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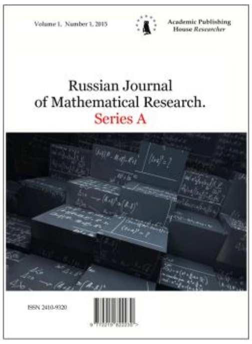

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Construction of Coefficient Inequality for a New Subclass of Class of Starlike Analytic Functions

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Abstract

In this paper, we will discuss a newly constructed subclass of analytic starlike functions by which we will be obtaining sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to this subclasses.

Keywords: Univalent functions; Starlike functions; Close to convex functions and bounded functions.

MATHEMATICS SUBJECT CLASSIFICATION: 30C50

1. **Introduction:** Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([7], [8]) proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$**,** Fekete and Szegö[9] used Löwner's method to prove the following well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if\ \mu \leq 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1. \end{cases} \quad (1.2)$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ (See Chhichra[1], Babalola[6]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and satisfying the condition

$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}.$ (1.3)

We denote by $\mathcal{K}$, the class of univalent convex functions $h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$ and satisfying the condition

$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}.$ (1.4)

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}.$ (1.5)

The class of close to convex functions is denoted by $\mathbb{C}$ and was introduced by Kaplan [3] and it was shown by him that all close to convex functions are univalent.

$S^*(A,B) = \left\{f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\}$ (1.6)

$\mathcal{K}(A,B) = \left\{f(z) \in \mathcal{A}; \frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz}, -1 \leq B < A \leq 1, z \in \mathbb{E}\right\}$ (1.7)

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}(A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclass as $\left\{\mathbf{f(z)} \in \mathcal{A}; \frac{1}{2}\left(\frac{\mathbf{zf'(z)}}{\mathbf{f(z)}} + \left(\frac{\mathbf{zf'(z)}}{\mathbf{f(z)}}\right)^{\frac{1}{\alpha}}\right) = \frac{\mathbf{1+w(z)}}{\mathbf{1-w(z)}}; \mathbf{z} \in \mathbb{E}\right\}$ and we will denote this class as $f(z) \in \Sigma S^*[\alpha]$.

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z)); z \in \mathbb{E}$ and we write $f(z) \prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z) = \sum_{n=1}^{\infty} d_n z^n, w(0) = 0, |w(z)| < 1.$ (1.8)

It is known that $|d_1| \leq 1, |d_2| \leq 1 - |d_1|^2.$ (1.9)

2. **PRELIMINARY LEMMAS:** For $0 < c < 1$, we write $w(z) = \left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+Aw(z)}{1+Bw(z)} = 1 + (A-B)c_1 z + (A-B)(c_2 - Bc_1^2)z^2 + - - -$$
(2.1)

3. MAIN RESULTS

THEOREM 3.1: Let $f(z) \in f(z) \in \Sigma S^*[\alpha]$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2\alpha}{(\alpha+1)^3}[5\alpha^2 + 10\alpha - 3 - 8\mu\alpha(\alpha+1)]; if\ \mu \leq \frac{4\alpha^2+8\alpha-4}{8\alpha(\alpha+1)} & (3.1) \\ \frac{2\alpha}{\alpha+1} \quad ; \ if\ \frac{4\alpha^2+8\alpha-4}{8\alpha(\alpha+1)} \leq \mu \leq \frac{6\alpha^2+12\alpha-2}{8\alpha(\alpha+1)} & (3.2) \\ \frac{2\alpha}{(\alpha+1)^3}[8\mu\alpha(\alpha+1) - 65 - 10\alpha + 3] \ ; if\ \mu \geq \frac{6\alpha^2+12\alpha-2}{8\alpha(\alpha+1)} & (3.3) \end{cases}$$

The results are sharp.

Proof: By definition of $f(z) \in f(z) \in \Sigma S^*[\alpha]$, we have

$$\frac{1}{2}\left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}}\right) = \frac{1+w(z)}{1-w(z)}; w(z) \in \mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we get

$$1 + a_2 z\left(\frac{\alpha+1}{2\alpha}\right) + \frac{z^2}{2}\left[(2a_3 - a_2{}^2)\left(\frac{\alpha+1}{\alpha}\right) + \left(\frac{1-\alpha}{2\alpha^2}\right)a_2^2\right] + --- = (1 + 2c_1 z + 2(c_1^2 + c_2)z^2 + z^3(2c_3 + 4c_1c_{2+} c_1{}^3) + ---). \quad (3.5)$$

Identifying terms in (3.5), we get

$$a_2 = \frac{4c_1}{\alpha+1} \quad (3.6)$$

$$a_3 = \left(\frac{2\alpha}{\alpha+1}\right)\left[c_1^2 + c_2 + \frac{4c_1^2}{(\alpha+1)^2}[\alpha^2 + 2\alpha - 1]\right] \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_3 - \mu a_2^2 = c_1^2\left[\frac{2\alpha}{\alpha+1} + \frac{8\alpha(\alpha^2+2\alpha-1)}{(\alpha+1)^3} - \frac{16\mu\alpha^2}{(\alpha+1)^2}\right] + c_2\left[\frac{2\alpha}{\alpha+1}\right] \quad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3 - \mu a_2^2| \leq \left|\frac{2\alpha}{\alpha+1} + \frac{8\alpha(\alpha^2+2\alpha-1)}{(\alpha+1)^3} - \frac{16\mu\alpha^2}{(\alpha+1)^2}\right||c_1^2| + |c_2|\left|\frac{2\alpha}{\alpha+1}\right| \quad (3.9)$$

Using (1.11) in (3.9), we get

$$|a_3 - \mu a_2^2| \leq \left[\frac{2\alpha}{(\alpha+1)^3}\left[\left|\left(5\alpha^2 + 10\alpha - 3\right)\right| - 8\mu\alpha(\alpha+1)\right] - \frac{2\alpha}{\alpha+1}\right]|c_1|^2 + \frac{2\alpha}{\alpha+1} \quad (3.10)$$

<u>**Case I**</u>: $\mu \geq \frac{5\alpha^2+10\alpha-3}{8\alpha(\alpha+1)}$. (3.10) can be rewritten as

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2\alpha}{(\alpha+1)^3}[8\mu\alpha(\alpha+1) - (6\alpha^2 + 12\alpha - 2)]|c_1|^2 + \frac{2\alpha}{\alpha+1} \quad (3.11)$$

<u>**Subcase I (a)**</u>: $\mu \geq \frac{6\alpha^2+12\alpha-2}{8\alpha(\alpha+1)}$. Using (1.11), (3.11) becomes

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{(\alpha+1)^3}\left[8\mu\alpha(\alpha+1) - 5\alpha^2 - 10\alpha + 3\right]. \quad (3.12)$$

<u>**Subcase I (b)**</u>: $\mu < \frac{6\alpha^2+12\alpha-2}{8\alpha(\alpha+1)}$. We obtain from (3.11)

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{\alpha+1} \quad (3.13)$$

<u>**Case II**</u>: $\mu < \frac{5\alpha^2+10\alpha-3}{8\alpha(\alpha+1)}$

Preceding as in case I, we get

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{\alpha+1} + \frac{2\alpha}{(\alpha+1)^3}[4\alpha^2 + 8\alpha - 4 - 8\mu\alpha(\alpha+1)]|c_1|^2. \quad (3.14)$$

<u>**Subcase II (a)**</u>: $\mu \leq \frac{4\alpha^2+8\alpha-4}{8\alpha(\alpha+1)}$

(3.14) takes the form (3.15)

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{(\alpha+1)^3}\left[5\alpha^2 + 10\alpha - 3 - 8\mu\alpha(\alpha+1)\right] \quad (3.16)$$

<u>**Subcase II (b)**</u>: $\mu > \frac{4\alpha^2+8\alpha-4}{8\alpha(\alpha+1)}$

Preceding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \frac{2\alpha}{\alpha+1} \quad (3.17)$$

<u>Combining (3.12), (3.16) and (3.17), the theorem is proved</u>.

Extremal function for (3.1) and (3.3) is defined by

$$f_1(z) = (1 + az)^b$$

Where $a = \frac{\{(2\alpha+\beta-3\alpha\beta)^2-(1-\alpha)\beta(\beta-3)+4\alpha(1-\beta)(\beta+2)\}a_2^2-4(3\alpha+\beta-4\alpha\beta)a_3}{(2\alpha+\beta-3\alpha\beta)a_2}$

And $b = \frac{(2\alpha+\beta-3\alpha\beta)^2a_2^2}{\{(2\alpha+\beta-3\alpha\beta)^2-(1-\alpha)\beta(\beta-3)+4\alpha(1-\beta)(\beta+2)\}a_2^2-4(3\alpha+\beta-4\alpha\beta)a_3}$

Extremal function for (3.2) is defined by $f_2(z) = z(1 + Bz^2)^{\frac{A-B}{2B}}$.

Corollary 3.2: Putting $\alpha = 1, \beta = 0$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu, if \mu \leq 1; \\ \frac{1}{3} if 1 \leq \mu \leq \frac{4}{3}; \\ \mu - 1, if \mu \geq \frac{4}{3} \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.

Corollary 3.3: Putting $A = 1, B = -1$ *and* $\alpha = 0, \beta = 1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, if \mu \leq \frac{1}{2}; \\ 1 if \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, if \mu \geq 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent starlike functions.

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