{(1-as)}[\rho(x^*, x_n) + b\rho(x_{n-1}, x_n)$$
$$+ c\rho(x_{n-1}, x^*)]$$
$$\leq [\rho(x^*, x_n) + bp^n\rho(x_0, x_1)$$
$$+ c\rho(x_{n-1}, x^*)]$$

Taking lim $n \to \infty$, we get

$$\lim_{n\to\infty} \rho(x^*, T(x^*)) = 0$$
$$\Rightarrow x^* = T(x^*)$$

$\Rightarrow x^*$ is the fixed point of *T*.

Now, for the uniqueness of fixed point.

Let *x* and *y* be two fixed points of *T*

$$\Rightarrow x = T(x),\ y = T(y)$$

$$\rho(x, y) = \rho(T(x), T(y))$$
$$\leq a\rho(x, T(x)) + b\rho(y, T(y)) + c\rho(x, y)$$

$\Rightarrow \rho(x, y) \leq c\rho(x, y)$. which is a contradiction. The proof is complete.

Now we will discuss the extension of the following theorem given by Hardy and Rogers [13] to the b-metric space as our second result in theorem 3.4.

Theorem 3.3. *Let (M,ρ) be a metric space and $T: M \to M$ a mapping satisfies the following condition for all $x, y \in M$.*

(i)

$$\rho(Tx,Ty) \le a.\rho(x,Tx) + b.\rho(y,Ty) + c.\rho(x,Ty) + e.\rho(y,Tx) + f.\rho(x,y),$$

where a, b, c, e, f are nonnegative and we set α = a + b + c + d + e + f. Then

(a) If M is complete metric space and α < 1 then T has a unique fixed point.

(b) If (i) is modified to the condition. $x \ne y$ then this implies

$$\rho(Tx,Ty) \le a.\rho(x,Tx) + b.\rho(y,Ty) + c.\rho(x,Ty) + e.\rho(y,Tx) + f.\rho(x,y),$$

and in this case we assume M is compact. T is continuous and α = 1, then T has a unique fixed point.

Here we have studied the extension of theorem 3.3 in the b-metric space.

Theorem 3.4. Let (M,ρ) be a complete b-metric space and a mapping $T: M \to M$ satisfying the following condition for all $x, y \in M$.

$$\rho(Tx,Ty) \le a\rho(x,Tx) + b\rho(y,Ty) + c\rho(x,Ty) + e\rho(y,Tx) + f\rho(x,y) \tag{3.2}$$

where a, b, c, e, f are nonnegative and we set α = a + b + c + e + f, such that $a \in (0, \frac{1}{2s})$. for $s \ge 1$ then T has a unique fixed point.

Before going to prove this theorem we require following lemma 3.1 [13].

Lemma 3.1. *Let the condition 3.2 hold on (M,ρ) for a self map T on it. Then if $a \in (0, \frac{1}{2s})$ there exist $\rho < \frac{1}{2s}$ such that*

$$\rho(Tx,T^2x) \le \beta\rho(x,Tx). \tag{3.3}$$

Proof. Let y = Tx in (3.2) and simplify to get

$$\rho(Tx,T^2x) \le \frac{a+f}{1-b}\rho(x,Tx) + \frac{c}{1-b}\rho(x,T^2x) \tag{3.4}$$

Now using triangular inequality $\rho(x,T^2x) \le s[\rho(x,Tx) + \rho(Tx,T^2x)]$ so from 3.4 we obtain

$$\frac{1}{s}\rho(T^2x,x) - \rho(Tx,x) \le \frac{a+f}{1-b}\rho(x,Tx) + \frac{c}{1-b}\rho(x,T^2x)$$

on simplifying

$$\rho(T^2x,x) \le \frac{(1+a+f-b)s}{1-b-c.s}\rho(x,Tx) \tag{3.5}$$

Now substituting inequality (3.5) into (3.4), we get

$$\rho(Tx,T^2x) \le \left(\frac{a+f+c.s}{1-b-c.s}\right)\rho(x,Tx) \tag{3.6}$$

using symmetry, we can exchange a with b and c with e in (3.6) to obtain

$$\rho(Tx,T^2x) \le \left(\frac{b+f+e.s}{1-b-e.s}\right)\rho(x,Tx) \tag{3.7}$$

and then

$$\beta = \min\left(\frac{a+f+c.s}{1-b-c.s}, \frac{b+f+e.s}{1-b-e.s}\right) \tag{3.8}$$

satisfies the conclusion of this lemma.

Proof of Theorem 3.4. Let $x_0 \in M$ and $\{x_n\}$ be a sequence in *M*, such that

$$x_n = Tx_{n-1} = T^n x_0$$

Now using lemma 3.1 we can show that

$$\rho(x_{n+1}, x_n) \le \beta^n \rho(x_0, x_1)$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in *M*.

Let *m*, *n* > 0, with *m* > *n*

$$\begin{aligned}\rho(x_n,x_m) &\le s[\rho(x_n,x_{n+1}) + \rho(x_{n+1},x_m)] \\ &\le s\rho(x_n,x_{n+1}) + s^2\rho(x_{n+1},x_{n+2}) \\ &\quad + s^3\rho(x_{n+2},x_{n+3}) + \ldots \\ &\le s\beta^n\rho(x_0,x_1) + s^2\beta^{n+1}\rho(x_0,x_1) \\ &\quad + s^3\beta^{n+2}\rho(x_0,x_1) + \ldots\end{aligned}$$

when taking lim $n \to \infty$ we get

$$\lim_{n\to\infty} \rho(x_n,x_m) = 0$$

$\Rightarrow \{x_n\}$ is a Cauchy sequence in *M*. Since *M* is complete, we consider that $\{x_n\}$ converges to x^*.

Now, we show that x^* is fixed point of *T*.

we have

$$\begin{aligned}&\rho(x^*,T(x^*)) \le s[\rho(x^*,x_n) + \rho(x_n,T(x^*))] \\ &\le s[\rho(x^*,x_n) + \rho(T(x_{n-1}),T(x^*))] \\ &\le s[\rho(x^*,x_n) + a\rho((x_{n-1},T(x_{n-1})) + b\rho((x^*,T(x^*)) \\ &\quad + c\rho(x_{n-1},T(x^*)) + e\rho(x^*,T(x_{n-1})) + f\rho(x_{n-1},x^*) \\ &\Rightarrow \rho(x^*,T(x^*)) \le s[a\rho(x_{n-1},x_n) + b\rho(x^*,T(x^*)) \\ &\quad + c\rho(x_{n-1},T(x^*)) + (e+1)\rho(x^*,x_n) + f\rho(x_{n-1},x^*)]\end{aligned}$$

Taking lim $n \to \infty$ we get

$$\rho(x^*,T(x^*)) \le s(b+c)\rho(x^*,T(x^*))$$

which contradicts unless $x^* = T(x^*)$.

Now, we show the uniqueness of fixed point.

Let *x* and *y* be two fixed points of *T*.

$$\begin{aligned}&\Rightarrow x = T(x),\ y = T(y) \\ &\rho(x,y) = \rho(T(x),T(y)) \\ &\qquad \le a\rho(x,T(x)) + b\rho(y,T(y)) + c\rho(x,T(y)) \\ &\qquad + e\rho(y,T(x)) + f\rho(x,y)\end{aligned}$$

$\le (c+e+f)\rho(x,y)$ which is a contradiction. The proof is complete.

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