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On Interval Valued Generalized Difference Classes Defined by Orlicz Function

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Abstract In this paper, using the difference operator and Orlicz functions, we introduce and examine some generalized difference sequence spaces of interval numbers. We prove completeness properties of these spaces. Further, we investigate some inclusion relations related to these spaces.

Keywords: *sequence space, interval numbers, difference sequence, completeness*

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1. Introduction

The work of interval arithmetic was originally introduced by Dwyer [3] in 1951. The development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [15] and Moore and Yang [16]. Furthermore, Moore and others [3]; [4]; [10] and [17] have developed applications to differential equations. Chiao in [2] introduced sequence of interval numbers and defined usual convergence of sequences of interval numbers. Şengönül and Eryilmaz in [20] introduced and studied bounded and convergent sequence spaces of interval numbers. Recently Esi studied strongly λ-and strongly almost λ-convergent sequences spaces of the interval numbers in [5], respectively. Also, Esi studied some new type sequence space of the interval numbers in [6,7] and lacunary sequence spaces for interval numbers in [8]. In Hazarika [11] introduced the notion of λ-ideal convergent interval valued di¤erence classes defined by Musielak-Orlicz function.

Kizmaz [12] introduced the notion of di¤erence sequence spaces as follows:

$$X(\Delta)=\left\{x=(x_k):(\Delta x_k)\in X\right\}$$

for $X=l_\infty$, c and c_0. Later on, the notion was generalized by Et and Çolak [9] as follows:

$$X\left(\Delta^n\right)=\left\{x=(x_k):\left(\Delta^n x_k\right)\in X\right\}$$

for $X=l_\infty$, c and c_0, where $\Delta^n x=\left(\Delta^n x_k\right)=\left(\Delta^{n-1}x_k-\Delta^{n-1}x_{k+1}\right)$, $\Delta^0 x=x$ and also this generalized difference notion has the following binomial representation:

$$\Delta^n x_k=\sum_{i=0}^{n}(-1)^i\binom{n}{i}x_{k+i}\quad \textit{for all } k\in\mathbb{N}.$$

Recall in [18], [13] that an Orlicz function M is continuous, convex, non-decreasing function define for $x>0$ such that $M(0)=0$ and $M(x)>0$ for $x>0$ and $M(x)\to 0$ as $x\to\infty$. If convexity of Orlicz function is replaced by $M(x+y)\le M(x)+M(y)$ then this function is called the modulus function and characterized by Ruckle [19]. An Orlicz function M is said to satisfy Δ_2*-condition* for all values u, if there exists $K>0$ such that $M(2u)\le KM(u), u\ge 0$. Subsequently, the notion of Orlicz function was used to defined sequence spaces by Altin et. al., [1], Tripathy and Mahanta [21], Tripathy et. al., [22], Tripathy and Sarma [23] and many others.

2. Preliminaries

A set consisting of a closed interval of real numbers $\bar{x}$ such that $a\le x\le b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties. We denote the set of all real valued closed intervals by IR. Any elements of IR is called closed interval and denoted by $\bar{x}$. That is $\bar{x}=\left\{x\in\mathbb{R}:a\le x\le b\right\}$. An interval number $\bar{x}$ is a closed subset of real numbers (see [2]). Let x_l and x_r be first and last points of the interval number $\bar{x}$, respectively. For $\bar{x}=\left[x_{1_l},x_{1_r}\right]$, $\bar{y}=\left[x_{2_l},x_{2_r}\right]\in IR$, we have

$$\bar{x}=\bar{y}\Leftrightarrow x_{1_l}=x_{2_l}, x_{1_r}=x_{2_r},$$

$$\bar{x}+\bar{y}=\left\{x\in\mathbb{R}:x_{1_l}+x_{2_l}\le x\le x_{1_r}+x_{2_r}\right\},$$

and if $\alpha \ge 0$, then

$$\alpha\bar{x}=\left\{x\in\mathbb{R}:\alpha x_{1_l}\le x\le \alpha x_{1_r}\right\}$$

and if $\alpha < 0$, then

$$\alpha\bar{x}=\left\{x\in\mathbb{R}:\alpha x_{1_r}\le x\le \alpha x_{1_l}\right\},$$

$$\bar{x}\cdot\bar{y}=\left\{\begin{array}{l}x\in\mathbb{R}:\\ \min\{x_{1l}\cdot x_{2l},x_{1l}\cdot x_{2r},x_{1r}\cdot x_{2l},x_{1r}\cdot x_{2r},\}\le x\\ \le\max\{x_{1l}\cdot x_{2l},x_{1l}\cdot x_{2r},x_{1r}\cdot x_{2l},x_{1r}\cdot x_{2r},\}\end{array}\right\}.$$

The set of all interval numbers IR is a complete metric space under the mertic d defined by

$$d(\bar{x},\bar{y})=\max\left\{\left|x_{1_l}-x_{2_l}\right|,\left|x_{1_r}-x_{2_r}\right|\right\}\ [15].$$

In the special case $\bar{x}=[a,a]$ and $\bar{y}=[b,b]$, we obtain usual metric of $\mathbb{R}$.

Let us define transformation $f:\mathbb{N}\to\mathbb{R}$ by $k\to f(k)=\bar{x}=(\bar{x}_k)$. Then $\bar{x}=(\bar{x}_k)$ is called sequence of interval numbers and $\bar{x}_k$ is called k^{th} term of the interval numbers sequence $\bar{x}=(\bar{x}_k)$. The set of all sequences of the interval numbers denoted by $\bar{w}$ cf. [2].

A sequence $\bar{x}=(\bar{x}_k)$ of interval numbers is said to be convergent to the interval number $\bar{x}_0$ if for each $\varepsilon>0$ there exists a positive integer k_0 such that $d(\bar{x}_k,\bar{x}_0)<\varepsilon$ for all $k\ge k_0$ and we denote it by $\lim_k \bar{x}_k=\bar{x}_0$.

Thus, $\lim_k \bar{x}_k=\bar{x}_0 \Leftrightarrow \lim_k x_{k_l}=x_{0_l}$ and $\lim_k x_{k_r}=x_{0_r}$ [2].

A sequence space $\bar{E}$ is said to be *solid (or normal)* if $\alpha\bar{x}=(\alpha_k\bar{x}_k)\in\bar{E}$ whenever $\bar{x}=(\bar{x}_k)\in\bar{E}$ for all sequences $\alpha=(\alpha_k)$ of scalars with $|\alpha_k|\le 1$ for all $k\in\mathbb{N}$.

A sequence space $\bar{E}$ is said to be *symmetric* if $\bar{x}=(\bar{x}_k)\in\bar{E}$ implies $(\bar{x}_{(\pi)k})\in\bar{E}$ where π is a permutation of $\mathbb{N}$.

A sequence space $\bar{E}$ is said to be *monotone* if $\bar{E}$ contains the canonical pre-images of all its step spaces.

Let $K=(k_1<k_2<\ldots<)\subset\mathbb{N}$ and $\bar{E}$ be a sequence space. A K-step set of $\bar{E}$ is a class of sequences $\lambda_K^{\bar{E}}=\left\{(\bar{x}_{k_n})\in\bar{E}:(\bar{x}_k)\in\bar{E}\right\}$. A canonical pre-image of a sequence $(\bar{x}_{k_n})\in\lambda_K^{\bar{E}}$ is a sequence $\bar{y}=(\bar{y}_k)\in\bar{E}$ defined as follows:

$$\bar{y}_k=\begin{cases}\bar{x}_k, & \text{if } k\in K;\\ \bar{0}, & \text{otherwise.}\end{cases}$$

A canonical pre-image of a step set $\lambda_K^{\bar{E}}$ is a set of canonical pre-images of all elements in $\lambda_K^{\bar{E}}$, i.e. $\bar{y}=(\bar{y}_k)$ is in canonical pre-image $\lambda_K^{\bar{E}}$ if and only $\bar{y}=(\bar{y}_k)$ is canonical pre-image of some $\bar{x}=(\bar{x}_k)\notin\lambda_K^{\bar{E}}$.

A sequence space $\bar{E}$ is said to be *sequence algebra* if $\bar{x}\otimes\bar{y}=(\bar{x}_k\otimes\bar{y}_k)\in\bar{E}$ whenever $\bar{x}=(\bar{x}_k)$, $\bar{y}=(\bar{y}_k)\in\bar{E}$.

A sequence space $\bar{E}$ is said to be *convergence free* if $\bar{y}=(\bar{y}_k)\in\bar{E}$ whenever $\bar{x}=(\bar{x}_k)\in\bar{E}$ and $\bar{y}_k=0$ whenever $\bar{x}_k=\bar{0}$, where $\bar{0}=[0,0]$ is the zero element.

Remark 2.1. A sequence space $\bar{E}$ is solid implies $\bar{E}$ is monotone.

3. Main Results

In this paper we introduce and examine some generalized difference sequences of interval numbers using the Orlicz functions.

Definition 3.1. Let $\bar{x}=(\bar{x}_k)$ be a sequence of interval numbers and M be an Orlicz function. We define the following sequence spaces:

$$\bar{c}(\Delta^n,M)=\left\{\begin{array}{l}\bar{x}=(\bar{x}_k):\lim\limits_{k\to\infty}M\left(\dfrac{d(\Delta^n\bar{x}_k,\bar{x}_0)}{r}\right)=0,\\ \textit{for some } r>0 \textit{ and } \bar{x}_0\in IR\end{array}\right\},$$

$$\bar{c}_0(\Delta^n,M)=\left\{\begin{array}{l}\bar{x}=(\bar{x}_k):\lim\limits_{k\to\infty}M\left(\dfrac{d(\Delta^n\bar{x}_k,\bar{0})}{r}\right)=0,\\ \textit{for some } r>0\end{array}\right\},$$

$$\bar{\ell}_\infty(\Delta^n,M)=\left\{\begin{array}{l}\bar{x}=(\bar{x}_k):\sup\limits_k M\left(\dfrac{d(\Delta^n\bar{x}_k,\bar{0})}{r}\right)<\infty,\\ \textit{for some } r>0\end{array}\right\},$$

where

$$\Delta^n\bar{x}_k=\sum_{i=0}^{n}(-1)^i\binom{n}{i}\bar{x}_{k+i}.$$

Throughout the paper, X will denote any one of the notation $\bar{c},\bar{c}_0$ and $\bar{\ell}_\infty$.

Theorem 3.1. *$\bar{\ell}_\infty(\Delta^n,M)$ and $\bar{c}(\Delta^n,M)$ are complete metric spaces with the metric*

$$\rho(\bar{x},\bar{y})=\sum_{k=1}^{n}d(\bar{x}_k,\bar{y}_k)+\inf\left\{r>0:\sup_k M\left(\frac{d(\Delta^n\bar{x}_k,\Delta^n\bar{y}_k)}{r}\right)\le 1\right\}.$$

Proof. Let $(\bar{x}_k^s)$ be any Cauchy sequence in $\bar{\ell}_\infty(\Delta^n,M)$ where $\bar{x}^s=(\bar{x}_k^s)=(\bar{x}_1^s,\bar{x}_1^s\ldots,\bar{x}_k^s,\ldots)$ $\in\bar{\ell}_\infty(\Delta^n,M)$ for each $s\in\mathbb{N}$. Then for given $\varepsilon>0$. For a fixed $X_0>0$ and choose $a>0$ such that $M\left(\frac{aX_0}{2}\right)\ge 1$. Then there exists $n_0\in\mathbb{N}$ such that

$$\rho\left(\overline{x}^s,\overline{x}^t\right)=\sum_{k=1}^{n} d\left(\overline{x}_k^s,\overline{x}_k^t\right)$$
$$+\inf\left\{r>0:\sup_k M\left(\frac{d\left(\Delta^n\overline{x}_k^s,\Delta^n\overline{x}_k^t\right)}{r}\right)\le 1\right\}<\frac{\varepsilon}{aX_0}, \quad (3.1)$$
$$for\ s,t\ge n_0.$$

Hence

$$\sum_{k=1}^{n} d\left(\overline{x}_k^s,\overline{x}_k^t\right)<\varepsilon,\ for\ s,t\ge n_0.$$
$$\Rightarrow d\left(\overline{x}_k^s,\overline{x}_k^t\right)<\frac{\varepsilon}{n},\ for\ s,t\ge n_0.$$

Then $\left(\overline{x}_k^s\right)$ is a Cauchy sequence in IR and so $\left(\overline{x}_k^s\right)$ is a convergent sequence in IR. Let $\lim_{t\to\infty}\overline{x}_k^t=\overline{x}_k$. Again from (3.1)

$$\sup_k M\left(\frac{d\left(\Delta^n\overline{x}_k^s,\Delta^n\overline{x}_k^t\right)}{r}\right)\le 1 \quad for\ s,t\ge n_0.$$
$$M\left(\frac{d\left(\Delta^n\overline{x}_k^s,\Delta^n\overline{x}_k^t\right)}{\rho\left(\overline{x}^s,\overline{x}^t\right)}\right)\le 1\le M\left(\frac{aM_0}{2}\right)$$
$$for\ s,t\ge n_0,\ and\ k\in N.$$
$$d\left(\Delta^n\overline{x}_k^s,\Delta^n\overline{x}_k^t\right)<\frac{\varepsilon}{2},\ for\ s,t\ge n_0.$$

Hence $\left(\Delta^n\overline{x}_k^s\right)$ is a Cauchy sequence in IR for all $k\in\mathbb{N}$ and so $\left(\Delta^n\overline{x}_k^s\right)$ is a convergent sequence in IR for all $k\in\mathbb{N}$. Let $\lim_{t\to\infty}\Delta^n\overline{x}_k^t=\overline{x}_k$ for all $k\in\mathbb{N}$.

For $k=1$, we have

$$\lim_{t\to\infty}\Delta^n\overline{x}_1^t=\lim_{t\to\infty}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\overline{x}_{1+i}^t=\overline{x}_1.$$

Similarly we have

$$\lim_{t\to\infty}\Delta^n\overline{x}_k^t=\lim_{t\to\infty}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\overline{x}_{k+i}^t=\overline{x}_k$$
$$for\ \ all\ \ k=1,2,\ldots,n.$$

Thus $\lim_{t\to\infty}\overline{x}_{1+k}^t$ exists. Let $\lim_{t\to\infty}\overline{x}_{1+k}^t=\overline{x}_{1+k}$. Proceeding in this way inductively we conclude that $\lim_{t\to\infty}\overline{x}_k^t=\overline{x}_k$ for all $k\in\mathbb{N}$. Using continuity of M, we have

$$\sup_k M\left(\frac{d\left(\Delta^n\overline{x}_k^s,\Delta^n\overline{x}_k\right)}{r}\right)\le 1 \quad for\ s\ge n_0.$$
$$\inf\left\{r>0:\sup_k M\left(\frac{d\left(\Delta^n\overline{x}_k^s,\Delta^n\overline{x}_k\right)}{r}\right)\le 1\right\}<\varepsilon,\ for\ s\ge n_0.$$

Thus for all $s\ge n_0$, we obtain that

$$\sum_{k=1}^{n} d\left(\overline{x}_k^s,\overline{x}_k\right)+\inf\left\{r>0:\sup_k M\left(\frac{d\left(\Delta^n\overline{x}_k^s,\Delta^n\overline{x}_k\right)}{r}\right)\le 1\right\}<2\varepsilon.$$

That is

$$\rho\left(\overline{x}^s,\overline{x}\right)<2\varepsilon,\ i.e.,\overline{x}^s\to\overline{x}\ \ as\ \ s\to\infty.$$

Then the inequality

$$\rho\left(\overline{x},\overline{0}\right)\le\rho\left(\overline{x},\overline{x}^s\right)+\rho\left(\overline{x}^s,\overline{0}\right),\ for\ s\ge n_0$$

implies that $x\in\overline{\ell}_\infty\left(\Delta^n,M\right)$. This completes the proof.

Theorem 3.2. *The classes of interval numbers of sequences* $\overline{c}_0\left(\Delta^n,M\right)$ *and* $\overline{c}\left(\Delta^n,M\right)$ *are nowhere dense subsets of* $\overline{\ell}_\infty\left(\Delta^n,M\right)$.

Proof. From Theorem 3.1. we have $\overline{c}_0(\Delta)$ and $\overline{c}_0\left(\Delta^n,M\right)$ are closed subsets of the complete metric space $\overline{\ell}_\infty\left(\Delta^n,M\right)$. Also $\overline{c}_0\left(\Delta^n,M\right)$ and $\overline{c}\left(\Delta^n,M\right)$ are proper subsets which follows from the following example.

Example 3.1. Let $n=1$ and $M(x)=x$. Consider the interval sequence $\overline{x}=(\overline{x}_k)$ defined as follows:

$$\overline{x}_k=\begin{cases}[0,1] & for\ k\ even;\\ \left[-\left(1+\frac{1}{k}\right),-1\right] & for\ k\ odd\end{cases}$$

and

$$\Delta\overline{x}_k=\begin{cases}\left[1,2+\frac{1}{k+1}\right] & for\ k\ even;\\ \left[-\left(2+\frac{1}{k}\right),-1\right] & for\ k\ odd.\end{cases}$$

Thus $(\overline{x}_k)\in\overline{c}\left(\Delta^n,M\right)\supset\overline{c}_0\left(\Delta^n,M\right)$ but $(\overline{x}_k)\in\overline{\ell}_\infty\left(\Delta^n,M\right)$. Hence the result.

Theorem 3.3. $X\left(\Delta^{n-1},M\right)\subset X\left(\Delta^n,M\right)$ for $X=\overline{c},\overline{c}_0,\overline{\ell}_\infty$ *and the inclusions are strict.*

Proof. We give the proof for the inequality $\overline{c}\left(\Delta^{n-1},M\right)\subset\overline{c}\left(\Delta^n,M\right)$ only. The rest of the results follows similar way. Let $\overline{x}=(\overline{x}_k)\in\overline{c}\left(\Delta^{n-1},M\right)$. Then for some $r>0$, we have

$$\lim_k M\left(\frac{d\left(\Delta^{n-1}\overline{x}_k,\overline{x}_0\right)}{r}\right)=0\ \ for\ \ some\ \ \overline{x}_0\in IR. \quad (3.2)$$

Since

$$\Delta^n\overline{x}_k=\Delta^{n-1}\overline{x}_k-\Delta^{n-1}\overline{x}_{k+1}+\overline{x}-\overline{x}.$$

Then from the equation (3.2) and the continuity of M, the result follows from the following relation

$$M\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)$$
$$\leq\frac{1}{2}M\left(\frac{d\left(\Delta^{n-1}\overline{x}_k,\overline{x}_0\right)}{r}\right)+\frac{1}{2}M\left(\frac{d\left(\Delta^{n-1}\overline{x}_{k+1},\overline{x}_0\right)}{r}\right).$$

This shows that $\left(\overline{x}_k\right)\in\overline{c}\left(\Delta^n,M\right)$.

To show that the inclusions are strict, consider the following examples.

Example 3.2. Let $M(x)=x$ and $n=1$. Consider the sequence of interval numbers $\overline{x}=\left(\overline{x}_k\right)$ defined by

$$\overline{x}_k=\left[1,1+\frac{1}{k}\right]\quad for\ \ all\quad k\in\mathbb{N}$$

i.e. $\overline{x}_k\to\overline{1}$ as $k\to\infty$ and $\Delta\overline{x}_k\to\overline{0}$ as $k\to\infty$. Thus $\overline{x}=\left(\overline{x}_k\right)\notin\overline{c}_0$, but $\overline{x}=\left(\overline{x}_k\right)\in\overline{c}_0\left(\Delta\right)$. Hence the inclusion is strict.

Example 3.3. Let $M(x)=x$ and $n=1$. Consider the sequence of interval numbers $\overline{x}=\left(\overline{x}_k\right)$ defined by

$$\overline{x}_k=\left[k,k+1\right]\quad for\ \ all\quad k\in\mathbb{N}.$$

Then $\Delta\overline{x}_k=-1$. Thus $\overline{x}=\left(\overline{x}_k\right)=\overline{c}\left(\Delta^n\right)$. Hence the inclusion is strict.

Theorem 3.4. *Let M_1 and M_2 be two Orlicz functions. Then*

(i) $X\left(\Delta^n,M_2\right)\subset X\left(\Delta^n,M_1\cdot M_2\right)$

(ii) $X\left(\Delta^n,M_1\right)\cap X\left(\Delta^n,M_2\right)\subset X\left(\Delta^n,M_1+M_2\right),$
$\quad for\quad X=\overline{c},\overline{c}_0,\overline{\ell}_\infty$

Proof. (i) We prove the result for $X=\overline{c}$ and the rest of the cases will follow similarly. Let $\overline{x}=\left(\overline{x}_k\right)=\overline{c}\left(\Delta^n,M_2\right)$. Then for $r>0$ we have

$$\lim_k M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)=0\ \ for\ some\ \overline{x}_0\in IR. \quad (3.3)$$

Let $0<\varepsilon<1$ and δ with $0<\delta<1$ such that $M_1(t)<\varepsilon$ for $0<t<\delta$. We write

$$D_1=\left\{k\in N:M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)\leq\delta\right\},$$

$$D_2=\left\{k\in N:M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)>\delta\right\}.$$

Then for

$$M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)>\delta$$

we have

$$M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)<M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)\delta^{-1}$$
$$<1+\left[\left|M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)\delta^{-1}\right|\right]$$

where $k\in D_2$ and $\left[\left|a\right|\right]$ denotes the integer part of a. Given $\varepsilon>0$ by the definition of Orlicz function M for $M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)>\delta$ we have

$$M_1\left(M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)\right)$$
$$\leq\left(1+\left[\left|M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)\delta^{-1}\right|\right]\right)M_1(1)$$
$$\leq 2M_1(1)\left(M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)\delta^{-1}\right)<\varepsilon$$

for $k\in D_2$ and $k\geq n_1\in\mathbb{N}$, using (3.3).

Again for

$$M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)\leq\delta$$

we have

$$M_1\left(M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)\right)<\varepsilon,$$

for $k\in D_1$ and $k\geq n_2\in\mathbb{N}$, using (3.3).

Thus for $k>\max\left\{n_1,n_2\right\}$ we have

$$M_1\left(M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)\delta^{-1}\right)<\varepsilon.$$

Hence $\overline{x}=\left(\overline{x}_k\right)=\overline{c}\left(\Delta^n,M_1\cdot M_2\right)$. Thus

$$\overline{c}\left(\Delta^n,M_2\right)\subset\overline{c}\left(\Delta^n,M_1\cdot M_2\right).$$

(ii) It will follows from the following inequality

$$(M_1+M_2)\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)$$
$$\le M_1\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{x}_0\right)}{r}\right)+M_2\left(\frac{d\left(\Delta^n\overline{x}_k,\overline{A}_0\right)}{r}\right).$$

The proof of the following result is also routine work.

Theorem 3.5. *Let M_1 and M_2 be two Orlicz functions satisfying Δ_{2-}condition.* If $\beta=\lim_{t\to\infty}\frac{M_2(t)}{t}\ge 1$, then $X\left(\Delta^n,M_1\right)=X\left(\Delta^n,M_1\cdot M_2\right)$, where $X=\overline{c},\overline{c}_0\overline{\ell}_\infty$.

Theorem 3.6. *The classes of sequences of interval numbers $\overline{c}\left(\Delta^n,M\right)$ and $\overline{\ell}_\infty\left(\Delta^n,M\right)$ are not sequence algebra in general.*

Proof. The result follows from the following example.

Example 3.4. Let $n=1$ and $M(x)=x$. Consider the two sequences of interval numbers $\overline{x}=(\overline{x}_k)$, $\overline{y}=(\overline{y}_k)$ defined by

$$\overline{x}_k=[k-1,k+1],\overline{y}_k=[k-1,k].$$

Therefore for all $k\in\mathbb{N}$, we have

$$\Delta\overline{x}_k=-\overline{1},\Delta\overline{y}_k=-\overline{1}.$$

Thus $\overline{x}=(\overline{x}_k),\overline{y}=(\overline{y}_k)\in\overline{c}\left(\Delta^n,M\right)\subset\overline{\ell}_\infty\left(\Delta^n,M\right)$.

Now, we have

$$\Delta(\overline{x}_k\otimes\overline{y}_k)$$
$$=\left[(k-1)^2,k(k+1)\right]-\left[k^2,(k+1)(k+2)\right]$$
$$=\left[-(k+1),k\right],$$

i.e. $(\overline{x}_k\otimes\overline{y}_k)\notin\overline{\ell}_\infty(\Delta^n,M)(\supset\overline{c}(\Delta^n,M))$ This completes the proof.

Theorem 3.7. *The classes of interval numbers of sequences $\overline{c}\left(\Delta^n,M\right),\overline{c}_0\left(\Delta^n,M\right)$ and $\overline{\ell}_\infty\left(\Delta^n,M\right)$ are not convergence free.*

Proof. Let $n=1$ and $M(x)=x$. Consider the interval sequence $\overline{x}=(\overline{x}_k)$ defined as follows:

$$\overline{x}_k=\begin{cases}\overline{0}=[0,0], & \text{for } k=i^2,i\in\mathbb{N};\\ \left[0,\frac{1}{k}\right] & \text{otherwise}\end{cases}$$

and

$$\Delta\overline{x}_k=\begin{cases}\left[-\frac{1}{k+1},0\right], & \text{for } k=i^2,i\in\mathbb{N};\\ \left[0,\frac{1}{k}\right] & \text{for } k=i^2,i>1,i\in\mathbb{N}.\end{cases}$$

Hence $\Delta\overline{x}_k\to\overline{0}$ as $k\to\infty$. Thus $\overline{x}=(\overline{x}_k)\in\overline{c}_0\left(\Delta^n,M\right)\subset\overline{c}\left(\Delta^n,M\right)\subset\overline{\ell}_\infty\left(\Delta^n,M\right)$. Let $\overline{y}=(\overline{y}_k)$ defined as follows:

$$\overline{y}_k=\begin{cases}\overline{0}=[0,0], & \text{for } k=i^2,i\in\mathbb{N};\\ [0,k] & \text{otherwise}\end{cases}$$

and

$$\Delta\overline{y}_k=\begin{cases}\left[-(k+1),0\right], & \text{for } k=i^2,i\in\mathbb{N};\\ [0,k], & \text{for } k=i^2,i>1,i\in\mathbb{N};\\ \left[-(k+1),k\right], & \text{otherwise.}\end{cases}$$

Thus $\overline{y}=(\overline{y}_k)\notin\overline{\ell}_\infty\left(\Delta^n,M\right)\left(\supset\overline{c}\left(\Delta^n,M\right)\supset\overline{c}_0\left(\Delta^n,M\right)\right)$. Therefore the classes of interval numbers $\overline{c}\left(\Delta^n,M\right),\overline{c}_0\left(\Delta^n,M\right)$ and $\overline{\ell}_\infty\left(\Delta^n,M\right)$ are not convergence free.

Theorem 3.8. *The classes of interval numbers $\overline{c}\left(\Delta^n,M\right),\overline{c}_0\left(\Delta^n,M\right)$ and $\overline{\ell}_\infty\left(\Delta^n,M\right)$ are neither monotone nor solid.*

Proof. Let $n=1$ and $M(x)=x$. Consider the interval sequence $\overline{x}=(\overline{x}_k)$ defined by:

$$\overline{x}_k=\left[1,1+\frac{1}{k}\right]\quad \text{for all } k\in\mathbb{N}$$

and

$$\Delta\overline{x}_k=\left[0,\frac{1}{k(k+1)}\right]\quad \text{for all } k\in\mathbb{N}$$

i.e. $\Delta\overline{x}_k\to\overline{0}$ as $k\to\infty$. Thus $\overline{x}=(\overline{x}_k)\in\overline{c}_0\left(\Delta^n,M\right)\subset\overline{c}\left(\Delta^n,M\right)$.

Let $J=\{k\in\mathbb{N}:k=2i-1,i\in N\}$ be a subset of $\mathbb{N}$ and let $\overline{\left(\overline{c}_0\left(\Delta^n,M\right)_J\right)}$ be the canonical pre-image of the J-step set $\overline{c}_0\left(\Delta^n,M\right)_J$ of $\overline{c}_0\left(\Delta^n,M\right)$, defined as follows: $\overline{y}=(\overline{y}_k)\in\overline{\left(\overline{c}_0\left(\Delta^n,M\right)_J\right)}$ is the canonical pre-image of $\overline{x}=(\overline{x}_k)\in\overline{c}_0\left(\Delta^n,M\right)$ implies

$$\overline{y}_k=\begin{cases}\overline{x}_k & \text{for } k\in J;\\ \overline{0} & \text{for } k\notin J\end{cases}$$

Now

$$\overline{y}_k=\begin{cases}\left[1,1+\frac{1}{k}\right] & \text{for } k\in J;\\ \overline{0} & \text{for } k\notin J\end{cases}$$

and

$$\Delta\overline{y}_k=\begin{cases}\left[1,1+\frac{1}{k}\right] & \text{for } k\in J;\\ \left[-\left(1+\frac{1}{k+1}\right),-1\right] & \text{for } k\notin J\end{cases}$$

Thus $\overline{y}=(\overline{y}_k)\notin\overline{c}\left(\Delta^n,M\right)\left(\supset\overline{c}_0\left(\Delta^n,M\right)\right)$. Therefore the classes of interval numbers $\overline{c}\left(\Delta^n,M\right)$ and

$\overline{c}_0\left(\Delta^n, M\right)$ are not monotone. By the Remark 2.1, these spaces are not solid.

Now let's define the sequence $\overline{x} = \left(\overline{x}_k\right)$ by

$$\overline{x}_k = \left[k-1, k+1\right] \quad \textit{for all} \quad k \in \mathbb{N}$$

and $\Delta\overline{x}_k = -\overline{1}$, thus $\overline{x} = \left(\overline{x}_k\right) \in \overline{\ell}_\infty\left(\Delta^n, M\right)$.

Let $J = \left\{k \in \mathbb{N} : k = 2i-1, i \in \mathbb{N}\right\}$ be a subset of $\mathbb{N}$ and let $\overline{\overline{\ell}_\infty\left(\Delta^n, M\right)_J}$ be the canonical pre-image of the J-step set $\overline{\ell}_\infty\left(\Delta^n, M\right)_J$ of $\overline{\ell}_\infty\left(\Delta^n, M\right)$, defined as follows: $\overline{y} = \left(\overline{y}_k\right) \in \overline{\overline{\ell}_\infty\left(\Delta^n, M\right)_J}$ is the canonical pre-image of $\overline{x} = \left(\overline{x}_k\right) \in \overline{\ell}_\infty\left(\Delta^n, M\right)$ implies

$$\overline{y}_k = \begin{cases} \overline{x}_k, & \textit{for } k \in J; \\ \overline{0} & \textit{for } k \notin J. \end{cases}$$

Now

$$\overline{y}_k = \begin{cases} \left[k-1, k+1\right] & \textit{for } k \in J; \\ \overline{0} & \textit{for } k \notin J \end{cases}$$

and

$$\Delta\overline{y}_k = \begin{cases} \left[k-1, k+1\right] & \textit{for } k \in J; \\ \left[-\left(k+2\right), -\left(k+1\right)\right] & \textit{for } k \notin J. \end{cases}$$

Therefore $\overline{y} = \left(\overline{y}_k\right) \notin \overline{\ell}_\infty\left(\Delta^n, M\right)$ and $\overline{\ell}_\infty\left(\Delta^n, M\right)$ is not monotone. By the Remark 2.1, this space is not solid.

Theorem 3.9. *The classes of interval numbers* $\overline{c}\left(\Delta^n, M\right), \overline{c}_0\left(\Delta^n, M\right)$ *and* $\overline{\ell}_\infty\left(\Delta^n, M\right)$ *are not symmetric.*

Proof. The result follows from the following example.

Example 3.5. Let $n = 1$ and $M\left(x\right) = x$. Consider the interval sequence $\overline{x} = \left(\overline{x}_k\right)$ defined by

$$\overline{x}_k = \left[k, k+\frac{1}{2}\right] \quad \textit{for all} \quad k \in \mathbb{N}$$

and $\Delta\overline{x}_k = -\overline{1}$.

Thus $\overline{x} = \left(\overline{x}_k\right) \in \overline{\ell}_\infty\left(\Delta^n, M\right)$. Let the sequence of interval numbers $\overline{y} = \left(\overline{y}_k\right)$ be a rearrangement of the sequence of interval numbers $\overline{x} = \left(\overline{x}_k\right)$ defined as follows:

$$\overline{y} = \left(\overline{y}_k\right) = \begin{Bmatrix} \overline{x}_1, \overline{x}_2, \overline{x}_4, \overline{x}_3, \overline{x}_9, \overline{x}_5, \overline{x}_{16}, \\ \overline{x}_6, \overline{x}_{25}, \overline{x}_7, \overline{x}_{36}, \overline{x}_8, \overline{x}_{49}, \ldots \end{Bmatrix}$$

i.e.

$$\Delta\overline{y}_k = \begin{cases} \overline{x}_{\left(\frac{k+1}{2}\right)^2}, \textit{ for all } k \textit{ odd}; \\ \overline{x}_{\left(m+\frac{k}{2}\right)}, \textit{ for all } k \textit{ even and} \\ m \textit{ satifies } m\left(m-1\right) < k \le m\left(m+1\right). \end{cases}$$

Then for all k odd and $m \in \mathbb{N}$; satisfying $m\left(m-1\right) < \frac{k+1}{2} \le m\left(m+1\right)$, we have

$$\Delta\overline{y}_k = \left[\left(m+\frac{k}{2}\right) - \left(\frac{k+2}{2}\right)^2 - \frac{1}{2}, \left(m+\frac{k}{2}\right) - \left(\frac{k+2}{2}\right)^2 + \frac{1}{2}\right]$$

From the last two equation, it is clear that $\left(\Delta\overline{B}_k\right)$ is unbounded, thus $\overline{y} = \left(\overline{y}_k\right) \notin \overline{\ell}_\infty\left(\Delta^n, M\right)$. Therefore the class $\overline{\ell}_\infty\left(\Delta^n, M\right)$ is not symmetric.

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