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APPLIED HYBRID METHODS TO SOLVING INITIAL-VALUE PROBLEM TO INTEGRO-DIFFERENTIAL EQUATIONS OF VOLTERRA TYPE

In the chronology of scientific papers is noticeable that scientists study integro-differential equations involved later than the study of differential and integral equations. Consequently, the differential equations are fundamental investigated, than integral and integro-differential equations. Therefore, many experts are replace the solving of integral and integro-differential equations to solving of differential equations. Extending this idea, constructed here a general hybrid method for solving integro-differential equations, and investigates the equivalence of the above-mentioned equations. Constructed here the concrete stable hybrid method with high accuracy by using mesh points in a minimal amount, but rather $k = 1$ *(by using one mesh point) constructed one step method of the degree* $p = 6$.

Keywords: nonlinear integro-differential equations, hybrid method, ordinary differential equations, multistep methods, relation between order and degree for hybrid methods.

Consider to solving of the following Volterra integro-differential equation of first order:

$$y' = f(x, y) + \lambda \int_{x_0}^{x} K(x, s, y(s))ds,$$
$$x_0 \le s \le x \le X. \qquad (1)$$

Suppose that equation (1) has a unique solution defined on the segment $[x_0, X]$ and satisfying the following initial condition:

$$y(x_0) = y_0. \qquad (2)$$

To determine the numerical solution of the problem (1)–(2) assume that the continuous on totality of arguments, functions $f(x, y)$ and $K(x, s, y)$ defined in the domains

$$G = \{x_0 \le x \le X, |y + y_0| \le a\}$$

and

$$\bar{G} = \{x_0 \le s \le x \le X, |y - y_0| \le a\}$$

respectively, and also have continuous partial derivatives to up some order $p+1$, inclusively. The segment $[x_0, X]$ with a constant step-size h > 0 is divided into N equal parts and mesh points define in the form:

$$x_i = x_0 + ih \ (i = 0, 1, 2, ..., N).$$

To calculate the approximate values of the solution problem (1)–(2) used certain formulas for, which are denoted by y_i an approximate, but through $y(x_i)$ the exact value of the solution of problem (1)–(2) at the mesh points

$$x_i = x_0 + ih \ (i = 0, 1, 2, ..., N).$$

Beginning with V.Volterra' s work (see [1]), published in 1887 to present time, scientists engage for the investigation of approximate solutions of problem (1)–(2), constructed methods for the solving of equation (1) (see, for example [2]–[7]). But construct an effective method satisfying certain requirements is one of the based questions of modern computational mathematics. Therefore, scientists are often turning to the construction of numerical methods for the solving of problem (1)–(2), which has some additional properties. One of such methods is the hybrid methods, which applying to solve the problem (1)–(2) offer by Makroglou and developed in the works [8]–[9]. Here be in progress these researches constructed of stable hybrid methods with high accuracy and also constructed the specific methods with a certain accuracy, which are illustrated on the model problems.

In the case $\lambda = 0$ the equation (1) is converted to a differential equation to the study, which engaged many well-known scientists: N.Tusi, I.Newton, A.C.Clairaut, G.W.Leibniz, L.Euler, J.L.Lagrange, A.L.Cauchy, J.C.Adams, C.Runge, W.Kutta, etc. For the investigation of numerical solution of ordinary differential equations they was construct numerous methods with different properties. Therefore, for solving integral and integro-differential equations are often used numerical methods of differential equations. This approach is explained with the present of ordinary differential equation by the integral equation of the next from:

$$y(x) = y(x_0) + \int_{x_0}^{x} f(s, y(s))ds,$$
$$x_0 \le x \le X, \qquad (3)$$

which is obtained from the differential equation by integrating on the segment $[x_0, x]$. If equation (3) rewrite in a more general form:

$$y(x) = g(x) + \int_{x_0}^{x} K(x, s, y(s))ds, \qquad (4)$$

then we can receive equation of type (1) from it by differentiation. Given these connections between the equations (1), (3) and (4), here to consider application of the following hybrid method

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i} + h\sum_{i=0}^{k} \gamma_i y'_{n+i+\nu_i},$$
$$(|\nu_i| < 1;\ i = 0, 1, ..., k) \qquad (5)$$

to the solving of the problem (1)–(2). Note that the method (5) is applied to the solving of initial value problem for ordinary differential equations of first and second order (see [10]–[12]), and also to the solving of equation (4). The hybrid method used by Makroglou to solve the problem (1)–(2), may be received from the method (5) in particularly for $\gamma_i = 0$ $(i < m)$, $\gamma_m \ne 0$, $\gamma_j = 0$ $(m < j \le k)$.

1. Application of hybrid methods to solving Volterra integro-differential equations.

As is known, one of the first numerical methods for solving equation (1) is constructed and investigated by V.Volterra. For this purpose, Volterra used the method of quadratures, is consisted in a replacement an integral by some integral sum, which in one variant has the following form:

$$\int_{x_0}^{x_n} K(x_n, s, y(s))ds =$$
$$= h\sum_{i=0}^{n} a_i K(x_n, x_i, y_i) + R_n, \qquad (6)$$

here the quantities a_i (i = 0, 1, 2, ..., n) are the real numbers, but R_n – is the remainder term. If to the solving of the problem (1)–(2) apply k-step method with constant coefficients and taking into account the method of quadratures defined by the formula (6), then we have:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} +$$
$$+ h^2\sum_{i=0}^{k}\sum_{j=0}^{n+i}\bar{\beta}_i^{(j)} a_j K(x_{n+i}, x_j, y_j). \qquad (7)$$

Here, $\bar{\beta}_i^{(j)}$ (i, j = 0, 1, 2, ..., k) are the coefficients make up coefficients of the quadrature formula, and coefficients of k-step method, but α_i, β_i (i = 0, 1, ..., k) the coefficients of the k-step method. It is easy to remark that the while crossing from the current mesh point to the next amount of computational work is increased, since the second sum in method (7) depends on the variable n. For the relieve from these lack in [9] for solving of the equation (4) proposed the following method:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\alpha_i g_{n+i} +$$
$$+ h\sum_{j=0}^{k}\sum_{i=0}^{k}\beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}). \qquad (8)$$

Note that depending on the properties of the integral kernel some of the coefficients $\beta_i^{(j)}$ (i, j = 0, 1, 2, ..., k), will be equal to zero. If suppose that, the kernel of the integral is defined in the ε-expansion of domains $\bar{G}$ then the method (8) can be applied to the solving of equation (4). Otherwise, the coefficients $\beta_i^{(j)}$ (i, j = 0, 1, 2, ..., k) must satisfy the condition $\beta_i^{(j)} = 0$ ($i > j$). Note that for using of the method (8) must be known quantities y_0, y_1, ..., y_{k-1}. By the method (8) one can calculate the values of variables y_{n+k}. It is known that usually for solving problem (1)–(2) uses stable methods, but among the stable multistep methods the implicit methods are the most popular. However, when using them are meet with finding solutions of nonlinear equations, which is not always succeed. Usually in such cases, experts use iterative methods, or methods of predictior-correctior. It is easy to show that the predictior-correctior methods in particular, may also receive from the iterative methods. But to relieve of these shortcomings of implicit methods here is proposed to use the explicit hybrid methods. Therefore, we consider hybrid methods, and their applications to the solving of the integro-differential equations. Hybrid methods can be constructed by different ways. In the work [9] consider the following hybrid method with constant coefficients:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y'_{n+i+\nu_i},$$
$$(|\nu_i| < 1;\ i = 0,1,2,...,k). \qquad (9)$$

For the ν_i = 0 (i = 0, 1, 2, ..., k) from the formula (9) follows a well-known multistep methods with constant coefficients. Here we consider the case when there is $\nu_k^2 + \nu_{k-1}^2 + ... + \nu_0^2 \neq 0$. Usually, in the concrete methods with the maximum degree the quantity ν_k satisfies the condition $-1 < \nu_k < 0$. However, in this case we obtain explicit hybrid methods. For example, from the method (9) receive the next hybrid method with the maximum degree for $k = 1$:

$$y_{n+1} = y_n + h(y'_{n+\beta} + y'_{n+1-\beta})/2,$$
$$(\beta = 1/2 - \sqrt{3}/6). \qquad (10)$$

As the remark above explicit method (10) is obtained from equation (9) for k = 1 and have order accuracy p = 4. This method is unique in a class methods which has the degree of accuracy p = 4. For the construction of hybrid implicit methods, consider the following generalization of the method (9):

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y'_{n+i} + h\sum_{i=0}^{k}\gamma_i y'_{n+i+\nu_i},$$
$$(|\nu_i| < 1;\ i = 0,1,2,...,k). \qquad (11)$$

Obviously, if $\beta_k \neq 0$, and $-1 < \nu_k < 0$ then the method (11) is implicit. Appling implicit methods to solve scientific and engineering problems has some difficulties. Therefore, usually for the construction of concrete methods considered the case $\beta_k = 0$. Now consider the applications of the method (11) to the solving of problem (1)–(2). To this end, consider the following difference:

$$y(x_{n+1}) - y(x_n) = g(x_{n+1}) - g(x_n) +$$
$$+ h\int_{x_0}^{x_n} K'_x(\xi_n, s, y(s))ds +$$
$$+ \int_{x_n}^{x_{n+1}} K(x_{n+1}, s, y(s))ds, \qquad (12)$$

here $x_n < \xi_n < x_{n+1}$.

It is obvious that from the equality (4), one can be write the following:

$$y'(x) = g'(x) + K(x, x, y(x)) +$$
$$+ \int_{x_0}^{x} K'_x(x, s, y(s))ds,\ \ x_0 \le x \le X. \qquad (13)$$

Here we replace x by the ξ_n i.e $x = \xi_n$. Then we have:

$$h\int_{x_0}^{x_n} K'_x(\xi_n, s, y(s))ds =$$
$$= h(y'(\xi_n) - g'(\xi_n)) -$$
$$-hK(\xi_n, \xi_n, y(\xi_n)) -$$
$$-h\int_{x_n}^{\xi_n} K'_x(\xi_n, s, y(s))ds.$$

If taking into account obtained in (12), then receive the following:

$$y(x_{n+1}) - y(x_n) =$$
$$= g(x_{n+1}) - g(x_n) +$$
$$+ h(y'(\xi_n) - g'(\xi_n)) -$$
$$-hK(\xi_n, \xi_n, y(\xi_n)) +$$
$$+ \int_{x_n}^{\xi_n} K(x_n, s, y(s))ds +$$
$$+ \int_{\xi_n}^{x_{n+1}} K(x_{n+1}, s, y(s))ds. \qquad (14)$$

As is well known to the calculation of the integral one can apply different quadrature formulas as a method of the rectangle formulas and a trapezoid method. However, the method of quadratures can be defined as a linear combination of these methods. Then, generalizing the proposed scheme, we can write:

$$\int_{x_0}^{x_n} f(x)dx \approx h\sum_{i=0}^{n}\beta_i f_i + h\sum_{i=0}^{n}\gamma_i f_{i+\nu_i},$$
$$(|\nu_i| < 1;\ i = 0,1,2,...,n). \qquad (15)$$

If we take $\nu_i = 1/2$ (i = 0, 1, 2, ..., n) then after choosing the suitable coefficients from (15) we obtain a linear combination of generalized methods of the rectangle formulas and trapezoids (see, for example, [13, p. 184-186]. Similar schemes for the solving of ordinary differential equations are used by many authors (see, for example [14], [15]). Replacing the derivatives of functions by its values at different mesh points, applying interpolation polynomials Lagrange to calculation quantity $K(\xi_n, \xi_n, y(\xi_n))$. By using them and formula (15) in equality (14) one obtains the following formula:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\alpha_i g_{n+i} +$$
$$+h\sum_{j=0}^{k}\sum_{i=0}^{k}\beta_i^{(j)}K(x_{n+j}, x_{n+i}, y_{n+i}) +$$
$$+h\sum_{j=0}^{k}\sum_{i=0}^{k}\gamma_i^{(j)}K(x_{n+j+m_j}, x_{n+i+\nu_i}, y_{n+i+\nu_i}),$$
$$(\left|m_j\right| < 1;\ j = 0,1,2,...,k). \qquad (16)$$

The coefficients α_i, $\beta_i^{(j)}$, $\gamma_i^{(j)}$ $(i, j = 0, 1, 2, ..., k)$ are some real numbers, but $\alpha_k \neq 0$. Consider the determination of the coefficients. For this purpose we consider a special case and put $K(x, s, y) \equiv z(s, y)$. Then from (16) we have:

$$\sum_{i=0}^{k}\alpha_i(y_{n+i} - g_{n+i}) =$$
$$= h\sum_{i=0}^{k}\beta_i z_{n+i} + h\sum_{i=0}^{k}\gamma_i z_{n+i+\nu_i}, \qquad (17)$$

where

$$\beta_i = \sum_{j=0}^{k}\beta_i^{(j)},$$
$$\gamma_i = \sum_{j=0}^{k}\gamma_i^{(j)} (i = 0,1,2,...,k). \qquad (18)$$

From equations (4) we have:

$$y' - g' = z(x, y).$$

As is follows from here the method (17) coincides with the method (5).

To determine the coefficients α_i, β_i, γ_i $(i, j = 0, 1, 2, ..., k)$ of the method (17) using the method of undetermined coefficients and in the result receive the following nonlinear system of algebraic equations (see, for example [10]):

$$\sum_{i=0}^{k}\alpha_i = 0;$$
$$\sum_{i=0}^{k}\left(\frac{i^l}{l!}\alpha_i - \frac{i^{l-1}}{(l-1)!}\beta_i + \frac{(i+\nu_i)^{l-1}}{(l-1)!}\gamma_i\right) = 0,$$
$$l = 1,2,..., p. \qquad (19)$$

In this system number of equations is equal $p+1$, and the number of unknowns is equal $4k+4$. The quantity k is known, so determining the values of the quantity p used the values of the quantities k. One can show that between the quantities k and p has the following relation:

$$p \leq 4k + 2. \qquad (20)$$

For the application of the method (16) to the solving of problem (1)–(2), the problem (1)–(2) rewrite in the following form:

$$y' = f(x, y) + \varphi(x), y(x_0) = y_0, \qquad (21)$$

$$\varphi(x) = \int_{x_0}^{x} K(x, s, y(s))ds. \qquad (22)$$

Then the method (16) is apply to the solving of equation (22), and to solving problem (21) we apply the method (5) and choose the coefficients so that the coefficients in these methods coincides by the taking into account conditions (18). Note that if the method (5) is converges, then its coefficients satisfies the following conditions:

A: The coefficients α_i, β_i, γ_i, ν_i $(i = 0, 1, 2, ..., k)$ are some real numbers, moreover, $\alpha_k \neq 0$.

B: Characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k}\alpha_i\lambda^i;$$
$$\sigma(\lambda) \equiv \sum_{i=0}^{k}\beta_i\lambda^i;$$
$$\gamma(\lambda) \equiv \sum_{i=0}^{k}\gamma_i\lambda^{i+\nu_i}.$$

have no common multipliers different from the constant.

C: $\sigma(1) + \gamma(1) \neq 0$ and $p \geq 0$.

Consider the construction of specific methods and put $k = 2$. Then for the determining of the coefficients we obtain the following system of nonlinear equations:

$$\beta_2 + \beta_1 + \beta_0 + \gamma_2 + \gamma_1 + \gamma_0 = 2\alpha_2 + \alpha_1;$$
$$2\beta_2 + \beta_1 + l_2\gamma_2 + l_1\gamma_1 + l_0\gamma_0 = (2^2\alpha_2 + \alpha_1)/2;$$
$$2^2\beta_2 + \beta_1 + l_2^2\gamma_2 + l_1^2\gamma_1 + l_0^1\gamma_0 = (2^3\alpha_2 + \alpha_1)/3;$$
$$2^3\beta_2 + \beta_1 + l_2^3\gamma_2 + l_1^3\gamma_1 + l_0^3\gamma_0 = (2^4\alpha_2 + \alpha_1)/4;$$
$$2^4\beta_2 + \beta_1 + l_2^4\gamma_2 + l_1^4\gamma_1 + l_0^4\gamma_0 = (2^5\alpha_2 + \alpha_1)/5;$$
$$2^5\beta_2 + \beta_1 + l_2^5\gamma_2 + l_1^5\gamma_1 + l_0^5\gamma_0 = (2^6\alpha_2 + \alpha_1)/6;$$
$$2^6\beta_2 + \beta_1 + l_2^6\gamma_2 + l_1^6\gamma_1 + l_0^6\gamma_0 = (2^7\alpha_2 + \alpha_1)/7;$$
$$2^7\beta_2 + \beta_1 + l_2^7\gamma_2 + l_1^7\gamma_1 + l_0^7\gamma_0 = (2^8\alpha_2 + \alpha_1)/8;$$
$$2^8\beta_2 + \beta_1 + l_2^8\gamma_2 + l_1^8\gamma_1 + l_0^8\gamma_0 = (2^9\alpha_2 + \alpha_1)/9. \qquad (23)$$

By solving these system, we find the values of the coefficients of the method (5), and the coefficients of the method of the type (16) determined from the system (18). Consequently, if the method of the type (16) has the maximum degree and it·s coefficients are defined by the solution of the system (18), then it will not be unique with the maximum degree, because the system (18) has the solution more than one.

If put $\alpha_2 = 1$, $\alpha_1 = 0$, $\alpha_0 = -1$ in this system, then by solving the received system of nonlinear algebraic equations, we have:

$$\beta_2 = 64/180, \beta_1 = 98/180, \beta_0 = 18/180,$$
$$\gamma_2 = 18/180, \gamma_1 = 98/180, \gamma_0 = 64/180,$$
$$l_2 = 1 + \sqrt{21}/14, l_1 = 1, l_0 = 1 - \sqrt{21}/14.$$

Hence we get the following method:

$$y_{n+2} = y_n + h(64y'_{n+2} + 98y'_{n+1} + 18y'_n)/180 +$$
$$+h(18y'_{n+l_2} + 98y'_{n+1} + 64y'_{n+\gamma_0})/180. \qquad (24)$$

Remark, that this method is symmetric (so that $\nu_0 = -\nu_2$). But there is the nun symmetric method with the degree $p = 9$.

It is clear, that for using the method (24), it is necessary to determine the values of the quantities $y_{n+\gamma0}$ and $y_{n+\gamma2}$. To illustrate the above mentioned, consider the case $k = 1$ and put $\beta_1 = \beta_2 = 0$. Then by solving the system (23), we obtain By using this solution in the formula (9), one receive the method (10), for using which can be suggested the following algorithm, if is known the value $y_{1/2}$.

Step 1. Calculate the values $y_{n+1/2\pm\alpha}$ $(\alpha = \sqrt{3}/6)$ by the following method

$$y_{n+1/2+\gamma} = y_{n+1/2} + h\Big((4\gamma^3 + 6\gamma^2)y'_{n+1} -$$
$$-(8\gamma^3 - 24\gamma)y'_{n+1/2} + (4\gamma^3 - 6\gamma^2)y'_n\Big)\Big/24,$$

for g = $\pm\sqrt{3}/6$.

Step 2. Calculate the value of the quantity y_{n+1} by the method (10).

Step 3. Calculate the values of the quantity $y_{n+3/2}$ by the following methods

$$\hat{y}_{n+3/2} = y_{n+1} + h(23y'_{n+1} - 16y'_{n+1/2} + 5y'_n)\big/24,$$
$$y_{n+3/2} = y_{n+1} + h(9\hat{y}'_{n+3/2} + 19y'_{n+1} - 5y'_{n+1} + y'_n)\big/48.$$

Note that for solving some problems by this algorithm for calculating values of the quantity $y_{i+3/2}$ can be used the following method:

$$y_{n+3/2} = y_{n+1/2} +$$
$$+h(y'_{n+3/2} + 4y'_{n+1} + y'_{n+1/2})\big/6.$$

To illustrate applying this algorithm to solving problem (1)–(2) consider the following examples:

1. $y' = 1 + y - x\exp(-x^2) -$
$$-2\int_0^x xt\exp(-y^2(t))dt,$$
$y(0) = 0$, $0 \leq x \leq 1$.

The exact solution is $y(x) = x$.

2. $y' = -x^3\big/3 + 4\exp(-y)/3 +$
$$+(4/3)\int_1^x (1/x)s^2\exp(y(s))ds,$$
$y(1) = 0$, $1 \leq x \leq 2$.

The exact solution is $y(x) = \ln x$.

3. $y' = \int_0^x \cos s ds,$

$y(0) = -1, \; 0 \le x \le 2,$

the exact solution is $y(x) = -\cos x$.

4. $y' = \int_0^x xs \cos s^2 ds,$

$y(0) = -1/4, \; 0 \le x \le 2.$

The exact solution is $y(x) = -\cos x^2/4$.

For solving these problems, we are using above mentioned algorithm. Note that the example 1 is solved in the work [2], the examples 1–3 are solved in the work [7], the example 3 is solved in the work [3], and the example 4 is solved in the work [3]. Here all the examples solved by the hybrid method (10) and the receive results are putting in table 1 and also to solving some of these problems, here used the trapezoid method and the receive results are putting in the table 2, in which we used the next notation:

Method I – Predictor-corrector method consist is Euler and Trapezoid method applying to solving system consists only of ODE.

Method II – The same predictor-corrector method applying to solving system consists only of the integral equations.

Method III – The same predictor-corrector method applying to solving system consists of ODE and integral equation.

Conclusion. By the above mentioned are shown some of the advantages of the hybrid methods. Constructed concrete hybrid methods with the high accuracy. And also, in the simple case, have constructed an algorithm for using to solving of the problems of type (1)–(2). Note that the proposed algorithm for the using of the method (10) has a simple structure, which makes easy to using and to modifying it. Naturally, each method has some advantages and shortcomings. The main advantage of hybrid methods is their high accuracy, and the main shortcomings is the using of variables with the values $y_{n+\gamma}$ in irrational points (with $\gamma = \pm 1/2 - \sqrt{3}/6$, or $\gamma = \pm 1 - \sqrt{21}/14$). To overcome these shortcomings, here are using the methods of predictor-corrector type. It is easy to understand that the proposed algorithm can be modified by using more precise methods. We here describe a scheme by which to ensure applying of the hybrid method to solving problem (1). We believe that the study of hybrid methods is one of the promising directions, for the construction most accurate numerical methods, which also confirms by the above solved problems.

References:

1. V.Volterra. Theory of functional and of integral and integro-differensial equations, (in Russian), Moscow, Nauka, 1982, 304 p.

2. P.Linz Linear Multistep methods for Volterra Integro-Differensial equations, Journal of the Association for Computing Machinery, Vol.16, No.2, April 1969, pp. 295-301.

3. Feldstein A, Sopka J.R. Numerical methods for nonlinear Volterra integro-differensial equations, SINUM 11, 826-846.

4. H.Brunner. Imlicit Runge-Kutta Methods of Optimal oreder for Volterra integro-differensial equation. Methematics of computation, Volume 42, Number 165, January 1984, pp. 95-109.

5. Makroglou A.A. Block – by-block method for the numerical solution of Volterra delay integro-differensial equations, Computing 3, 1983, 30, №1, p. 49-62.

6. Bulatov M.B. Chistakov E.B. Chislennoe resheniye inteqro-differensialnix sisitem s virojdennoy matrisey pered proizvodnoy mnoqoshaqovimi metodami. Dif. Equations, 2006, 42, №9, pp. 1218-1255.

7. A.Makroglou. Hybrid methods in the numerical solution of Volterra integro-differensial equations. Journal of Numerical Analysis 2, 1982, pp. 21-35.

8. G.Y.Mehdiyeva, Ibrahimov V.R., M.N.Imanova. Application of the forward jumping Method to the solving of Volterra integral equation. Conference in Numerical Analysis, Chania, Crete, Greece. 2010, p. 106-111.

Table 1

Number of the examples	Value of the variable x	Maximal error for the method from [2]	Maximal error for the first method from [7]	Maximal error for the second method from [7]	Maximal error for the method from [3]	Maximal error for the hybrid method (10)
I h = 0,05	1,0	0,12 E-05	0,16 E-04	0,77 E-06		0,11 E-05
II h = 1/32	3,0				0,18 E-06	0,21 E-04
III h = 1/8	2,0		0,86 E-05	0,16 E-05	0,2 E-05	0,52 E-06
IV h = 1/8	2,0		0,43 E-03	0,58 E-04	0,129 E-03	0,2 E-04

Table 2

		Method I	Method II	Method III
I h = 0,05	1,0	0,28 E-03	0,24 E-03	0,52 E-04
II h = 1/32	2,0	0,12 E-03	0,33 E-03	0,70 E-04
III h = 0,05	1,0	0,19 E-03	0,19 E-03	0,19 E-03
IV h = 1/32	2,0	0,68 E-03	0,58 E-03	0,58 E-03

9. G.Y.Mehdiyeva, Ibrahimov V.R., M.N.Imanova On one application of hybrid methods for solving Volterre Integro-differential equations. World Academy of Science, engineering and Technology, Dubai, 2012, 1197-1201.

10. G.Y.Mehdiyeva, Ibrahimov V.R., M.N.Imanova, R.R.Mirzayev. On a way for constructing hybrid methods. Space and time – coordinate system of human development. Materials digest of VIIIth International Research and Practice Conference (Kiev, London, August 25 – September 1, 2011), p. 91-95.

11. J.O.Ehigie, S.A.Okunuga, A.B.Sofoluwe, M.A.Akanbi. On generalized 2-step continuous linear multistep method of hybrid type for the integration of second order ordinary differential equations, Archives of Applied Research, 2010, 2(6), pp.362-372.

12. G.Y.Mehdiyeva, Ibrahimov V.R., Imanova M.N. Application of multi step methods to the solving second order ordinary differential equation. The Third international Conference "Problems of Cybernetics and Informatics", Baku, Azerbaijan, 2010, p. 255-258.

13. I.S.Berezin, N.P.Jidkov. Methodi vichisleniy (russian). T.1, M. «Nauka», 1966, 632 c.

14. Hamming R.W. Numerical methods for scientists and engineers. McGraw-Hill, New York, 1962.

15. Ibrahimov V.R. On some properties of the Richardson extrapolation. Dif.eq. № 12, 1990, 2170-2173.