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q-Hardy-Littlewood-Type Maximal Operator with Weight Related to Fermionic p-Adic q-Integral on Z_p

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Abstract The q-extension of Hardy-littlewood-type maximal operator in accordance with q-Volkenborn integral in the p-adic integer ring was recently studied [11]. A generalization of Jang's results was given by Araci and Acikgoz [1]. By the same motivation of their papers, we aim to give the definition of the weighted *q*-Hardy-littlewood-type maximal operator by means of *fermionic* p-adic q-invariant distribution on Z_p. Finally, we derive some interesting properties involving this-type maximal operator.

Keywords: *fermionic p-adic q-integral on Z_p, hardy-littlewood theorem, p-adic analysis, q-analysis*

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1. Introduction

The concept of p-adic numbers was originally invented by Kurt Hensel who is German mathematician, around the end of the nineteenth century [12]. In spite of their being already one hundred years old, these numbers are still today enveloped in an aura of mystery within the scientific community and also play a vital and important role in mathematics.

The fermionic p-adic q-integral in the p-adic integer ring was originally constructed by Kim [2,6] who introduced Lebesgue-Radon-Nikodym Theorem with respect to fermionic p-adic q-integral on $\mathbb{Z}_p$. The fermionic p-adic q-integral on $\mathbb{Z}_p$ is used in mathematical physics for example the functional equation of the q-zeta function, the q-stirling numbers and q-mahler theory of integration with respect to the ring $\mathbb{Z}_p$ together with Iwasawa's p-adic q-L function.

In [11], Jang defined q-extension of Hardy-Littlewood-type maximal operator by means of q-Volkenborn integral on $\mathbb{Z}_p$. Afterwards, in [1], Araci and Acikgoz added a weight into Jang's q-Hardy-Littlewood-type maximal operator and derived some interesting properties by means of Kim's p-adic q-integral on $\mathbb{Z}_p$. Now also, we shall consider weighted q-Hardy-Littlewood-type maximal operator on the fermionic p-adic q-integral on $\mathbb{Z}_p$. Moreover, we shall analyse q-Hardy-Littlewood-type maximal operator via the fermionic p-adic q-integral on $\mathbb{Z}_p$.

Assume that p be an odd prime number. Let $\mathbb{Q}_p$ be the field of p-adic rational numbers and let $\mathbb{C}_p$ be the completion of algebraic closure of $\mathbb{Q}_p$.

Thus,

$$\mathbb{Q}_p = \left\{ x = \sum_{n=-k}^{\infty} a_n p^n \ : \ 0 \le a_n < p \right\}.$$

Then $\mathbb{Z}_p$ is an integral domain to be

$$\mathbb{Z}_p = \left\{ x = \sum_{n=0}^{\infty} a_n p^n \ : \ 0 \le a_n \le p-1 \right\},$$

or

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \le 1 \right\}.$$

In this paper, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ as an indeterminate.

The p-adic absolute value $|.|_p$, is normally defined by

$$|x|_p = \frac{1}{p^r},$$

where $x = p^r \frac{s}{t}$ with $(p,s) = (p,t) = (s,t) = 1$ and $r \in \mathbb{Q}$.

A p-adic Banach space B is a $\mathbb{Q}_p$-vector space with a lattice B^0 ($\mathbb{Z}_p$-module) separated and complete for p-adic topology, ie.,

$$B^0 \simeq \varprojlim_{n \in \mathbb{N}} B^0 / p^n B^0.$$

For all $x \in B$, there exists $n \in \mathbb{Z}$, such that $x \in p^n B^0$. Define

$$v_B(x) = \sup_{n \in \mathbb{N} \cup \{+\infty\}} \{n \,:\, x \in p^n B^0\}.$$

It satisfies the following properties:

$$v_B(x+y) \geq \min(v_B(x), v_B(y)),$$
$$v_B(\beta x) = v_p(\beta) + v_B(x), \text{ if } \beta \in \mathbb{Q}_p.$$

Then, $\|x\|_B = p^{-v_B(x)}$ defines a norm on B, such that B is complete for $\|.\|_B$ and B^0 is the unit ball.

A measure on $\mathbb{Z}_p$ with values in a p-adic Banach space B is a continuous linear map

$$f \mapsto \int f(x)\mu = \int_{\mathbb{Z}_p} f(x)\mu(x)$$

from $C^0(\mathbb{Z}_p, \mathbb{C}_p)$, (continuous function on $\mathbb{Z}_p$) to B. We know that the set of locally constant functions from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ is dense in $C^0(\mathbb{Z}_p, \mathbb{C}_p)$ so.

Explicitly, for all $f \in C^0(\mathbb{Z}_p, \mathbb{C}_p)$, the locally constant functions

$$f_n = \sum_{i=0}^{p^n-1} f(i) 1_{i+p^n\mathbb{Z}_p} \to f \text{ in } C^0.$$

Now if $\mu \in \mathbf{D}_0(\mathbb{Z}_p, \mathbb{Q}_p)$, set $\mu(i+p^n\mathbb{Z}_p) = \int_{\mathbb{Z}_p} 1_{i+p^n\mathbb{Z}_p}\mu$. Then $\int_{\mathbb{Z}_p} f\mu$ is given by the following Riemann sums

$$\int_{\mathbb{Z}_p} f\mu = \lim_{n\to\infty} \sum_{i=0}^{p^n-1} f(i)\mu(i+p^n\mathbb{Z}_p).$$

T. Kim defined μ_{-q} as follows:

$$\mu_{-q}(\xi + dp^n\mathbb{Z}_p) = \frac{(-q)^{\xi}}{[dp^n]_{-q}}$$

and this can be extended to a distribution on $\mathbb{Z}_p$. This distribution yields an integral in the case $d = 1$.

So, q-Volkenborn integral was defined by T. Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(\xi)\, d\mu_q(\xi) = \lim_{n\to\infty} \frac{1}{[p^n]_{-q}} \sum_{\xi=0}^{p^n-1} (-1)^{\xi} f(\xi) q^{\xi} \quad (1.1)$$

where $[x]_q$ is a q-extension of x which is defined by

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ cf. [1,2,4,5,6,7,11].

Let d be a fixed positive integer with $(p,d) = 1$. We now set

$$X = X_d = \varprojlim_n \mathbb{Z}/dp^n\mathbb{Z},$$
$$X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$
$$a + dp^n\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{p^n}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^n$. For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$,

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) = \int_X f(x)\, d\mu_{-q}(x) \quad \text{see [10]}$$

By means of q-Volkenborn integral, we consider below strongly p-adic q-invariant distribution μ_{-q} on $\mathbb{Z}_p$ in the form

$$\left| \begin{array}{l} [p^n]_{-q}\, \mu_{-q}(a + p^n\mathbb{Z}_p) \\ -[p^{n+1}]_{-q}\, \mu_{-q}(a + p^{n+1}\mathbb{Z}_p) \end{array} \right| < \delta_n,$$

where $\delta_n \to 0$ as $n \to \infty$ and δ_n is independent of a. Let $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, for any $a \in \mathbb{Z}_p$, we assume that the weight function $\omega(x)$ is defined by $\omega(x) = \omega^x$ where $\omega \in \mathbb{C}_p$ with $|1-\omega|_p < 1$. We define the weighted measure on $\mathbb{Z}_p$ as follows:

$$\mu_{f,-q}^{(\omega)}(a + p^n\mathbb{Z}_p) = \int_{a+p^n\mathbb{Z}_p} \omega^{\xi} f(\xi)\, d\mu_{-q}(\xi) \quad (1.2)$$

where the integral is the fermionic p-adic q-integral on $\mathbb{Z}_p$. From (1.2), we note that $\mu_{f,-q}^{(\omega)}$ is a strongly weighted measure on $\mathbb{Z}_p$. Namely,

$$\left| [p^n]_{-q}\, \mu_{f,-q}^{(\omega)}(a + p^n\mathbb{Z}_p) - [p^{n+1}]_{-q}\, \mu_{f,-q}^{(\omega)}(a + p^{n+1}\mathbb{Z}_p) \right|_p$$
$$= \left| \sum_{x=0}^{p^n-1} (-1)^x \omega^x f(x) q^x - \sum_{x=0}^{p^n} (-1)^x \omega^x f(x) q^x \right|_p$$
$$\leq \left| \frac{f(p^n)(-1)^{p^n} \omega^{p^n} q^{p^n}}{p^n} \right|_p |p^n|_p$$
$$\leq Cp^{-n}$$

Thus, we get the following proposition.

Proposition 1. *For* $f, g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$*, then, we have*

$$\mu^{(\omega)}_{\alpha f+\beta g,-q}\left(a+p^n\mathbb{Z}_p\right)$$
$$=\alpha\mu^{(\omega)}_{f,-q}\left(a+p^n\mathbb{Z}_p\right)+\beta\mu^{(\omega)}_{g,-q}\left(a+p^n\mathbb{Z}_p\right).$$

where α,β are positive constants. Also, we have

$$\left|\begin{array}{l}\left[p^n\right]_{-q}\mu^{(\omega)}_{f,-q}\left(a+p^n\mathbb{Z}_p\right)\\ -\left[p^{n+1}\right]_{-q}\mu^{(\omega)}_{f,-q}\left(a+p^{n+1}\mathbb{Z}_p\right)\end{array}\right|\leq Cp^{-n}$$

where C is positive constant.

Let $\mathbf{P}_q(x)\in\mathbb{C}_p\left[[x]_q\right]$ be an arbitrary q-polynomial. Now also, we indicate that $\mu^{(\omega)}_{\mathbf{P},-q}$ is a strongly weighted fermionic p-adic q-invariant measure on $\mathbb{Z}_p$. Without a loss of generality, it is sufficient to evidence the statement for $\mathbf{P}(x)=[x]^k_q$.

$$\mu^{(\omega)}_{\mathbf{P},-q}\left(a+p^n\mathbb{Z}_p\right)$$
$$=\lim_{m\to\infty}\frac{1}{\left[p^m\right]_{-q}}\sum_{i=0}^{p^{m-n}-1}w^{a+ip^n}\left[a+ip^n\right]^k_q(-q)^{a+ip^n}\quad(1.3)$$

where

$$\left[a+ip^n\right]^k_q$$
$$=\sum_{j=0}^{k}\binom{k}{j}[a]^{k-j}_q q^{aj}\left[p^n\right]^j_q[i]^j_{q^{p^n}}\quad(1.4)$$
$$=[a]^k_q+k[a]^{k-1}_q q^a\left[p^n\right]_q[i]_{q^{p^n}}+\ldots+q^{ak}\left[p^n\right]^k_q[i]^k_{q^{p^n}}$$

and

$$w^{a+ip^n}=w^a\sum_{l=0}^{ip^n}\binom{ip^n}{l}(w-1)^l\equiv w^a\left(\operatorname{mod}p^n\right).\quad(1.5)$$

By (1.5), we have

$$(-q)^{a+ip^n}$$
$$=(-q)^a\sum_{l=0}^{ip^n}\binom{ip^n}{l}(-1)^l(q+1)^l\quad(1.6)$$
$$\equiv(-q)^a\left(\operatorname{mod}p^n\right).$$

By (1.3), (1.4), (1.5) and (1.6), we have the following

$$\mu^{(\omega)}_{\mathbf{P},-q}\left(a+p^n\mathbb{Z}_p\right)$$
$$\equiv(-1)^a\omega^a q^a[a]^k_q\left(\operatorname{mod}p^n\right)$$
$$\equiv(-1)^a\omega^a q^a\mathbf{P}(a)\left(\operatorname{mod}p^n\right).$$

For $x\in\mathbb{Z}_p$, let $x\equiv x_n\left(\operatorname{mod}p^n\right)$ and $x\equiv x_{n+1}\left(\operatorname{mod}p^{n+1}\right)$, where x_n, $x_{n+1}\in\mathbb{Z}$ with $0\leq x_n<p^n$ and $0\leq x_{n+1}<p^{n+1}$
Then, we procure the following

$$\left|\begin{array}{l}\left[p^n\right]_{-q}\mu^{(\omega)}_{\mathbf{P},-q}\left(a+p^n\mathbb{Z}_p\right)\\ -\left[p^{n+1}\right]_{-q}\mu^{(\omega)}_{\mathbf{P},-q}\left(a+p^{n+1}\mathbb{Z}_p\right)\end{array}\right|\leq Cp^{-n},$$

where C is positive constant and $n>>0$.

Let $UD\left(\mathbb{Z}_p,\mathbb{C}_p\right)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$ with sup-norm

$$\|f\|_\infty=\sup_{x\in\mathbb{Z}_p}|f(x)|_p.$$

The difference quotient $\Delta_1 f$ of f is the function of two variables given by

$$\Delta_1 f(m,x)=\frac{f(x+m)-f(x)}{m}$$

for all $x,m\in\mathbb{Z}_p, m\neq 0$

A function $f:\mathbb{Z}_p\to\mathbb{C}_p$ is said to be a Lipschitz function if there exists a constant $M>0$ (the Lipschitz constant of f) such that

$$\left|\Delta_1 f(m,x)\right|\leq M\text{ for all }m\in\mathbb{Z}_p\setminus\{0\}\text{ and }x\in\mathbb{Z}_p.$$

The $\mathbb{C}_p$ linear space consisting of all Lipschitz function is denoted by $Lip\left(\mathbb{Z}_p,\mathbb{C}_p\right)$. This space is a Banach space with the respect to the norm $\|f\|_1=\|f\|_\infty\vee\|\Delta_1 f\|_\infty$ (for more information, see [3-9]). The objective of this paper is to introduce weighted q-Hardy Littlewood-type maximal operator on the fermionic p-adic q-integral on $\mathbb{Z}_p$. Also, we show that the boundedness of the weighted q-Hardy-littlewood-type maximal operator in the p-adic integer ring.

2. The Weighted q-Hardy-Littlewood-Type Maximal Operator

In view of (1.2) and the definition of fermionic p-adic q-integral on $\mathbb{Z}_p$, we now consider the following theorem.

Theorem 1. *Let $\mu^{(w)}_{-q}$ be a strongly fermionic p-adic q-invariant on $\mathbb{Z}_p$ and $f\in UD\left(\mathbb{Z}_p,\mathbb{C}_p\right)$. Then for any $n\in\mathbb{Z}$ and any $\xi\in\mathbb{Z}_p$, we have*

$$\int_{a+p^n\mathbb{Z}_p}\omega^\xi f(\xi)(-q)^{-\xi}d\mu_{-q}(\xi)$$
$$(1)\quad=\frac{(-1)^a\omega^a}{\left[p^n\right]_{-q}}\int_{\mathbb{Z}_p}\omega^\xi f\left(a+p^n\xi\right)(-q)^{-p^n\xi}d\mu_{-q^{p^n}}(\xi)$$

(2) $\int_{a+p^n\mathbb{Z}_p}\omega^{\xi}d\mu_{-q}(\xi)=\frac{\omega^a(-q)^a}{[p^n]_{-q}}\frac{2}{1+\omega^{p^n}q^{p^n}}$

Proof. (1) By using (1.1) and (1.2), we see the following applications:

$$\int_{a+p^n\mathbb{Z}_p}\omega^{\xi}f(\xi)(-q)^{-\xi}d\mu_{-q}(\xi)$$

$$=\lim_{m\to\infty}\frac{1}{[p^{m+n}]_{-q}}\sum_{\xi=0}^{p^m-1}\begin{bmatrix}\omega^{a+p^n\xi}f(a+p^n\xi)\\ \times(-q)^{-(a+p^n\xi)}\\ \times q^{a+p^n\xi}(-1)^{a+p^n\xi}\end{bmatrix}$$

$$=(-1)^a\omega^a\lim_{m\to\infty}\begin{bmatrix}\frac{1}{[p^m]_{-q^{p^n}}[p^n]_{-q}}\\ \times\sum_{\xi=0}^{p^m-1}\omega^{\xi}(-q)^{-p^n\xi}\\ \times f(a+p^n\xi)\left(-q^{p^n}\right)^{\xi}\end{bmatrix}$$

$$=\frac{(-1)^a\omega^a}{[p^n]_{-q}}\int_{\mathbb{Z}_p}\begin{bmatrix}\omega^{\xi}f(a+p^n\xi)\\ \times(-q)^{-p^n\xi}d\mu_{-q^{p^n}}(\xi)\end{bmatrix}.$$

(2) By the same method of (1), then, we easily derive the following

$$\int_{a+p^n\mathbb{Z}_p}\omega^{\xi}d\mu_{-q}(\xi)$$

$$=\lim_{m\to\infty}\frac{1}{[p^{m+n}]_{-q}}\sum_{\xi=0}^{p^m-1}\omega^{a+\xi p^n}(-q)^{a+\xi p^n}$$

$$=\frac{\omega^a(-q)^a}{[p^n]_{-q}}\lim_{m\to\infty}\frac{1}{[p^m]_{-q^{p^n}}}\sum_{\xi=0}^{p^m-1}\left(\omega^{p^n}\right)^{\xi}\left(-q^{p^n}\right)^{\xi}$$

$$=\frac{\omega^a(-q)^a}{[p^n]_{-q}}\lim_{m\to\infty}\frac{1+\left(\omega^{p^n}q^{p^n}\right)^{p^m}}{1+\omega^{p^n}q^{p^n}}$$

$$=\frac{\omega^a(-q)^a}{[p^n]_{-q}}\frac{2}{1+\omega^{p^n}q^{p^n}}$$

Since $\lim_{m\to\infty}q^{p^m}=1$ for $|1-q|_p<1$, our assertion follows.

We are now ready to introduce the definition of the weighted q-Hardy-littlewood-type maximal operator related to fermionic p-adic q-integral on $\mathbb{Z}_p$ with a strong fermionic p-adic q-invariant distribution μ_{-q} in the p-adic integer ring.

Definition 1. *Let $\mu_{-q}^{(\omega)}$ be a strongly fermionic p-adic q-invariant distribution on $\mathbb{Z}_p$ and $f\in UD(\mathbb{Z}_p,\mathbb{C}_p)$. Then, q-Hardy-littlewood-type maximal operator with weight related to fermionic p-adic q-integral on $a+p^n\mathbb{Z}_p$ is defined as*

$$\mathtt{M}_{p,q}^{(\omega)}f(a)$$
$$=\sup_{n\in\mathbb{Z}}\frac{1}{\mu_{1,-q}^{(w)}(\xi+p^n\mathbb{Z}_p)}\int_{a+p^n\mathbb{Z}_p}\omega^{\xi}(-q)^{-\xi}f(\xi)d\mu_{-q}(\xi)$$

for all $a\in\mathbb{Z}_p$.

We recall that famous Hardy-littlewood maximal operator $\mathtt{M}_{\mu}$, which is defined by

$$\mathtt{M}_{\mu}f(a)=\sup_{a\in Q}\frac{1}{\mu(Q)}\int_Q|f(x)|d\mu(x),\qquad(2.1)$$

where $f:\mathbb{R}^k\to\mathbb{R}^k$ is a locally bounded Lebesgue measurable function, μ is a Lebesgue measure on $(-\infty,\infty)$ and the supremum is taken over all cubes Q which are parallel to the coordinate axes. Note that the boundedness of the Hardy-Littlewood maximal operator serves as one of the most important tools used in the investigation of the properties of variable exponent spaces (see [11]). The essential aim of Theorem 1 is to deal mainly with the weighted q-extension of the classical Hardy-Littlewood maximal operator in the space of p-adic Lipschitz functions on $\mathbb{Z}_p$ and to find the boundedness of them. By means of Definition 1, then, we state the following theorem.

Theorem 2. *Let $f\in UD(\mathbb{Z}_p,\mathbb{C}_p)$ and $x\in\mathbb{Z}_p$, we get*

(1)
$$\mathtt{M}_{p,q}^{(\omega)}f(a)=\frac{(-1)^a}{2q^a}\sup_{n\in\mathbb{Z}}\left(1+\omega^{p^n}q^{p^n}\right)\int_{\mathbb{Z}_p}\omega^{\xi}f(x+p^n\xi)(-q)^{-p^n\xi}d\mu_{-q^{p^n}}(\xi)$$

(2)
$$\left|\mathtt{M}_{p,q}^{(\omega)}f(a)\right|_p\le\left|\frac{(-1)^a}{2q^a}\right|_p\sup_{n\in\mathbb{Z}}\left|1+\omega^{p^n}q^{p^n}\right|_p\|f\|_1\left\|\left(\frac{-q^{p^n}}{\omega}\right)^{-(.)}\right\|_{L^1}$$

where $\left\|\left(\frac{-q^{p^n}}{\omega}\right)^{-(.)}\right\|_{L^1}=\int_{\mathbb{Z}_p}\left(\frac{-q^{p^n}}{\omega}\right)^{-\xi}d\mu_{-q^{p^n}}(\xi)$.

Proof. (1) Because of Theorem 1 and Definition 1, we see

$$\mathtt{M}_{p,q}^{(\omega)}f(a)=\sup_{n\in\mathbb{Z}}\frac{1}{\mu_{1,-q}^{(\omega)}(\xi+p^n\mathbb{Z}_p)}\int_{a+p^n\mathbb{Z}_p}\omega^{\xi}(-q)^{-\xi}f(\xi)d\mu_{-q}(\xi)$$

$$=\frac{(-1)^a}{2q^a}\sup_{n\in\mathbb{Z}}\left(1+\omega^{p^n}q^{p^n}\right)\int_{\mathbb{Z}_p}\omega^{\xi}f\left(x+p^n\xi\right)(-q)^{-p^n\xi}d\mu_{-q^{p^n}}(\xi).$$

(2) On account of (1), we can derive the following

$$\left|\mathbf{M}_{p,q}^{(\omega)}f(a)\right|_p$$
$$=\left|\frac{(-1)^a}{2q^a}\sup_{n\in\mathbb{Z}}\left(1+\omega^{p^n}q^{p^n}\right)\int_{\mathbb{Z}_p}\omega^{\xi}f\left(x+p^n\xi\right)(-q)^{-p^n\xi}d\mu_{-q^{p^n}}(\xi)\right|_p$$
$$\leq\left|\frac{(-1)^a}{2q^a}\right|_p\sup_{n\in\mathbb{Z}}\left|\left(1+\omega^{p^n}q^{p^n}\right)\int_{\mathbb{Z}_p}\omega^{\xi}f\left(x+p^n\xi\right)(-q)^{-p^n\xi}d\mu_{-q^{p^n}}(\xi)\right|_p$$
$$\leq\left|\frac{(-1)^a}{2q^a}\right|_p\sup_{n\in\mathbb{Z}}\left|1+\omega^{p^n}q^{p^n}\right|_p\int_{\mathbb{Z}_p}\left|f\left(a+p^n\xi\right)\right|_p\left|\left(\frac{-q^{p^n}}{\omega}\right)^{-\xi}\right|_p d\mu_{-q^{p^n}}(\xi)$$
$$\leq\left|\frac{(-1)^a}{2q^a}\right|_p\sup_{n\in\mathbb{Z}}\left|1+\omega^{p^n}q^{p^n}\right|_p\|f\|_1\int_{\mathbb{Z}_p}\left|\left(\frac{-q^{p^n}}{\omega}\right)^{-\xi}\right|_p d\mu_{-q^{p^n}}(\xi)$$
$$=\left|\frac{(-1)^a}{2q^a}\right|_p\sup_{n\in\mathbb{Z}}\left|1+\omega^{p^n}q^{p^n}\right|_p\|f\|_1\left\|\left(\frac{-q^{p^n}}{\omega}\right)^{-(\cdot)}\right\|_{L^1}.$$

Thus, we complete the proof of theorem.

We note that Theorem 2 (2) shows the supnorm-inequality for the q-Hardy-Littlewood-type maximal operator with weight on $\mathbb{Z}_p$, on the other hand, Theorem 2 (2) shows the following inequality

$$\left\|\mathbf{M}_{p,q}^{(\omega)}f\right\|_\infty=\sup_{x\in\mathbb{Z}_p}\left|\mathbf{M}_{p,q}^{(\omega)}f(x)\right|_p\leq\mathbf{K}\|f\|_1\left\|\left(\frac{-q^{p^n}}{\omega}\right)^{-(\cdot)}\right\|_{L^1}\tag{2.2}$$

where $\mathbf{K}=\left|\frac{(-1)^a}{2q^a}\right|_p\sup_{n\in\mathbb{Z}}\left|1+\omega^{p^n}q^{p^n}\right|_p$. By the equation (2.2), we get the following Corollary, which is the boundedness for weighted q-Hardy-Littlewood-type maximal operator with weight on $\mathbb{Z}_p$.

Corollary 1. *$\mathbf{M}_{p,q}^{(\omega)}$ is a bounded operator from $UD\left(\mathbb{Z}_p,\mathbb{C}_p\right)$ into $L^\infty\left(\mathbb{Z}_p,\mathbb{C}_p\right)$, where $L^\infty\left(\mathbb{Z}_p,\mathbb{C}_p\right)$ is the space of all p-adic supnorm-bounded functions with the*

$$\|f\|_\infty=\sup_{x\in\mathbb{Z}_p}\left|f(x)\right|_p,$$

for all $f\in L^\infty\left(\mathbb{Z}_p,\mathbb{C}_p\right)$.

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