Turkish Journal of Analysis and Number Theory, 2013, Vol. 1, No. 1, 26-35
Available online at http://pubs.sciepub.com/tjant/1/1/7

DOI:10.12691/tjant-1-1-7

New Results Involving a Class of Generalized Hurwitz-Lerch Zeta Functions and Their Applications

H. M. Srivastava[1,*], Min-Jie Luo[2], R. K. Raina[3]

[1]Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia, Canada
[2]Department of Applied Mathematics, Donghua University, Shanghai, People's Republic of China
[3]Department of Mathematics, M. P. University of Agriculture and Technology, Rajasthan, India
*Corresponding author: harimsri@math.uvic.ca

Abstract In this paper, we study a certain class of generalized Hurwitz-Lerch zeta functions. We derive several new and useful properties of these generalized Hurwitz-Lerch zeta functions such as (for example) their partial differential equations, new series and Mellin-Barnes type contour integral representations involving Fox's *H*-function and a pair of summation formulas. More importantly, by considering their application in Number Theory, we construct a new continuous analogue of Lippert's Hurwitz measure. Some statistical applications are also given.

Keywords: *Hurwitz-Lerch zeta function, arithmetic density of number theory, partial differential equations, series and Mellin-Barnes type contour integral representations, Fox's H-function, summation formulas, generalized Hurwitz meausure, probability density function, moment generating function*

Cite This Article: H. M. Srivastava, Min-Jie Luo, and R. K. Raina, "New Results Involving a Class of Generalized Hurwitz-Lerch Zeta Functions and Their Applications." *Turkish Journal of Analysis and Number Theory* 1, no. 1 (2013): 26-35. doi: 10.12691/tjant-1-1-7.

1. Introduction, Definitions and Preliminaries

We begin by recalling the familiar general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$, which is defined by (see, for example, [2]; see also [18,19,20,21,22]).

$$\Phi(z,s,a):=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s} \tag{1.1}$$

($a\in\mathbb{C}\setminus\mathbb{Z}_0^-;s\in\mathbb{C}$, when $|z|<1$; $\mathcal{R}(s)>1$ when $|z|=1$).

Special cases of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ include (for example) the Riemann Zeta function $\zeta(s)$ and the Hurwitz (or generalized) Zeta function $\zeta(s,a)$ defined by (see, for details, [2, Chapter I] and [21, Chapter 2])

$$\zeta(s):=\sum_{n=1}^{\infty}\frac{1}{n^s}=\Phi(1,s,1)=\zeta(s,1)\quad(\mathcal{R}(s)>1) \tag{1.2}$$

and

$$\zeta(s,a):=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}=\Phi(1,s,a) \tag{1.3}$$
$$\left(\mathcal{R}(s)>1;a\in\mathbb{C}\setminus\mathbb{Z}_0^-\right),$$

respectively. Just as its aforementioned special cases $\zeta(s)$ and $\zeta(s,a)$, the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by (1.4) can be continued *meromorphically* to the whole complex s-plane, except for a simple pole at s = 1 with its residue 1. It is also known that [[2], Equation 1.11 (3)]

$$\Phi(z,s,a)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-at}}{1-ze^{-t}}dt=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(a-1)t}}{e^t-z}dt \tag{1.4}$$

($\mathcal{R}(a)>0;\mathcal{R}(s)>0$, when $|z|\leq 1(z\neq 1)$; $\mathcal{R}(s)>1$ when $z=1$).

Recently, the following modified (and slightly generalized) version of the integral in (1.4) was introduced and studied by Raina and Chhajed [[16], Equation (1.6)]:

$$\Theta_{\mu}^{\lambda}(z,s,a;b):=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}\exp\left(-at-\frac{b}{t^{\lambda}}\right)\left(1-ze^{-t}\right)^{-\mu}dt \tag{1.5}$$

$$\left(\min\{\mathcal{R}(a),\mathcal{R}(s)\}>0;\mathcal{R}(b)\geq 0;\lambda\geq 0;\mu\in\mathbb{C}\right)$$

where we have assumed *further* that

$$\mathcal{R}(s)>0\ \text{ when }\ b=0\ \text{ and }\ |z|\leq 1\ (z\neq 1)$$

or

$$\mathcal{R}(s-\mu)>0\ \text{ when }\ b=0\ \text{ and }\ z=1,$$

provided, of course, that the integral in (1.5) exists. As a matter of fact, the aforementioned investigation by Raina and Chhajed [16] was motivated by the following special case of the function $\Theta_{\mu}^{\lambda}(z,s,a;b)$ defined by (1.5):

$$\Theta_{\mu}^{\lambda}(z,s,a;0)=\Phi_{\mu}^{*}(z,s,a) \\ =\frac{1}{\Gamma(s)}\int_0^{\infty}\left(\frac{t^{s-1}e^{-at}}{\left(1-ze^{-t}\right)^{\mu}}\right)dt \quad (1.6)$$

($\mathcal{R}(a)>0;\mathcal{R}(s)>0$, when $|z|\le 1(z\ne 1)$; $\mathcal{R}(s-\mu)>0$ when $z=1$).

where the function $\Phi_{\mu}^{*}(z,s,a)$ defined by

$$\Phi_{\mu}^{*}(z,s,a):=\sum_{n=0}^{\infty}\frac{(\mu)_n}{(a+n)^s}\frac{z^n}{n!} \quad (1.7)$$

was studied by Goyal and Laddha [[4], Equation (1.5)]. Here, and in what follows, $(\lambda)_{\nu}\ (\lambda,\nu\in\mathbb{C})$ denotes the Pochhammer symbol (or the *shifted* factorial) which is defined, in terms of the familiar Gamma function, by

$$(\lambda)_{\nu}:=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \\ =\begin{cases}1 & (\nu=0;\lambda\in\mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n+1) & (\nu=n\in\mathbb{N};\lambda\in\mathbb{C}),\end{cases}$$

where it is understood *conventionally* that $(0)_0:=1$ and assumed *tacitly* that the G-quotient exists, $\mathbb{N}$ being the set of positive integers.

It may be of interest to observe in passing that, in terms of the *Riemann-Liouville fractional derivative operator* $\mathcal{D}_z^{\mu}$ defined by (see, for example, [3,7,17])

$$\mathcal{D}_z^{\mu}\{f(z)\}\begin{cases}\frac{1}{\Gamma(-\mu)}\int_0^{z}(z-t)^{-\mu-1}f(t)dt \\ (\mathcal{R}(\mu)<0) \\ \frac{d^m}{d_z^m}\left\{\mathcal{D}_z^{\mu-m}\{f(z)\}\right\} \\ (m-1\le\mathcal{R}(\mu)<m(m\in\mathbb{N})),\end{cases}$$

the series definitions in (1.1) and (1.7) readily yield

$$\Phi_{\mu}^{*}(z,s,a) \\ =\frac{1}{\Gamma(\mu)}\mathcal{D}_z^{\mu-1}\left\{z^{\mu-1}\Phi(z,s,a)\right\}(\mathcal{R}(\mu)>0), \quad (1.8)$$

which (as already remarked by Lin and Srivastava [8]) exhibits the interesting (and useful) fact that the function $\Phi_{\mu}^{*}(z,s,a)$ is essentially a Riemann-Liouville fractional derivative of the classical Hurwitz-Lerch function $\Phi(z,s,a)$.

One other special case of the function $\Theta_{\mu}^{\lambda}(z,s,a;b)$ defined by (1.5) occurs when we set $\lambda=\mu=1$ and $z=1$ in the definition (1.5). We thus obtain

$$\Theta_1^1(1,s,a;b)=\zeta_b(s,a): \\ =\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}\exp\left(-at-\frac{b}{t}\right)}{1-e^t}dt, \quad (1.9)$$

where $\zeta_b(s,a)$ is the extended Hurwitz zeta function defined in [1]. In fact, just as it is already pointed out in [10], the series representation (see [[16], Equation (2.1)]) given for the function $\Theta_{\mu}^{\lambda}(z,s,a;b)$ in (1.5) is incorrect. Obvious *further* specializations in (1.6) and (1.9) would immediately relate these functions with the Riemann zeta function $\zeta(s)$ and the Hurwitz (or generalized) zeta function $\zeta(s,a)$ defined by (1.2) and (1.3), respectively.

By using the series expansion of the binomial $\left(1-ze^{-t}\right)^{-\mu}$ occurring in the integrand of (1.5) and evaluating the resulting integral by means of the *corrected* version of a known integral formula [[13], Equation (1.53)] in terms of Fox's *H*-function defined by (1.12) below, the following series and Mellin-Barnes type contour integral representations of the function $\Theta_{\mu}^{\lambda}(z,s,a;b)$ defined by (1.5) were obtained in [10]:

$$\Theta_{\mu}^{\lambda}(z,s,a;b)=\frac{1}{\lambda\Gamma(s)}\sum_{n=0}^{\infty}\frac{(\mu)_n}{(a+n)^s} \\ \cdot H_{0,2}^{2,0}\left[(a+n)b^{\frac{1}{\lambda}}\,\middle|\,\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right]\frac{z^n}{n!}\ (\lambda>0) \quad (1.10)$$

and

$$\Theta_{\mu}^{\lambda}(z,s,a;b)=\frac{1}{2\pi i\lambda\Gamma(s)\Gamma(\mu)}\int_{-i\infty}^{i\infty}\frac{\Gamma(\mathfrak{s})\Gamma(\mu-\mathfrak{s})}{(a-\mathfrak{s})^s} \\ \cdot H_{0,2}^{2,0}\left[(a+\mathfrak{s})b^{\frac{1}{\lambda}}\,\middle|\,\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right](-z)^{-\mathfrak{s}}\,d\mathfrak{s}\ (\lambda>0), \quad (1.11)$$

it being assumed that each member of the assertions (1.10) and (1.11) exists.

Remark 1. The *H*-functions involved on the right-hand sides of (1.10) and (1.11) are particular cases of the celebrated Fox's *H*-function which is defined as follows.

Definition 1. The well-known Fox's *H*-function is defined here by (see, for details, [[13], Defintion 1.1]; see also [[6,11,23,24])

$$H_{p,q}^{m,n}(z)=H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_p,B_q)\end{matrix}\right] \\ =H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,A_1),\cdots,(a_p,A_p)\\(b_1,B_1),\cdots,(b_p,B_q)\end{matrix}\right] \\ =\frac{1}{2\pi i}\int_{\mathcal{L}}\Xi(\mathfrak{s})z^{-\mathfrak{s}}\,d\mathfrak{s} \quad (1.12)$$

where

$$\Xi(\mathfrak{s}) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j \mathfrak{s}) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j \mathfrak{s})}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j \mathfrak{s}) \prod_{j=n+1}^{p} \Gamma(a_j + A_j \mathfrak{s})} \quad (1.13)$$

Here

$$z \in \mathbb{C} \setminus \{0\} \text{ with } |\arg(z)| < \pi$$

an empty product is interpreted as 1, m, n, p and q are integers such that $1 \le m \le q$ and $0 \le n \le p$,

$$A_j > 0 \,(j = 1, \cdots, p) \text{ and } B_j > 0 \,(j = 1, \cdots, q),$$
$$\alpha_j \in \mathbb{C} \,(j = 1, \cdots, p) \text{ and } \beta_j \in \mathbb{C} \,(j = 1, \cdots, q),$$

and $\mathscr{L}$ is a suitable Mellin-Barnes type contour separating the poles of the gamma functions

$$\left\{\Gamma(b_j + B_j \mathfrak{s})\right\}_{j=1}^{m}$$

from the poles of the gamma functions

$$\left\{\Gamma(1 - a_j - A_j \mathfrak{s})\right\}_{j=1}^{n}.$$

In our present investigation, we consider certain statistical applications of the generalized Hurwitz-Lerch zeta function $\Theta_\mu^\lambda(z, s, a; b)$ defined by (1.5). We first derive a partial differential equation satisfied by the function in (1.11). We then obtain another series representation and related results for this generalized Hurwitz-Lerch zeta function. The results derived here are also applied in our investigations concerning the generalized Hurwitz-Lerch zeta measure and its related statistical concepts.

2. Differential Equation of the Generalized Hurwitz-Lerch Zeta Function $\Theta_\mu^\lambda(z, s, a; b)$

In this section, we will show that the generalized Hurwitz-Lerch zeta function $\Theta_\mu^\lambda(z, \alpha, a, b)$ satisfies a partial differential equation when the parameter λ is given by

$$\lambda = \frac{1}{m} \quad (m \in \mathbb{N}).$$

We first prove the following lemma which will be used in the proof of our main theorem.

Lemma (Derivative Property). *The following derivative formulas hold true:*

$$\Theta_{\mu+1}^\lambda(z, s, a+1; b) = \frac{1}{\mu} \frac{d}{dz}\left\{\Theta_\mu^\lambda(z, s, a; b)\right\} \quad (\lambda > 0) \quad (2.1)$$

and

$$\Theta_{\mu+1}^{\frac{1}{m}}\left(z, s, a+1; b^{\frac{1}{m}}\right) = \frac{1}{\mu} \frac{d}{dz}\left\{\Theta_\mu^{\frac{1}{m}}\left(z, s, a; b^{\frac{1}{m}}\right)\right\} \quad (m \in \mathbb{N}). \quad (2.2)$$

Proof. The proofs of the derivative formulas (2.1) and (2.2) are direct. For example, by applying the series representation (1.10), one easily finds that

$$\begin{aligned} &\frac{1}{\mu} \frac{d}{dz}\left\{\Theta_\mu^\lambda(z, s, a; b)\right\} \\ &= \frac{1}{\lambda \mu \Gamma(s)} \sum_{n=1}^{\infty} \frac{(\mu)_n}{(a+n)^s} \cdot H_{0,2}^{2,0}\left[(a+n) b^{\frac{1}{\lambda}} \,\middle|\, \overline{(s,1), \left(0, \frac{1}{\lambda}\right)}\right] \frac{z^{n-1}}{(n-1)!} \\ &= \frac{1}{\lambda \mu \Gamma(s)} \sum_{n=0}^{\infty} \frac{(\mu)_{n+1}}{(a+1+n)^s} \cdot H_{0,2}^{2,0}\left[(a+1+n) b^{\frac{1}{\lambda}} \,\middle|\, \overline{(s,1), \left(0, \frac{1}{\lambda}\right)}\right] \frac{z^n}{n!} \\ &= \Theta_{\mu+1}^\lambda(z, s, a+1; b) \quad (\lambda > 0) \end{aligned}$$

which is precisely the first result (2.1) asserted by the Lemma. The second assertion (2.2) follows immediately from (2.1) upon setting

$$\lambda = \frac{1}{m} \quad (m \in \mathbb{N}) \text{ and } b \to b^{\frac{1}{m}} \quad (m \in \mathbb{N}). \quad \square$$

Our first main result is contained in the following theorem.

Theorem 1. *The generalized Hurwitz-Lerch zeta function $\Theta_\mu^{\frac{1}{m}}(z, s, a; b)$ $(m \in \mathbb{N})$ satisfies the following partial differential equation:*

$$\begin{bmatrix} (-1)^{m+1} m^m \mathfrak{D}_b \\ -(a+1) b^m \theta_z \end{bmatrix} \left\{\Theta_\mu^{\frac{1}{m}}(z, s, a; b)\right\} = 0, \quad (2.3)$$

where the differential operators $\mathfrak{D}_b$ and θ_z are given by

$$\mathfrak{D}_b := \theta_b(\theta_b - s)\left(\theta_b - \frac{1}{m}\right) \cdots \left(\theta_b - \frac{m-1}{m}\right) \quad (2.4)$$

and

$$\theta_z := z \frac{\partial}{\partial_z},$$

respectively.

Proof. We first rewrite the H-function occurring in the Mellin-Barnes type contour integral representation (1.11) as follows:

$$\begin{aligned} &H_{0,2}^{2,0}\left[(a - \mathfrak{s}) b^{\frac{1}{\lambda}} \,\middle|\, \overline{(s,1), \left(0, \frac{1}{\lambda}\right)}\right] \\ &= \frac{1}{2\pi i} \int_{\mathfrak{L}} \Gamma(s+w) \Gamma\left(\frac{w}{\lambda}\right) \left[(a - \mathfrak{s}) b^{\frac{1}{\lambda}}\right]^{-w} dw, \end{aligned} \quad (2.5)$$

where $\mathfrak{L}$ is a suitable Mellin-Barnes type contour integral in the complex w-plane. By setting

$$\frac{1}{\lambda} = m \quad (m \in \mathbb{N}) \text{ and } b \to b^{\frac{1}{m}} \quad (m \in \mathbb{N})$$

in the above equation (2.5) and then applying the well-known (Gauss-Legendre) multiplication formula:

$$\Gamma(m\mathfrak{z}) = (2\pi)^{\frac{1-m}{2}} m^{m\mathfrak{z}-\frac{1}{2}} \prod_{j=1}^{m} \Gamma\left(\mathfrak{z}+\frac{j-1}{m}\right) \quad (2.6)$$
$$\left(\mathfrak{z} \neq 0, -\frac{1}{m}, -\frac{2}{m}, \cdots; m \in \mathbb{N}\right),$$

we find that

$$\begin{aligned} &H_{0,2}^{2,0}\left[(a-\mathfrak{s})^b \,\middle|\, \overline{(s,1),(0,m)}\right] \\ &= \frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{L}} \Gamma(s+w)\Gamma(mw)\left[(a-\mathfrak{s})^b\right]^{-w} dw \\ &= \frac{(2\pi)^{\frac{1-m}{2}}}{2\pi\mathrm{i}\sqrt{m}} \int_{\mathfrak{L}} \Gamma(s+w)\prod_{j=1}^{m}\Gamma\left(w+\frac{j-1}{m}\right)\left[(a-\mathfrak{s})bm^{-m}\right]^{-w} dw \\ &= \frac{(2\pi)^{\frac{1-m}{2}}}{\sqrt{m}} G_{0,m+1}^{m+1,0}\left((a-\mathfrak{s})bm^{-m} \,\middle|\, \overline{s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}}\right), \end{aligned} \quad (2.7)$$

where

$$G_{0,m+1}^{m+1,0}\left((a-\mathfrak{s})bm^{-m}\right)$$

is a very specialized case of Meijer's G-function $G_{p,q}^{m,n}(\mathfrak{z})$ which, in turn, is a special case of Fox's H-function defined by (1.12), that is, we have the following relationship (see, for details, [14]; see also [2,12,15]):

$$\begin{aligned} G_{p,q}^{m,n}(\mathfrak{z}) &= G_{p,q}^{m,n}\left(\mathfrak{z} \,\middle|\, \begin{matrix} (a_j)_{j=1}^{p} \\ (b_j)_{j=1}^{q} \end{matrix}\right) \\ &= H_{p,q}^{m,n}\left[\mathfrak{z} \,\middle|\, \begin{matrix} (a_1,1),\cdots,(a_p,1) \\ (b_1,1),\cdots,(b_p,1) \end{matrix}\right], \end{aligned} \quad (2.8)$$

We know that the function W defined by

$$W := G_{p,q}^{m,n}\left(\mathfrak{z} \,\middle|\, \begin{matrix} (a_j)_{j=1}^{p} \\ (b_j)_{j=1}^{q} \end{matrix}\right)$$

satisfies the following differential equation of order $\max(p,q)$ (see, for example, [[2], Equation 5.4(1)]):

$$\begin{bmatrix} (-1)^{p-m-n}\mathfrak{z}\left(\vartheta_{\mathfrak{z}}-a_1+1\right)\cdots\left(\vartheta_{\mathfrak{z}}-a_p+1\right) \\ -\left(\vartheta_{\mathfrak{z}}-b_1\right)\cdots\left(\vartheta_{\mathfrak{z}}-b_q\right) \end{bmatrix} W = 0,$$

where

$$\vartheta_{\mathfrak{z}} = \mathfrak{z}\frac{d}{d\mathfrak{z}}.$$

Hence, clearly, the function given by (2.7) satisfies the following differential equation:

$$\begin{bmatrix} (-1)^{m+1}(a-\mathfrak{s})m^{-m}b \\ -\theta_b(\theta_b-s)\left(\theta_b-\frac{1}{m}\right)\cdots\left(\theta_b-\frac{m-1}{m}\right) \end{bmatrix} \quad (2.9)$$
$$\bullet G_{0,m+1}^{m+1,0}\left((a-\mathfrak{s})bm^{-m} \,\middle|\, \overline{s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}}\right) = 0,$$

where, as already stated in Theorem 1,

$$\theta_b = b\frac{\partial}{\partial b}.$$

Now, if we write [see also Equation (2.4)]

$$\mathfrak{D}_b := \theta_b(\theta_b-s)\left(\theta_b-\frac{1}{m}\right)\cdots\left(\theta_b-\frac{m-1}{m}\right) \quad \left(\theta_b := b\frac{\partial}{\partial b}\right),$$

then the equation (2.9) becomes

$$\begin{aligned} &\mathfrak{D}_b\left\{G_{0,m+1}^{m+1,0}\left((a-\mathfrak{s})bm^{-m} \,\middle|\, \overline{s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}}\right)\right\} \\ &= (-1)^{m+1}(a-\mathfrak{s})m^{-m}b \\ &\bullet G_{0,m+1}^{m+1,0}\left((a-\mathfrak{s})bm^{-m} \,\middle|\, \overline{s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}}\right). \end{aligned} \quad (2.10)$$

Applying the differential operator $\mathfrak{D}_b$ to the function $\Theta_{\mu+1}^{\frac{1}{m}}\left(-\mathfrak{z},s,a;b^{\frac{1}{m}}\right)$ given by (1.11) with

$$\mathfrak{z} \to -\mathfrak{z}, \lambda = \frac{1}{m}(m \in \mathbb{N}) \text{ and } b \to b^{\frac{1}{m}} \quad (m \in \mathbb{N}),$$

we find by using (2.10) that

$$\begin{aligned} &\mathfrak{D}_b\left\{\Theta_{\mu}^{\frac{1}{m}}\left(-\mathfrak{z},s,a;b^{\frac{1}{m}}\right)\right\} \\ &= \frac{\sqrt{m}(2\pi)^{\frac{1-m}{2}}}{2\pi\mathrm{i}\Gamma(s)\Gamma(\mu)} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\Gamma(\mathfrak{s})\Gamma(\mu-\mathfrak{s})}{(a-\mathfrak{s})^s} \\ &\bullet \mathfrak{D}_b\left\{G_{0,m+1}^{m+1,0}\left((a-\mathfrak{s})bm^{-m} \,\middle|\, \overline{s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}}\right)\right\}\mathfrak{z}^{-\mathfrak{s}} d\mathfrak{s} \\ &= \frac{(-1)^{m+1} m^{\frac{1}{2}-m}(2\pi)^{\frac{1-m}{2}} b}{2\pi\mathrm{i}\Gamma(s)\Gamma(\mu)} \int_{\xi-\mathrm{i}\infty}^{\xi+\mathrm{i}\infty} \frac{\Gamma(\mathfrak{s})\Gamma(\mu-\mathfrak{s})}{(a-\mathfrak{s})^s}(a-\mathfrak{s}) \\ &\bullet G_{0,m+1}^{m+1,0}\left((a-\mathfrak{s})bm^{-m} \,\middle|\, \overline{s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}}\right)\mathfrak{z}^{-\mathfrak{s}} d\mathfrak{s} \\ &=: (-1)^{m+1} m^{-m} b\left(a\mathscr{I}_1 - \mathscr{I}_2\right), \end{aligned} \quad (2.11)$$

where the first integral $\mathscr{I}_1$ is actually the generalized Hurwitz-Lerch zeta function given by

$$\mathscr{I}_1 = \Theta_{\mu}^{\frac{1}{m}}\left(-\mathfrak{z},s,a;b^{\frac{1}{m}}\right). \quad (2.12)$$

The evaluation of the second integral $\mathscr{I}_2$ given by

$$\mathscr{I}_2 := \frac{\sqrt{m}\,(2\pi)^{\frac{1-m}{2}}}{2\pi\mathrm{i}\Gamma(s)\Gamma(\mu)}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\Gamma(\mathfrak{s}+1)\Gamma(\mu-\mathfrak{s})}{(a-\mathfrak{s})^s} \cdot\left\{G_{0,m+1}^{m+1,0}\left((a-\mathfrak{s})bm^{-m}\left|\overline{s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}}\right.\right)\right\}\mathfrak{z}^{-\mathfrak{s}}\,d\mathfrak{s} \tag{2.13}$$

is more complicated. Since the residues of $\Gamma(\mathfrak{s}+1)$ at the poles $\mathfrak{s}=-k$ $(k\in\mathbb{N})$ are computed by

$$\operatorname*{Res}_{\mathfrak{s}=-k}\{\Gamma(\mathfrak{s}+1)\} = \lim_{\mathfrak{s}\to-k}(\mathfrak{s}+k)\Gamma(\mathfrak{s}+1) = \frac{(-1)^{k-1}}{(k-1)!}, \tag{2.14}$$

the Residue Theorem implies that

$$\begin{aligned}\mathscr{I}_2 &= \frac{\sqrt{m}(2\pi)^{\frac{1-m}{2}}}{\Gamma(s)\Gamma(\mu)}\sum_{k=1}^{\infty}\frac{\Gamma(\mu+k)}{(a+k)^s}\mathfrak{z}^k\operatorname*{Res}_{\mathfrak{s}=-k}\{\Gamma(\mathfrak{s}+1)\}\cdot G_{0,m+1}^{m+1,0}\left((a+k)bm^{-m}\left|\overline{s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}}\right.\right)\\ &= \frac{\sqrt{m}(2\pi)^{\frac{1-m}{2}}}{\Gamma(s)\Gamma(\mu)}\sum_{k=1}^{\infty}\frac{\Gamma(\mu+k)}{(a+k)^s}\frac{(-1)^{k-1}\mathfrak{z}^k}{(k-1)!}\cdot G_{0,m+1}^{m+1,0}\left((a+k)bm^{-m}\left|\overline{s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}}\right.\right)\\ &= \frac{\mathfrak{z}\sqrt{m}(2\pi)^{\frac{1-m}{2}}}{\Gamma(s)\Gamma(\mu)}\sum_{k=0}^{\infty}\frac{\Gamma(\mu+k+1)}{(a+1+k)^s}\frac{(-\mathfrak{z})^k}{k!}\cdot G_{0,m+1}^{m+1,0}\left((a+1+k)bm^{-m}\left|\overline{s,0,\frac{1}{m},\frac{2}{m},\cdots,\frac{m-1}{m}}\right.\right)\\ &= \frac{m\mu\mathfrak{z}}{\Gamma(s)}\sum_{k=0}^{\infty}\frac{(\mu+1)_k}{(a+1+k)^s}\frac{(-\mathfrak{z})^k}{k!}\cdot H_{0,2}^{2,0}\left[(a+1+k)b\left|\overline{(s,1),(0,m)}\right.\right]\\ &= \mu\mathfrak{z}\Theta_{\mu+1}^{\frac{1}{m}}\left(-\mathfrak{z},s,a+1;b^{\frac{1}{m}}\right).\end{aligned} \tag{2.15}$$

Applying (2.2) in (2.15), we get

$$\mathscr{I}_2 = -\mathfrak{z}\frac{d}{d\mathfrak{z}}\left\{\Theta_{\mu}^{\frac{1}{m}}\left(-\mathfrak{z},s,a;b^{\frac{1}{m}}\right)\right\}. \tag{2.16}$$

Now, upon substituting from (2.12) and (2.16) into (2.11), we obtain

$$\begin{aligned}&\mathfrak{D}_b\left\{\Theta_{\mu}^{\frac{1}{m}}\left(-\mathfrak{z},s,a;b^{\frac{1}{m}}\right)\right\}\\ &= (-1)^{m+1}m^{-m}ab\Theta_{\mu}^{\frac{1}{m}}\left(-\mathfrak{z},s,a;b^{\frac{1}{m}}\right)\\ &\quad+(-1)^{m+1}m^{-m}b\mathfrak{z}\frac{\partial}{\partial\mathfrak{z}}\left\{\Theta_{\mu}^{\frac{1}{m}}\left(-\mathfrak{z},s,a;b^{\frac{1}{m}}\right)\right\},\end{aligned} \tag{2.17}$$

which, after a little simplification, becomes

$$\left[(-1)^{m+1}m^{-m}\mathfrak{D}_b - ab - b\theta_{\mathfrak{z}}\right]\left\{\Theta_{\mu}^{\frac{1}{m}}\left(-\mathfrak{z},s,a;b^{\frac{1}{m}}\right)\right\} = 0\quad\left(\theta_{\mathfrak{z}} := \mathfrak{z}\frac{\partial}{\partial\mathfrak{z}}\right). \tag{2.18}$$

Finally, by setting

$$b\to b^m\quad(m\in\mathbb{N})\quad\text{and}\quad\mathfrak{z}\to-\mathfrak{z}$$

in the last equation (2.18), we arrive at the desired result (2.3) asserted by Theorem 1. □

It is interesting to consider a special case of Theorem 1 when m = 1. Thus, if we write

$$\Theta_{\mu}^{1}(\mathfrak{z},s,a;b) =: \Theta_{\mu}(\mathfrak{z},s,a;b), \tag{2.19}$$

then we have the following corollary.

Corollary 1. *The generalized Hurwitz-Lerch zeta function* $\Theta_{\mu}(\mathfrak{z},s,a;b)$ *satisfies the following partial differential equation:*

$$\left[b\frac{\partial}{\partial b}\left(b\frac{\partial}{\partial b}-s\right)\left(b\frac{\partial}{\partial b}-1\right)-(a+1)b\mathfrak{z}\frac{\partial}{\partial\mathfrak{z}}\right]\Theta_{\mu}(\mathfrak{z},s,a;b)=0 \tag{2.20}$$

Furthermore; the function $\Theta_{\mu}(1,s,a;b)$*, considered as an analytic function of the variable b satisfies the following relation:*

$$\begin{aligned}&b\frac{\partial}{\partial b}\left(b\frac{\partial}{\partial b}-s\right)\left(b\frac{\partial}{\partial b}-1\right)\Theta_{\mu}(1,s,a;b)\\ &=(a+1)b\mu\Theta_{\mu+1}(1,s,a+1;b)\end{aligned} \tag{2.21}$$

3 Further Series Representations and Related Results

In this section, we first give a new series representation of the generalized Hurwitz-Lerch zeta function $\Theta_{\mu}^{\lambda}(\mathfrak{z},s,a;b)$ involving the familiar Laguerre polynomials of order (index) α and degree n in x, which are generated by

$$(1-t)^{-\alpha-1}\exp\left(-\frac{xt}{1-t}\right)=\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^n\quad(|t|<1;\alpha\in\mathbb{C}). \tag{3.1}$$

Indeed, upon setting

$$t\to1-t^{\lambda}\quad\text{and}\quad x=b$$

in (3.1), we get

$$\exp\left(-\frac{b}{t^{\lambda}}\right)=t^{\lambda(\alpha+1)}e^{-b}\sum_{n=0}^{\infty}L_n^{(\alpha)}(b)\left(1-t^{\lambda}\right)^n. \tag{3.2}$$

We now make use of (3.2) and the series expansions of the binomials

$$\left(1-t^{\lambda}\right)^n\quad\textit{and}\quad\left(1-\mathfrak{z}e^{-t}\right)^{-\mu}$$

occurring in the integrand of (1.5). By evaluating the resulting Eulerian integral, we thus arrive at the series representations given by Theorem 2 below.

Theorem 2. *Each of the following series representations holds true for the generalized Hurwitz-Lerch zeta function* $\Theta_\mu^\lambda(z,s,a;b)$:

$$\Theta_\mu^\lambda(z,s,a;b)=\frac{e^{-b}}{\Gamma(s)}\sum_{n,\ell=0}^{\infty}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{\mu+\ell-1}{\ell}\cdot\Gamma\big(s+\lambda(\alpha+j+1)\big)L_n^{(\alpha)}(b)\frac{z^\ell}{(a+\ell)^{s+\lambda(\alpha+j+1)}} \tag{3.3}$$

$$\big(\mathcal{R}(a)>0;\mathcal{R}(s+\lambda\alpha)>-\lambda\big)$$

and

$$\Theta_\mu^\lambda(z,s,a;b)=\frac{e^{-b}}{\Gamma(s)}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\Gamma\big(s+\lambda(\alpha+j+1)\big)\cdot L_n^{(\alpha)}(b)\Phi_\mu^*\big(z,s+\lambda(\alpha+j+1),a\big) \tag{3.4}$$

$$\big(\mathcal{R}(a)>0;\mathcal{R}(s+\lambda\alpha)>-\lambda\big),$$

provided that each member of the assertions (3.3) and (3.4) exists, $\Phi_\mu^*(z,s,a)$ being given by (1.7).

Proof. As already outlined above, our demonstration of the first assertion (3.3) of Theorem 2 is based essentially upon the representation (3.2) and the following well-known Eulerian integral:

$$\int_0^\infty t^{\rho-1}e^{-\sigma t}dt=\frac{\Gamma(\rho)}{\sigma^\rho}\quad\big(\min\{\mathcal{R}(\rho),\mathcal{R}(\sigma)\}>0\big). \tag{3.5}$$

The second assertion (3.4) follows from the first assertion (3.3) when we interpret the ℓ-series in (3.4) by means of the definition (1.7).

In our derivation of each of the summation formulas (3.3) and (3.4), it is assumed that the required inversions of the order of summation and integration are justified by absolute and uniform convergence of the series and integrals involved. The final results (3.3) and (3.4) would thus hold true whenever each member of the assertions (3.3) and (3.4) of Theorem 2 exists. □

Remark 2. For the extended Hurwitz zeta function $\zeta_b(s,a)$ defined by (1.9), it is easily deduced from the assertion (3.4) of Theorem 2 when $\lambda=\mu=1$ and $z=1$ that

$$\zeta_b(s,a)=\frac{e^{-b}}{\Gamma(s)}\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\Gamma(s+\alpha+j+1)\cdot L_n^{(\alpha)}(b)\zeta(s+\alpha+j+1,a) \tag{3.6}$$

$$\big(\mathcal{R}(a)>0;\mathcal{R}(s+\alpha)>-1\big),$$

provided that each member of (3.6) exists, $\zeta_b(s,a)$ being the Hurwitz (or generalized) zeta function given by (1.3). The obvious *further* special case of (3.6) when $a=1$ and $\alpha=0$ would yield the *corrected* version of a known result (see [[1], Equation (7.78)]).

We now give a pair of summation formulas involving the generalized Hurwitz-Lerch zeta function $\Theta_\mu^\lambda(z,s,a;b)$.

Theorem 3. *Each of the following summation formulas holds true for the generalized Hurwitz-Lerch zeta function* $\Theta_\mu^\lambda(z,s,a;b)$:

$$2^{s-1}\left[\Theta_\mu^\lambda(-z,s,a;b)+\Theta_\mu^\lambda(z,s,a;b)\right]=\sum_{n=0}^{\infty}\frac{(-\mu)_{2n}}{(2n)!}\Theta_\mu^\lambda\left(z^2,s,\frac{a}{2}+n;2^\lambda b\right)z^{2n} \tag{3.7}$$

and

$$2^{s-1}\left[\Theta_\mu^\lambda(-z,s,a;b)-\Theta_\mu^\lambda(z,s,a;b)\right]=\sum_{n=0}^{\infty}\frac{(-\mu)_{2n+1}}{(2n+1)!}\Theta_\mu^\lambda\left(z^2,s,\frac{a+1}{2}+n;2^\lambda b\right)z^{2n+1}, \tag{3.8}$$

provided that each member of the assertions (3.7) and (3.8) exists.

Proof. Making use of the integral representation in (1.5) for the function $\Theta_\mu^\lambda(z,s,a;b)$, we get

$$\Theta_\mu^\lambda(-z,s,a;b)+\Theta_\mu^\lambda(z,s,a;b)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\exp\left(-at-\frac{b}{t^\lambda}\right)\frac{\left(1-ze^{-t}\right)^\mu+\left(1+ze^{-t}\right)^\mu}{\left(1-z^2e^{-2t}\right)^\mu}dt. \tag{3.9}$$

Since

$$\left(1-ze^{-t}\right)^\mu+\left(1+ze^{-t}\right)^\mu=2\sum_{n=0}^{\infty}\frac{(-\mu)_{2n}}{(2n)!}z^{2n}e^{-2nt}, \tag{3.10}$$

by substituting from (3.10) into (3.9) and interchanging the order of summation and integration, we find that

$$\begin{aligned}\Theta_\mu^\lambda(-z,s,a;b)+\Theta_\mu^\lambda(z,s,a;b)&=\frac{2}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{(-\mu)_{2n}}{(2n)!}z^{2n}\int_0^\infty t^{s-1}\exp\left(-(a+2n)t-\frac{b}{t^\lambda}\right)(1-z^2e^{-2t})^{-\mu}dt\\&=\frac{2^{1-s}}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{(-\mu)_{2n}}{(2n)!}z^{2n}\int_0^\infty \tau^{s-1}\exp\left(-\left(\frac{a}{2}+n\right)\tau-\frac{2^\lambda b}{\tau^\lambda}\right)(1-z^2e^{-t})^{-\mu}d\tau\\&=2^{1-s}\sum_{n=0}^{\infty}\frac{(-\mu)_{2n}}{(2n)!}\Theta_\mu^\lambda\left(z^2,s,\frac{a}{2}+n;2^\lambda b\right)z^{2n},\end{aligned} \tag{3.11}$$

which obviously proves the assertion (3.7) of Theorem 3.

The assertion (3.8) of Theorem 3 can be proven in a manner analogous to that detailed above. □

Remark 3. If we set $\mu=1$ in (3.7) and (3.8), the series occurring on their right-hand sides would terminate. Upon setting $z\to -z$ and $a\to 2a$, we thus obtain

$$2^{s-1}\left[\Theta_1^\lambda(z,s,2a;b)+\Theta_1^\lambda(-z,s,2a;b)\right]=\Theta_1^\lambda\left(z^2,s,a;2^\lambda b\right) \tag{3.12}$$

and

$$2^{s-1}\left[\Theta_1^\lambda(z,s,2a;b)-\Theta_1^\lambda(-z,s,2a;b)\right]=z\Theta_1^\lambda\left(z^2,s,a+\frac{1}{2};2^\lambda b\right). \tag{3.13}$$

In particular, if we set $z = 1$ in these last two summation formulas (3.12) and (3.13), we get

$$\begin{aligned}&\Theta_1^{\lambda}(-1, s, 2a; b)\\&= 2^{1-s}\Theta_1^{\lambda}\left(1, s, a; 2^{\lambda}b\right) - \Theta_1^{\lambda}(1, s, 2a; b)\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}&\Theta_1^{\lambda}(-1, s, 2a; b)\\&= \Theta_1^{\lambda}(1, s, 2a; b) - 2^{1-s}\Theta_1^{\lambda}\left(1, s, a+\frac{1}{2}; 2^{\lambda}b\right),\end{aligned} \tag{3.15}$$

respectively. In its *further* special case when $\lambda = 1$, the summation formula (3.14)can be shown to correspond to known results (see, for example, [[1], Theorem 7.9]; see also [1]).

4. A Generalization of the Hurwitz Measure

Suppose that $\chi_A(n)$ denote the characteristic function of the subset A of the set $\mathbb{N}$ of positive integers (or, in the language of probability theory, the indicator function of the event $A \subseteq \mathbb{N}$). Then it is well known that the following arithmetic density of number theory:

$$dens(A) = \lim_{k\to\infty}\frac{1}{k}\sum_{n=1}^{k}\chi_A(n) \tag{4.1}$$

does not define a measure on the set $\mathbb{N}$ of positive integers. In order to remedy this deficiency, Golomb [5] defined a probability on the sample space $\mathbb{N}$ as follows:

$$Q_s(A) = \frac{1}{\zeta(s)}\sum_{n=1}^{\infty}\frac{\chi_A(n)}{n^s}, \tag{4.2}$$

where $\zeta(s)$ denotes the Riemann zeta function defined by (1.2) and the characteristic (or indicator) function $\chi_A(\omega)$ is given by

$$\chi_A(n) = \begin{cases}1 & (\omega \in A)\\ 0 & (\omega \notin A)\end{cases}. \tag{4.3}$$

Furthermore, Golomb [5] showed that, if the subset A of $\mathbb{N}$ has an arithmetic density, then

$$\lim_{s\to 1} Q_s(A) = dens(A), \tag{4.4}$$

thereby allowing number-theoretic facts regarding densities of sets of positive integers to be proven by probabilistic means and then showing that such properties are preserved in the limit.

In an interesting sequel to Golomb's investigation [5], Lippert [9] gave an analogous definition of the probabilities P_s when the set $\mathbb{N}$ is replaced by the set of all real numbers greater than 1. Thus, for a Borel set $A \subseteq (1,\infty)$, Lippert's Hurwitz measure of the set A is defined by (see, for details, [[9], Definition 1]

$$P_s(A) = \frac{s}{\zeta(s)}\int_1^{\infty}\chi_A(a)\zeta(s+1, x)dx \tag{4.5}$$

or, equivalently, by

$$P_s(A) = \int_{x\in(1,\infty)}\chi_A(x)d\tilde{\mu}(x, s), \tag{4.6}$$

where, in terms of the Hurwitz (or generalized) zeta function $\zeta(s,a)$ defined by (1.3),

$$\begin{aligned}&\tilde{\mu}(x,s) := -\frac{\zeta(s,x)}{\zeta(s)} \quad \text{and}\\&d\tilde{\mu}(x,s) = -\frac{d\zeta(s,x)}{\zeta(s)} = s\frac{\zeta(s+1,x)}{\zeta(s)}dx.\end{aligned} \tag{4.7}$$

In this section, we propose to introduce a new continuous analogue of Lippert's Hurwitz measure in (4.5) by using a special case of the generalized Hurwitz-Lerch zeta function $\Theta_{\mu}^{\lambda}(z, s, a; b)$ defined by (1.5).

Definition 2. For a Borel set $A \subseteq (1,\infty)$, the generalized Hurwitz measure of the set A is defined by

$$P_s(A) = \frac{s}{\Theta_1^{\lambda}(1, s, 1; b)}\int_1^{\infty}\chi_A(a)\Theta_1^{\lambda}(1, s+1, a; b)da \tag{4.8}$$

or, equivalently, by

$$P_s(A) = \int_{a\in(1,\infty)}\chi_A(a)d\tilde{\mu}(a, s; b, \lambda), \tag{4.9}$$

where

$$\tilde{\mu}(a, s; b, \lambda) := -\frac{\Theta_1^{\lambda}(1, s, a; b)}{\Theta_1^{\lambda}(1, s, 1; b)} \tag{4.10}$$

and

$$\begin{aligned}d\tilde{\mu}(a, s; b, \lambda) &= -\frac{d\Theta_1^{\lambda}(1, s, a; b)}{\Theta_1^{\lambda}(1, s, 1; b)}\\&= \frac{s}{\Theta_1^{\lambda}(1, s, 1; b)}\Theta_1^{\lambda}(1, s+1, a; b)da,\end{aligned} \tag{4.11}$$

since it is easily seen from the definition (1.5) that

$$\frac{d}{da}\Theta_{\mu}^{\lambda}(z, s, a; b) = -s\Theta_{\mu}^{\lambda}(z, s+1, a; b). \tag{4.12}$$

In view of the following relationship:

$$\begin{aligned}&P_s((1,\infty)) = \int_1^{\infty} d\tilde{\mu}(a, s; b, \lambda)\\&= \lim_{a\to\infty}\tilde{\mu}(a, s; b, \lambda) - \tilde{\mu}(1, s; b, \lambda) = 1,\end{aligned}$$

the generalized Hurwitz measure $P_s(A)$ in (4.8) or (4.9) also defines a probability measure on $(1,\infty)$.

Remark 4. For $\lambda = 1$ and by letting $b \to 0$, we have

$$\lim_{b\to 0} H_{0,2}^{2,0}\left[ab\middle|\overline{(s,1),(0,1)}\right] = \Gamma(s), \tag{4.13}$$

which implies that

$$\begin{aligned}&\lim_{b\to 0}\tilde{\mu}(a, s; b, 1) = -\lim_{b\to 0}\frac{\Theta_1^{1}(1, s, a, b)}{\Theta_1^{1}(1, s, 1, b)}\\&= -\frac{\zeta(s,x)}{\zeta(s)} =: \tilde{\mu}(x, s).\end{aligned} \tag{4.14}$$

Thus, clearly, $\tilde{\mu}(x,s)$ can be continuously approximated by $\tilde{\mu}(a,s;b,1)$.

Proposition. *The measure* $\tilde{\mu}(a,s;b,\lambda)$ *satisfies the following difference equation:*

$$\tilde{\mu}(a+1,s;b,\lambda)-\tilde{\mu}(a,s;b,\lambda) = \frac{H_{0,2}^{2,0}\left[ab^{\frac{1}{\lambda}}\left|\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right.\right]}{\lambda a^s\Gamma(s)\Theta_1^\lambda(1,s,1;b)} \quad (s>1;a>0;b>0;\lambda>0). \tag{4.15}$$

Proof. From the series representation (1.10) of $\Theta_\mu^\lambda(z,s,a+1;b)$ (with $\mu=1$ and $z=1$), we have

$$\begin{aligned}\Theta_1^\lambda(1,s,a+1;b) &= \frac{1}{\lambda\Gamma(s)}\sum_{n=0}^{\infty}\frac{1}{(a+n+1)^s}H_{0,2}^{2,0}\left[(a+n+1)b^{\frac{1}{\lambda}}\left|\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right.\right]\\ &= \frac{1}{\lambda\Gamma(s)}\sum_{n=1}^{\infty}\frac{1}{(a+n)^s}H_{0,2}^{2,0}\left[(a+n)b^{\frac{1}{\lambda}}\left|\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right.\right]\\ &= \Theta_1^\lambda(1,s,a;b)-\frac{1}{\lambda a^s\Gamma(s)}H_{0,2}^{2,0}\left[ab^{\frac{1}{\lambda}}\left|\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right.\right].\end{aligned} \tag{4.16}$$

The difference equation (4.15) now follows on combining (4.10) and (4.16). □

Remark 5. For $\lambda=1$ and by letting $b\to 0$, the difference equation (4.15) reduces to the following form:

$$\tilde{\mu}(a+1,s)-\tilde{\mu}(a,s)=\frac{1}{a^s\zeta(s)}, \tag{4.17}$$

where $\tilde{\mu}(x,s)$ is given by (4.7).

For open events, the generalized Hurwitz measure $P_s(A)$ in (4.8) or (4.9) can be evaluated by using (4.9) and the above Proposition. The results are being stated as Theorem 4 below.

Theorem 4. *If* $A=(a,a+1)$, *then*

$$P_s(A)=P_s((a,a+1)) = \frac{H_{0,2}^{2,0}\left[ab^{\frac{1}{\lambda}}\left|\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right.\right]}{\lambda a^s\Gamma(s)\Theta_1^\lambda(1,s,1;b)}. \tag{4.18}$$

More generally; the generalized Hurwitz measure of an open set $A\subseteq(1,\infty)$ *is given by*

$$P_s(A)=\sum_{i\in I}P_s((a_i,b_i)) = \sum_{i\in I}\left(\frac{\Theta_1^\lambda(1,s,a_i;b)-\Theta_1^\lambda(1,s,b_i;b)}{\Theta_1^\lambda(1,s,1;b)}\right), \tag{4.19}$$

where

$$A=\bigcup_{i\in I}(a_i,b_i) \qquad (a_i,b_i\in[1,\infty);i\in I).$$

The following theorem shows that the generalized Hurwitz measure $P_s(A)$ in (4.8) or (4.9) basically inherits all properties of Lippert's Hurwitz measure given by (4.5) or (4.6).

Theorem 5. *Corresponding to the generalized Hurwitz measure given by (4.19), let*

$$A(\varepsilon)=\bigcup_{i\in\mathbb{N}}(i,i+\varepsilon) \qquad (\varepsilon\in[0,1]). \tag{4.20}$$

Then

$$\lim_{s\to 1}P_s(A(\varepsilon))=\varepsilon. \tag{4.21}$$

Proof. From (4.19), we have

$$P_s(A)=\sum_{i=1}^{\infty}\left(\frac{\Theta_1^\lambda(1,s,i;b)-\Theta_1^\lambda(1,s,i+\varepsilon;b)}{\Theta_1^\lambda(1,s,1;b)}\right). \tag{4.22}$$

By expanding the function $\Theta_1^\lambda(1,s,i+\varepsilon,b)$ by means of Taylor's series and using the derivative formula (4.12), we get

$$P_s(A)=\frac{1}{\Theta_1^\lambda(1,s,1;b)}\left(\begin{aligned}&\varepsilon s\sum_{i=1}^{\infty}\Theta_1^\lambda(1,s+1,i;b)\\ &-\frac{\varepsilon^2}{2}s(s+1)\sum_{i=1}^{\infty}\Theta_1^\lambda(1,s+2,i;b)+\cdots\end{aligned}\right). \tag{4.23}$$

We now consider each sum in (4.23) separately. We thus find that

$$\begin{aligned}\sum_{i=1}^{\infty}\Theta_1^\lambda(1,s+m,i;b) &= \frac{1}{\lambda\Gamma(s+m)}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{H_{0,2}^{2,0}\left[(i+n)b^{\frac{1}{\lambda}}\left|\overline{(s+m,1),\left(0,\frac{1}{\lambda}\right)}\right.\right]}{(i+n)^{s+m}}\\ &= \frac{1}{\lambda\Gamma(s+m)}\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}\frac{H_{0,2}^{2,0}\left[(j+n+1)b^{\frac{1}{\lambda}}\left|\overline{(s+m,1),\left(0,\frac{1}{\lambda}\right)}\right.\right]}{(j+n+1)^{s+m}}\end{aligned} \tag{4.24}$$

Since the number of non-negative integer solutions of the Diophantine equation $j+n=N$ is

$$\binom{N+1}{1}=N+1,$$

the double summation in (4.24) can be replaced by a single summation, that is,

$$\begin{aligned}\sum_{i=1}^{\infty}\Theta_1^\lambda(1,s+m,i;b) &= \frac{1}{\lambda\Gamma(s+m)}\sum_{N=0}^{\infty}\frac{H_{0,2}^{2,0}\left[(N+1)b^{\frac{1}{\lambda}}\left|\overline{(s+m,1),\left(0,\frac{1}{\lambda}\right)}\right.\right]}{(N+1)^{s+m-1}}\\ &= \Theta_1^\lambda(1,s+m-1,1;b).\end{aligned} \tag{4.25}$$

We thus obtain

$$\lim_{s\to 1}P_s(A)=\lim_{s\to 1}\left(\begin{aligned}&\varepsilon s\frac{\Theta_1^\lambda(1,s,1;b)}{\Theta_1^\lambda(1,s,1;b)}\\ &-\frac{\varepsilon^2}{2}s(s+1)\frac{\Theta_1^\lambda(1,s+1,1;b)}{\Theta_1^\lambda(1,s,1;b)}+\cdots\end{aligned}\right)$$

$$= \varepsilon - \frac{\varepsilon^2}{2} s(s+1) \lim_{s\to 1} \frac{\Theta_1^\lambda(1,s+1,1;b)}{\Theta_1^\lambda(1,s,1;b)} + \cdots. \quad (4.26)$$

We note that, when $s \to 1$, the series for $\Theta_1^\lambda(1,s,1;b)$ is divergent and the series for $\Theta_1^\lambda(1,s+1,1;b)$ is convergent. Therefore, all other terms vanish in (4.26) except the leading term. Consequently, we get

$$\lim_{s\to 1} P_s(A) = \varepsilon, \quad (4.27)$$

which completes the proof of Theorem 5. □

As in the theory of probability, we introduce the following definition.

Definition 3. A random variable ξ is said to be generalized Hurwitz distributed if its probability density function (p.d.f.) is given by

$$f_\xi(a) := \begin{cases} \dfrac{s\Theta_1^\lambda(1,s+1,a;b)}{\Theta_1^\lambda(1,s,1;b)} & (a \geq 1) \\ 0 & (otherwise). \end{cases} \quad (4.28)$$

Theorem 6. *Let ξ be a continuous random variable ξ with its p.d.f defined by (4.28). Then the moment generating function $M(z)$ of the random variable ξ is given by*

$$M(z) := \mathbb{E}_s\left[e^{z\xi}\right] = \sum_{n=0}^{\infty} \mathbb{E}_s\left[\xi^n\right] \frac{z^n}{n!} \quad (4.29)$$

with the moment $\mathbb{E}_s\left[\xi^n\right]$ of order n given by

$$\mathbb{E}_s\left[\xi^n\right] = \sum_{k=0}^{\infty} \frac{n!}{(n-k)!} \frac{\Gamma(s-k)}{\Gamma(s)} \frac{\Theta_1^\lambda(1,s-k,1;b)}{\Theta_1^\lambda(1,s,1;b)}. \quad (4.30)$$

Proof. The assertion in (4.29) follows easily by using the exponential series for $e^{z\xi}$. If we use integration by parts, we find from the definition that

$$\mathbb{E}_s\left[\xi^n\right] = \int_1^\infty a^n f_\xi(a) da$$
$$= \frac{s}{\Theta_1^\lambda(1,s,1;b)} \int_1^\infty a^n d\Theta_1^\lambda(1,s+1,a;b) da$$
$$= -\frac{1}{\Theta_1^\lambda(1,s,1;b)} \int_1^\infty a^n d\left(\Theta_1^\lambda(1,s,a;b)\right)$$
$$= -\frac{a^n\Theta_1^\lambda(1,s,a;b)}{\Theta_1^\lambda(1,s,1;b)}\Bigg|_1^\infty + \frac{n}{\Theta_1^\lambda(1,s,1;b)} \int_1^\infty a^{n-1}\Theta_1^\lambda(1,s,1;b) da$$
$$= -\lim_{a\to\infty} \frac{a^n\Theta_1^\lambda(1,s,a;b)}{\Theta_1^\lambda(1,s,1;b)} + 1$$
$$+ \frac{n}{\Theta_1^\lambda(1,s,1;b)} \int_1^\infty a^{n-1}\Theta_1^\lambda(1,s,a;b) da$$
$$= 1 + \frac{n}{\Theta_1^\lambda(1,s,1;b)} \int_1^\infty a^{n-1}\Theta_1^\lambda(1,s,a;b) da \qquad (n \in \mathbb{N}),$$
$$(4.31)$$

where we have also used the derivative property (4.12) and the following limit formula:

$$\lim_{a\to\infty} a^{n-1}\Theta_1^\lambda(1,s,a;b)$$
$$= \lim_{a\to\infty} \frac{a^n}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}\exp\left(-at - \frac{b}{t^\lambda}\right)}{1-e^{-t}} dt$$
$$= \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}\exp\left(\frac{b}{t^\lambda}\right)}{1-e^{-t}} \lim_{a\to\infty} a^n e^{-at} dt \quad (4.32)$$
$$= 0 \qquad (n \in \mathbb{N}).$$

Consequently, we have the following reduction formula for $\mathbb{E}_s\left[\xi^n\right]$:

$$\mathbb{E}_s\left[\xi^n\right] = 1 + \frac{\Theta_1^\lambda(1,s-1,1;b)}{\Theta_1^\lambda(1,s,1;b)} \frac{n}{s-1} \mathbb{E}_{s-1}\left[\xi^{n-1}\right] \quad (n \in \mathbb{N}).$$
$$(4.33)$$

By iterating the recurrence (4.33), we arrive at the desired result (4.30) asserted by Theorem 6. □

Remark 6. The assertion (4.30) of Theorem 6 provides a generalization of a known result [[9], Proposition 3].

References

[1] M. A. Chaudhry and S. M. Zubair, On a Class of Incomplete Gamma Functions with Applications, Chapman and Hall (CRC Press Company), Boca Raton, London, New York and Washington, D.C., 2001.

[2] A. Erd´elyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions, Vol. **I**, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[3] A. Erd´elyi,W. Magnus, F. Oberhettinger and F. G. Tricomi, Tables of Integral Transforms, Vol. **II**, McGraw-Hill Book Company, New York, Toronto and London, 1954.

[4] S. P. Goyal and R. K. Laddha, On the generalized Zeta function and the generalized Lambert function, Ga˙nita Sandesh **11** (1997), 99-108.

[5] S. W. Golomb, A class of probability distributions on the integers, J. Number Theory **2** (1970), 189-192.

[6] A. A. Kilbas and M. Saigo, H-Transforms: Theory and Applications, Chapman and Hall (CRC Press Company), Boca Raton, London, New York and Washington, D.C., 2004.

[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies, Vol. **204**, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

[8] S.-D. Lin and H. M. Srivastava, Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations, Appl. Math. Comput. **154** (2004), 725-733.

[9] R. A. Lippert, A probabilistic interpretation of the Hurwitz zeta function, Adv. Math. **97** (1993), 278-284.

[10] M.-J. Luo and R. K. Raina, Some new results related to a class of generalized Hurwitz zeta function, Ann. Polon. Math. (submitted for publication).

[11] A. M. Mathai and R. K. Saxena, The H-Functions with Applications in Statistics and Other Disciplines, Wiley Eastern Limited, New Delhi, 1978.

[12] A. M. Mathai and R.K. Saxena, Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences, Lecture Notes in Mathematics, Vol. **348**, Springer-Verlag, Berlin, Heidelberg and New York, 1973.

[13] A. M. Mathai, R. K. Saxena and H. J. Haubold, The H-Function: Theory and Applications, Springer, New York, Dordrecht, Heidelberg and London, 2010.

[14] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (Editors), NIST Handbook of Mathematical Functions [With 1 CD-ROM (Windows, Macintosh and UNIX)], U. S. Department

of Commerce, National Institute of Standards and Technology, Washington, D. C., 2010; Cambridge University Press, Cambridge, London and New York, 2010.

[15] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, Integrals and Series, Vol. **3**: More Special Functions, Gordon and Breach Science Publishers, London and New York, 1990.

[16] R. K. Raina and P. K. Chhajed, Certain results involving a class of functions associated with the Hurwitz zeta function, Acta Math. Univ. Comenianae **73** (2004), 89-100.

[17] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Translated from the Russian: Integrals and Derivatives of Fractional Order and Some of Their Applications (”Naukai Tekhnika”, Minsk, 1987), Gordon and Breach Science Publishers, Reading, Tokyo, Paris, Berlin and Langhorne (Pennsylvania), 1993.

[18] H. M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, Math. Proc. Cambridge Philos. Soc. **129** (2000), 77-84.

[19] H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, Appl. Math. Inform. Sci. **5** (2011), 390-444.

[20] H. M. Srivastava, Generating relations and other results associated with some families of the extended Hurwitz-Lerch Zeta functions, SpringerPlus **2** (2013), Article ID 2:67, 1-14.

[21] H. M. Srivastava and J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[22] H. M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[23] H. M. Srivastava, K. C. Gupta and S. P. Goyal, The H-Functions of One and Two Variables with Applications, South Asian Publishers, New Delhi and Madras, 1982.

[24] H. M. Srivastava and H. L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.