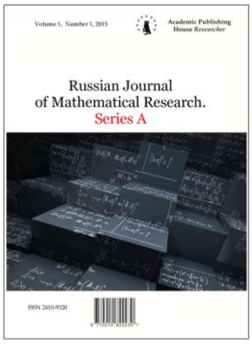

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The Distribution of the Distance Between Two Random Points in a Convex Set

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Abstract

A formula for calculation of the density function $f_\rho(x)$ of the distance between two independent points randomly and uniformly chosen in a bounded convex domain **D** is given. The obtained formula permits to find an explicit form of the density function $f_\rho(x)$ for the domains **D** with known chord length distributions. In particular, an application of the formula gives explicit expressions for $f_\rho(x)$ in the cases of a disc, a regular triangle, a rectangle and a regular pentagon.

Keywords: Chord length distribution function; kinematic measure; a set of segments; bounded convex domain.

Mathematics Subject Classification 2010: 60D05, 52A22, 53C65.

Introduction

Let **D** be a bounded, convex domain in the Euclidean plane, with the area $||\mathbf{D}||$ and the perimeter $|\partial \mathbf{D}|$. Let P_1 and P_2 be two points chosen at random, independently and with uniform distribution in **D**. We are going to find the probability that the distance $\rho(P_1, P_2)$ between P_1 and P_2 is equal to or less than x, that is we would like to find the distribution function $F_\rho(x)$ of $\rho(P_1, P_2)$. By definition,

$$F_\rho(x) = P\big((P_1, P_2) \in \mathbf{D}: \rho(P_1, P_2) \le x\big) = \frac{1}{||\mathbf{D}||^2} \iint_{\{(P_1,P_2):\ \rho(P_1,P_2)\le x\}} dP_1\, dP_2, \qquad (1.1)$$

where dP_i, $i = 1, 2$ is the Lebesgue measure in the plane $\boldsymbol{R}^2$.

From the expression of the area element in polar coordinates we have

$$dP_1\, dP_2 = r\, dP_1\, dr\ d\varphi, \qquad (1.2)$$

where φ is the angle between the line through the points P_1, P_2 and the reference direction in the plane. If we leave r fixed, then $dP_1\, d\varphi$ is the kinematic density for the segment $P_1 P_2$ of length r. Using (1.2) we can rewrite (1.1) in the form:

$$F_\rho(x) = \frac{1}{||\mathbf{D}||^2}\int_0^x r\,K(\mathbf{D}, r)dr, \qquad (1.3)$$

where $K(\boldsymbol{D}, r)$ is the kinematic measure of all oriented segments of length r that lie inside **D**.

Therefore, using (1.3) we obtain a relationship between the density function $f_\rho(x)$ of $\rho(P_1, P_2)$ and the kinematic measure $K(\boldsymbol{D}, r)$:

$$f_\rho(x) = \frac{xK(\boldsymbol{D}, x)}{||\boldsymbol{D}||^2}. \qquad (1.4)$$

Note that we can calculate the kinematic measure of all unoriented segments that lie inside **D** and then the result multiplied by 2.

The main formula for kinematic measure. Let S be a segment of length r. As it is well-known [1]-[3], the solution of the problem on finding the kinematic measure $K(\boldsymbol{D}, r)$ of segments with constant length r, contained in **D**, is not simple and essentially depends on the form of **D**. Explicit expressions for $K(\boldsymbol{D}, r)$ are known only in two cases [1], [2]. In the first case, **D** is the disc $\boldsymbol{C}_d$, and obviously $K(\boldsymbol{C}_d, r) = 0$ when $d \le r$, and

$$K(\boldsymbol{C}_d, r) = \frac{\pi d^2}{2}\left[\frac{\pi}{2} - \arcsin\frac{r}{d} - \frac{r}{d}\sqrt{1 - \frac{r^2}{d^2}}\right] \text{ when } d \ge r. \qquad (2.1)$$

In the second case, **D** is a rectangle $\boldsymbol{R}_{a,b}$ with sides a, b, and $K(\boldsymbol{R}_{a,b}, r) = 0$, if r is not less than the length $\sqrt{a^2 + b^2}$ of the diagonal of $\boldsymbol{R}_{a,b}$ [1], [2], and if $r \le \min(a, b)$, then

$$K(\boldsymbol{R}_{a,b}, r) = \pi ab - 2r(a + b) + r^2. \qquad (2.2)$$

Note that these formulae are for unoriented segments. Therefore, for oriented segments we have to multiply the formulae by 2.

In the paper [4], a formula for the kinematic measure $K(\boldsymbol{D}, r)$ of sets of segments with constant length r entirely contained in **D** is obtained. The obtained formula in [4] permits to calculate the mentioned kinematic measure $K(\boldsymbol{D}, r)$ by means of the chord length distribution function of **D**. The formula permits to find an explicit expression for the kinematic measure $K(\boldsymbol{D}, r)$ of domains **D** with known chord length distributions. In particular, using the obtained formula some explicit expressions for $K(\boldsymbol{D}, r)$ for a disc, a regular triangle, a rectangle and a regular pentagon are derived.

Let $S_1 = MS$ be the image of segment S under an Euclidean motion $M \in \boldsymbol{M}$, where $\boldsymbol{M}$ is the group of all Euclidean motions in the plane. For the locally compact group $\boldsymbol{M}$, there is a locally finite Haar measure, i.e. a locally finite, non identically zero Borel measure invariant both from the left and from the right. Segment S_1 can be defined by means of the two coordinates (g, t), where $g \in \boldsymbol{G}$ ($\boldsymbol{G}$ is the space of all straight lines in the plane $\boldsymbol{R}^2$) contains segment S_1, and t is the one-dimensional coordinate of the center of the interval S_1 on the line g. On the space $\boldsymbol{G} \times \boldsymbol{R}$, we define a measure $m(\cdot)$ by its element, in the following way:

$$m(dS_1) = dg\,dt,$$

where dg is a locally finite measure in the space **G**, which is invariant with respect to the group $\boldsymbol{M}$ and dt is the one-dimensional Lebesgue measure on g. The measure $m(\cdot)$ is said to be a kinematic measure on the group $\boldsymbol{M}$ [1], [3].

This section gives a main formula for calculating the kinematic measure $K(\boldsymbol{D},r)$ in the terms of distribution function of the chord length of the domain **D**. Obviously,

$K(\boldsymbol{D},r) = 0$, if $r \geq diam(\boldsymbol{D})$, where $diam(\boldsymbol{D})$ is the diameter of **D**, i.e.

$diam(\boldsymbol{D}) = \max\{\rho(x,y): \ x,y \in \boldsymbol{D}\}$,

where $\rho(x,y)$ is the distance between the points x and y. Therefore, only the case $r \leq diam(\boldsymbol{D})$ is considered throughout the paper. It is evident that in the mentioned case

$$K(\boldsymbol{D},r) = \iint_{\{(g,t):\ S_1(g,t)\subset \boldsymbol{D}\}} dg\, dt = \int_{[\boldsymbol{D}]} (\chi(g) - r)^+\, dg,$$

where $[\boldsymbol{D}] = \{g \in \boldsymbol{G}:\ g \cap \boldsymbol{D} \neq \emptyset\}$ is the set of straight lines crossing the domain **D**, $\chi(g) = g \cap \boldsymbol{D}$ is a chord in **D**, while

$$x^+ = \begin{cases} 0, & if\ x \leq 0 \\ x, & if\ x \geq 0. \end{cases}$$

Consequently,

$$K(\boldsymbol{D},r) = \int_{\chi(g)>r} \chi(g)dg - r\int_{\chi(g)>r} dg = \pi\,||\boldsymbol{D}|| - G(r) - r\,|\partial\boldsymbol{D}|[1 - F_{\boldsymbol{D}}(r)], \quad (2.3)$$

where

$$G(x) = \int_{\chi(g)\leq x} \chi(g)\,dg$$

and $F_{\boldsymbol{D}}(\cdot)$ is the chord length distribution function of the domain **D**, defined as

$$F_{\boldsymbol{D}}(y) = \int_{\chi(g)\leq y} dg.$$

We transform (2.3) for the measure $K(\boldsymbol{D},r)$ to a more suitable for applications form. To this end, we prove the following formula:

$$G(x) = |\partial\boldsymbol{D}| \int_0^x u\, f_{\boldsymbol{D}}(u)du, \quad (2.4)$$

where $f_{\boldsymbol{D}}(x)$ is the chord length density function for the domain **D**, i.e. $f_{\boldsymbol{D}}(x) = F_{\boldsymbol{D}}{}'(x)$ is the first derivative of the distribution function (see [11]). Further, for calculating the derivative of the function $G(x)$ we observe that

$$\frac{G(x+\Delta x)-G(x)}{\Delta x} = \frac{1}{\Delta x}\int_{x<\chi(g)\leq x+\Delta x} \chi(g)dg = (x + \theta\,\Delta x)|\partial\boldsymbol{D}|\frac{F_{\boldsymbol{D}}(x+\Delta x)-F_{\boldsymbol{D}}(x)}{\Delta x}. \quad (2.5)$$

Then, assuming that the distribution function $F_{\boldsymbol{D}}(x)$ possesses the density $f_{\boldsymbol{D}}(x)$, we let $\Delta x \to 0$ in (2.5) and get

$$G'(x) = |\partial\boldsymbol{D}|\, x\, f_{\boldsymbol{D}}(x),$$

which implies

$$G(x) = G(0) + |\partial\boldsymbol{D}| \int_0^x u f_{\boldsymbol{D}}(u)du = |\partial\boldsymbol{D}| \int_0^x u f_{\boldsymbol{D}}(u)du,$$

since $G(0) = \int_{\chi(g)\le 0}\chi(g)dg = 0$.

Now, we transform formula (2.4) by means of integration by parts:

$$G(x) = |\partial\boldsymbol{D}| \int_0^x u\, f_{\boldsymbol{D}}(u)du =$$

$$-|\partial\boldsymbol{D}| \int_0^x u\, d[1 - F_{\boldsymbol{D}}(u)] = -x\,|\partial\boldsymbol{D}|\,[1 - F_{\boldsymbol{D}}(x)] + |\partial\boldsymbol{D}| \int_0^x [1 - F_{\boldsymbol{D}}(u)]\, du.$$

Consequently,

$$G(x) = |\partial\boldsymbol{D}| \int_0^x [1 - F_{\boldsymbol{D}}(u)]\, du - x\,|\partial\boldsymbol{D}|\,[1 - F_{\boldsymbol{D}}(x)]. \tag{2.6}$$

At last, substituting (2.6) into formula (2.3) for $K(\boldsymbol{D},r)$ we come to the main formula of this section:

$$K(\boldsymbol{D},r) = \pi||\boldsymbol{D}|| - r\,|\partial\boldsymbol{D}| + |\partial\boldsymbol{D}| \int_0^r F_{\boldsymbol{D}}(u)\, du. \tag{2.7}$$

Thus, if the explicit form of the function $F_{\boldsymbol{D}}(u)$ for $0 \le u \le r$ is given, then one can derive an explicit expression for $K(\boldsymbol{D},r)$ by means of (2.7). The formula (2.7) have been obtained for unoriented segments. For oriented segments this formula should be multiplied by 2.

Substituting (2.7) into (1.3) (and multiply by 2) we obtain the main formula of the present paper:

$$f_\rho(x) = \frac{1}{||\boldsymbol{D}||^2}\left[2\pi\, x||\boldsymbol{D}|| - 2\,x^2\,|\partial\boldsymbol{D}| + 2\,x|\partial\boldsymbol{D}| \int_0^x F_{\boldsymbol{D}}(u)\, du\right], \tag{2.8}$$

where $F_{\boldsymbol{D}}(\cdot)$ is the chord length distribution function for the domain **D**.

The case of a disk. In the case of the disc $\boldsymbol{D} = \boldsymbol{C}_d$ with diameter of the length d, the chord length distribution function is of the form [9]

$$F_{C_d}(y) = \begin{cases} 0, & if \;\; y \le 0, \\ 1 - \sqrt{1 - \frac{y^2}{d^2}}, & if \;\; 0 \le y \le d \\ 1, & if \;\; y \ge d. \end{cases} \tag{3.1}$$

Consequently, substituting (3.1) into (2.7) we get

$$K(\boldsymbol{C}_d,r) = \frac{\pi^2 d^2}{2} - 2\pi\, d \int_0^l \sqrt{1 - \frac{u^2}{d^2}}\; du. \tag{3.2}$$

Since

$$\int_0^r \sqrt{1 - \frac{u^2}{d^2}}\; du = \frac{d}{2}\arcsin\frac{r}{d} + \frac{r}{2}\sqrt{1 - \frac{r^2}{d^2}}, \tag{3.3}$$

substituting (3.3) into (3.2) we get the kinematic measure $K(\boldsymbol{C}_d,r)$ for $l \le d$, i.e. formula (2.1). Substitution this result in (1.3) or (2.8) we obtain the density function of the distance between two points chosen in the disk of diameter d:

$$f_\rho(x) = \frac{8}{\pi d^2}\left[\pi\, x - 2x \arcsin\frac{x}{d} - \frac{2x^2}{d}\sqrt{1 - \frac{x^2}{d^2}}\right].$$

<u>Remark.</u> Above, the exact values of the density function $f_\rho(x)$ are calculated for a disc of diameter d. Note that if we know the explicit form of the chord length distribution function for a domain, using (2.8) we can calculate density function $f_\rho(x)$ of the distance between two random

points in **D**. In [10] the explicit form of the chord length distribution function is given for any regular polygon, and particularly the corresponding result for the regular hexagon can be seen in [8]. Consequently, density $f_p(x)$ can be calculated for any regular polygon by applying the result of [10].

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