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# Free vibration analysis of thick plates resting on Winkler elastic foundation

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**ABSTRACT**

The purpose of this study is to determine the effects of various parameters such as the aspect ratio, subgrade reaction modulus and thickness/span ratio on the frequency parameters of thick plates resting on Winkler-type elastic foundations. For this purpose, 4-noded (PBQ4) and 8-noded (PBQ8) Mindlin plate elements are adopted for the analysis using Winkler foundation model. Two different integration rules, namely the full integration (FI) and the selective reduced integration (SRI) techniques, are used to obtain stiffness matrix of plates. The results obtained in this study are compared with the results that are obtained by SAP2000 structural analysis software.

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## 1. Introduction

The dynamic response of plates resting on elastic foundation is an important tool in understanding the dynamic behaviour of many engineering problems. In general, the analysis of this problem is based on the incorporation of the foundation reaction into the corresponding differential equation of plates. Different models for the foundation plates have been proposed to provide more efficient and practical solutions. The simplest model is the Winkler model which is also called as one parameter model. This model can be considered as an idealization of the soil medium by a number of mutually independent springs and the vertical displacements of the plate are assumed to be proportional at every point to the contact pressure. Although it has some difficulties, Winkler model is widely used because of its simplicity.

In the classical plate theory which is known as Kirchhoff plate theory, the effect of the shear deformation through the plate thickness is ignored. However the effect of the shear deformation becomes important as the thickness of the plate increases. For this reason, it is obvious that shear deformation has to be taken into account especially for the thick plates. Mindlin plate elements are fundamentally simple to adopt for modelling the shear deformation behaviour of the thick plates. But Mindlin plate element has shear locking problem when a thin plate is considered. Reduced integration and selective reduced integration techniques have been proposed in many studies to avoid this problem.

Some of the studies on this topic is as follows. Shen et al. (2001) investigated effects of the parameters such as foundation stiffness, transverse shear deformation and plate aspect ratio on dynamic response of Reissner-Mindlin plates resting on Pasternak type elastic foundation. Malekzadeh (2009) presented an accurate solution procedure based on the three-dimensional elasticity theory for the free vibration analysis of thick functionally graded plates on two-parameter elastic foundation. Leung and Zhu (2005) presented an analytical trapezoidal hierarchical element for the transverse vibration of Mindlin plates resting on two parameter foundation using Legendre orthogonal polynomials to avoid shear locking problem. Omurtag et al. (1997) studied the free vibration analysis of Kirchhoff plates resting on elastic foundation using Gateaux differential. Zhou et al. (2006) investigated the effects of various thickness-radius ratios, foundation stiffness parameters and boundary conditions on the dynamic behavior of the thick circular plates on elastic foundation. Jedrysiak (2003) calculated

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frequencies of thin plates interacting with an elastic periodic foundation. Zhong and Yin (2008) investigated the free vibration behavior of plate on elastic foundation by finite integral transform method using Winkler foundation method. Tovstik (2009) studied vibration and stability of a prestressed plate on elastic foundation. Hsu (2010) numerically modelled the vibration behaviour of orthotropic plates on nonlinear elastic foundations.

In this study, 4-noded (PBQ4) and 8-noded (PBQ8) Mindlin plate elements that includes the effects of shear deformation are adopted for the free vibration analysis of the thick plates resting on Winkler foundation. A computer program using finite element method is coded in FORTRAN to calculate stiffness and mass matrices of the system and then MATLAB for Windows 5.3 is used to obtain the solution of the generalized eigenvalue problem.

## 2. Mathematical Model

The equation of motion for a plate-soil system subjected to free vibration without damping is

$$[M]\{\ddot{w}\} + [K]\{w\} = 0\,, \qquad (1)$$

where $[K]$ is the stiffness matrix of the plate-soil system, $[M]$ is the mass matrix of the plate-soil system, $w$ and $\ddot{w}$ are the displacement and acceleration of the plate, respectively.

The subsoil has a finite depth on a rigid base at the bottom. The total potential energy of plate-soil system without load can be expanded as

$$\Pi = \frac{1}{2}\int [B]^T [D][B] dA + \frac{1}{2}\int [w(x,y)]^T\, k[w(x,y)] dA\,, \quad (2)$$

where $[B]$ is strain displacement matrix, $[D]$ is the material matrix and $k$ is the subgrade reaction modulus of the Winkler-type foundation. In this study 4-noded (PBQ4) and 8-noded (PBQ8) quadrilateral rectangular finite elements based on Reissner-Mindlin theory are used to develop the element stiffness matrix. In these elements, nodal displacements at each node are

$$w, \varphi_x, \varphi_y\,, \qquad (3)$$

where $w$ is the transverse displacement, $\varphi_x$ and $\varphi_y$ are the rotations. The rotations $\varphi_x$ and $\varphi_y$ are independent, and are not related to $w$ by differentiation. Displacement shape functions are given as

$$[N_i] = [N_1 \;\; 0 \;\; 0 \;\; N_2 \;\; 0 \;\; 0 \;\; \ldots \;\; N_n \;\; 0 \;\; 0]\,, \qquad (4)$$

in which $n$ is equal to 4 for PBQ4 and 8 for PBQ8. Shape functions can be found in explicit forms Weaver and Johnston (1984). After the standard procedures of finite element method are applied, the stiffness matrices of the plate-soil system can be evaluated as

$$U = \frac{1}{2}\{w_e\}^t\left([k_p] + [k_w]\right)\{w_e\}\,, \qquad (5)$$

where $[k_p]$ and $[k_w]$ are stiffness matrix of the plate and stiffness matrix of the Winkler foundation, $\{w_e\}$ is the nodal displacement vector. More explanation is given in Özgan and Daloğlu (2007, 2009).

The Winkler foundation element stiffness matrices for PBQ4 and PBQ8 are given in explicit forms in Özgan and Daloğlu (2007).

The dynamics of elastic structures include the kinetic energy of the plate in addition to the strain energy. Evaluation of the mass matrix of plate-subsoil system is based on Hamilton's variational principle with the kinetic energy of

$$\pi_k = \frac{1}{2}\int_\Omega \{\dot{w}\}^T [\mu]\{\dot{w}\} d\,, \qquad (6)$$

where $\{\dot{w}\}$ represents the partial derivative of the vector of generalized displacement with respect to time variable and $[\mu]$ is the mass density matrix. See reference Özgan and Daloğlu (2009) for the evaluation of mass matrices.

After substituting $w = W \sin \omega t$ into the governing equation for a plate subjected to free vibration with no damping given Eq. (1), one can obtain

$$([K] - \omega^2[M])\{W\} = 0\pi_k = \frac{1}{2}\int_\Omega \{\dot{w}\}^T [\mu]\{\dot{w}\} d\,, \qquad (6)$$

where $\{W\}$is a vector of mode shape of vibration and $\lambda$ ($\lambda=\omega^2$, $\omega$ is the circular frequency) is the frequency parameter. The eigenvalue solution of this equation yields frequency parameters and the corresponding mode shapes (Kolar and Nemec, 1989).

## 3. Numerical Examples

The plate considered for the numerical analysis in this study has 10 m length in the *x* and *y* direction and 0.5 m thickness in the *z* direction. The modulus of elasticity and Poisson ratio of the plate are $27.10^9$ N/m$^2$ and 0.20 respectively. The subgrade reaction modulus of the Winkler-type foundation is $5 \cdot 10^6$ N/m$^3$. For the dynamic analysis, the mass density of the plate is taken as 2500 kg/m$^3$. 5 finite elements in the each direction for PBQ8 elements are enough for desired accuracy while 10 finite elements are required for PBQ4 elements. The example is analyzed by MZC element and SAP2000 structural analysis software in addition to the elements presented in this study and comparisons are made. MZC element based on Kirchhoff plate theory is commonly used for the thin plates due to it ignores shear deformation effects through the thickness of the plate. In the solution of SAP2000, the option of thick plate element is selected. Results are presented in Table 1 and Fig. 1.

As can be seen form figure, curves for PBQ4(FI) and PBQ4(SRI) become different from that of other solutions after 5$^{th}$ and 4$^{th}$ frequency parameter respectively. While PBQ4(FI) produces higher frequency values, PBQ4(SRI) produces lower frequency values than expected. It is seen that selective reduced integration in the PBQ4 element cannot be satisfactory for the free vibration analysis of the thick plate on Winkler-type foundation. PBQ8 elements are in good agreement with the solution of SAP2000 commercial software. But it is observed that PBQ8(SRI) element is getting closer to SAP2000 results compare to the solution by PBQ8(FI) element for the higher frequency parameters. Therefore, further analysis will be done by PBQ8(SRI) element.

The same example is analyzed for the various thickness/span ratio, aspect ratio and subgrade reaction modulus to demonstrate the effects of these parameters on the frequency parameters of the plate. Subgrade reaction modulus is taken as $5{\cdot}10^5$, $5{\cdot}10^6$ and $5{\cdot}10^7$ N/m$^3$. Aspect ratios are taken as 1.0, 1.5 and 2.0 for each subgrade reaction modulus as the length of the plate in the *x* direction is fixed. Thickness/span ratios are used as 0.05, 0.10 and 0.20 for each aspect ratio. Results obtained are presented in Table 2.

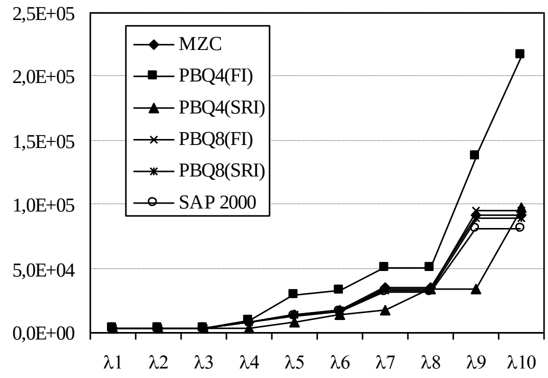


**Fig. 1.** The variation of the first ten frequency parameters of the plate.

**Table 1.** The first six frequency parameters of the plate.

| $l_y/l_x$ | $h/l_x$ | Finite Element | Frequency Parameters | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | $l_1$ | $l_2$ | $l_3$ | $l_4$ | $l_5$ | $l_6$ |
| 1.0 | 0.05 | MZC | 3997.50 | 3997.50 | 4000.00 | 8804.58 | 13951.08 | 17177.13 |
| | | PBQ4(FI) | 3990.02 | 3990.02 | 4000.00 | 8893.01 | 29689.66 | 32970.03 |
| | | PBQ4(SRI) | 3990.02 | 3990.02 | 4000.00 | 4000.00 | 8612.76 | 14022.42 |
| | | PBQ8(FI) | 3990.42 | 3990.42 | 4000.40 | 8676.00 | 13957.64 | 17252.34 |
| | | PBQ8(SRI) | 3990.02 | 3990.02 | 4000.00 | 8578.23 | 13794.58 | 16961.24 |
| | | SAP2000 | 4000.00 | 4000.00 | 4000.00 | 8619.60 | 13292.31 | 16380.24 |

**Table 2.** The first six frequency parameters of plates for various values of subgrade reaction modulus, aspect ratio and thickness/span ratio.

| $k$ (kN/m$^3$) | $l_y/l_x$ | $h/l_x$ | Frequency Parameters | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | $l_1$ | $l_2$ | $l_3$ | $l_4$ | $l_5$ | $l_6$ |
| 500 | 1.0 | 0.05 | 399.00 | 399.00 | 400.00 | 4949.59 | 10231.94 | 13396.65 |
| | | 0.10 | 198.02 | 198.02 | 200.00 | 17437.39 | 37666.67 | 49498.74 |
| | | 0.20 | 96.15 | 96.15 | 100.00 | 58636.52 | 136957.06 | 164973.65 |
| | 1.5 | 0.05 | 399.00 | 399.56 | 400.00 | 2436.71 | 2613.35 | 10882.97 |
| | | 0.10 | 198.02 | 199.11 | 200.00 | 7942.38 | 8851.53 | 39138.63 |
| | | 0.20 | 96.15 | 98.25 | 100.00 | 27278.64 | 31856.42 | 127158.72 |
| | 2.0 | 0.05 | 399.00 | 399.75 | 400.00 | 1103.32 | 1531.07 | 5747.27 |
| | | 0.10 | 198.02 | 199.50 | 200.00 | 2977.62 | 4521.19 | 20362.94 |
| | | 0.20 | 96.15 | 99.00 | 100.00 | 10663.40 | 15473.41 | 68531.00 |
| 5000 | 1.0 | 0.05 | 3990.02 | 3990.02 | 4000.00 | 8578.23 | 13794.58 | 16961.24 |
| | | 0.10 | 1980.20 | 1980.20 | 2000.00 | 19205.93 | 39396.66 | 51233.02 |
| | | 0.20 | 961.53 | 961.53 | 1000.00 | 59482.04 | 127744.03 | 165770.71 |
| | 1.5 | 0.05 | 3990.02 | 3995.56 | 4000.00 | 6024.68 | 6196.79 | 14460.22 |
| | | 0.10 | 1980.19 | 1991.15 | 2000.00 | 9718.92 | 10619.45 | 40895.87 |
| | | 0.20 | 961.53 | 982.53 | 1000.00 | 28136.04 | 32699.08 | 127989.70 |
| | 2.0 | 0.05 | 3990.02 | 3997.50 | 4000.00 | 4694.02 | 5120.43 | 9326.86 |
| | | 0.10 | 1980.20 | 1995.01 | 2000.00 | 4759.35 | 6300.29 | 22131.76 |
| | | 0.20 | 961.53 | 990.10 | 1000.00 | 11529.30 | 16334.51 | 69377.88 |
| 50000 | 1.0 | 0.05 | 39900.05 | 39900.05 | 40000.00 | 44414.56 | 49420.66 | 52606.77 |
| | | 0.10 | 19801.71 | 19801.71 | 20000.00 | 36891.06 | 56695.87 | 68575.33 |
| | | 0.20 | 9614.85 | 9614.85 | 10000.00 | 67936.66 | 135612.65 | 173740.39 |
| | 1.5 | 0.05 | 39900.05 | 39995.42 | 40000.00 | 41904.25 | 42030.80 | 50232.52 |
| | | 0.10 | 19801.71 | 19911.28 | 20000.00 | 27484.05 | 28298.20 | 58467.89 |
| | | 0.20 | 9614.85 | 9824.98 | 10000.00 | 36709.57 | 41127.01 | 136298.93 |
| | 2.0 | 0.05 | 39900.05 | 39974.83 | 40000.00 | 40600.62 | 41013.90 | 45122.41 |
| | | 0.10 | 19801.71 | 19949.92 | 20000.00 | 22576.27 | 24091.16 | 39819.77 |
| | | 0.20 | 9614.85 | 9900.71 | 10000.00 | 20187.69 | 24945.02 | 77846.19 |

Fig. 2 shows the variation of first six frequency parameters with the aspect ratio for $k$=5·10$^6$ N/m$^3$. As seen from figures, aspect ratio don't affect the frequency parameters of the plate for the first three frequency parameter but the frequency parameters of the plate decrease as the aspect ratio increase after 3$^{rd}$ frequency parameters. This decrease is greater for the larger thickness/span ratio. While the curves are concave for 4$^{th}$ and 5$^{th}$ frequency parameters, the curves become convex for 6$^{th}$ frequency parameters. Namely; while the decrease in the 4$^{th}$ and 5$^{th}$ frequency parameters of the plate with increasing aspect ratio decrease for larger values of aspect ratio, the opposite situation arises in the 6$^{th}$ frequency parameters of the plate

Fig. 3 shows the variation of first six frequency parameters with subgrade reaction modulus for $l_y/l_x$=1.0. Frequency parameters of the plate increase as the subgrade reaction modulus of the subsoil increases. The increase in the frequency parameters of the plate with the increasing subgrade reaction modulus becomes greater for larger value of subgrade reaction modulus of subsoil. First three frequency parameters of the plate are similar to each other but after 3$^{rd}$ frequency parameters, the increase in the frequency parameters with increasing subgrade reaction modulus of the subsoil decrease.

The frequency parameters of the plate decrease with increasing thickness/plate ratio for first three frequency parameters, but the frequency parameters increase with increasing thickness/span ratio after 3$^{rd}$ frequency parameters. It is noted that some exceptional circumstances arise for $k$=5·10$^7$ N/m$^3$.

In this study, mode shapes of the plate are also obtained for all parameters considered but only mode shapes corresponding to six lowest frequency parameters for $k$=5·10$^6$ N/m$^3$, $h/l_x$=0.05 and $l_y/l_x$=2.0 are presented since the presentation of all mode shapes would take up excessive space. These mode shapes are given in Fig. 4.

## 4. Conclusions

A 4-noded (PBQ4) and an 8-noded (PBQ8) Mindlin plate elements are adopted for the analysis of plates on elastic foundation using Winkler model, and the effects of subgrade reaction modulus of the subsoil, aspect ratio and thickness/span ratio on the frequency parameter of the plate are investigated. This analysis is carried out by using Matlab for Windows 5.3 for the solution of the generalized eigenvalue problem including stiffness and mass matrices that are evaluated by a computer program coded for the purpose using finite element method. Following conclusions can be drawn from the results obtained in the study.

- Results show that PBQ8 element can be used effectively for the free vibration analysis of the plate on Winkler-type elastic foundation.
- Aspect ratio don't affect the frequency parameters of the plate for the first three frequency parameter but the frequency parameters of the plate decrease with increasing the aspect ratio after 3$^{rd}$ frequency parameters.
- The frequency parameters of the plate increase with increasing subgrade reaction modulus of the subsoil.
- The frequency parameters of the plate decrease with increasing thickness/plate ratio for first three frequency parameters but in general, the frequency parameters increase with increasing thickness/span ratio after 3$^{rd}$ frequency parameters.

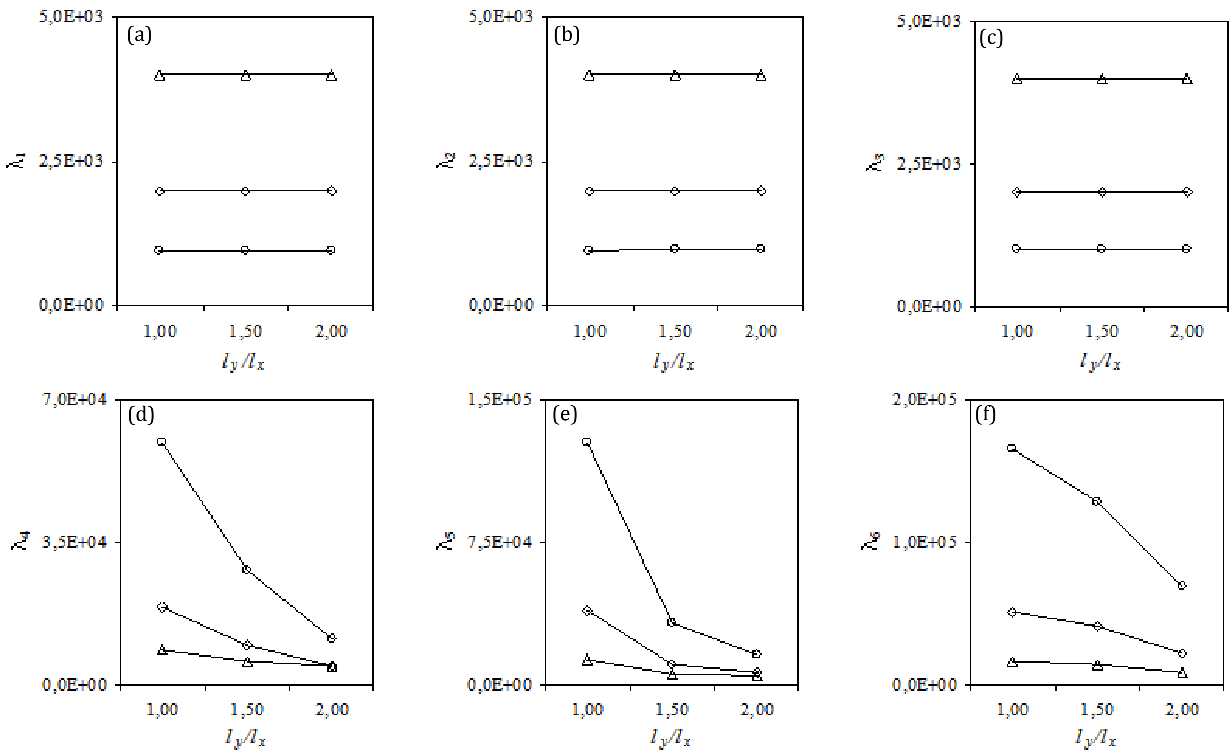


**Fig. 2.** The variation of the first six frequency parameters of the plate with various values of aspect ratio for $k$=5000 kN/m$^3$ (–Δ–, $h/l_x$ =0.05; –◊–, $h/l_x$ =0.10; –o–, $h/l_x$ =0.20).

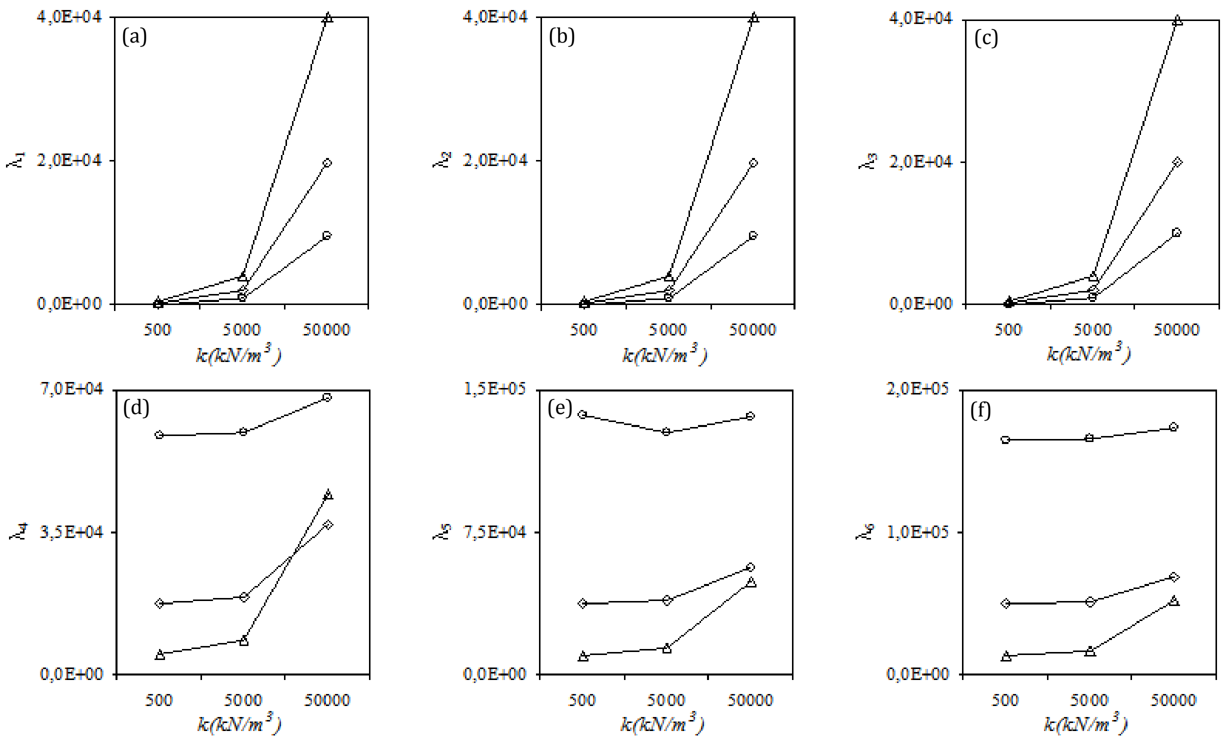


**Fig. 3.** The variation of the first six frequency parameters of the plate with various values of subgrade reaction modulus for aspect ratio=1.0 (–Δ–, $h/l_x$ =0.05; –◊–, $h/l_x$ =0.10; –o–, $h/l_x$ =0.20).

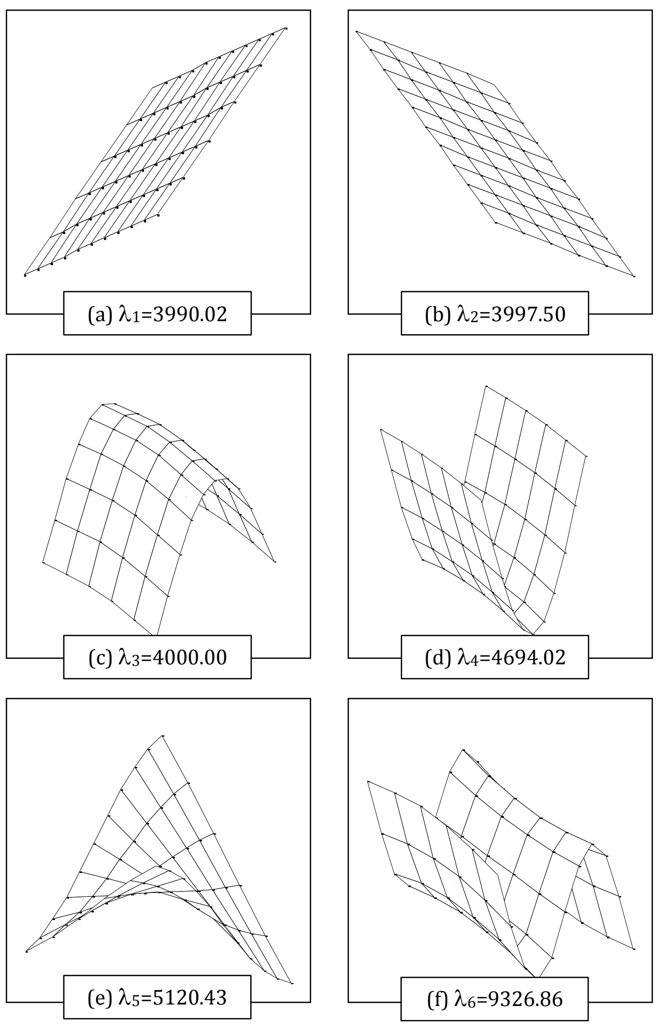


**Fig. 4.** First six mode shape for $k$=5000 kN/m$^3$, $l_y/l_x$=2.0 and $h/l_x$=0.05.

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