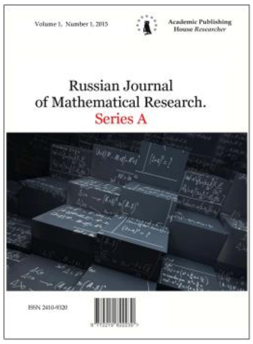

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Recognition of Convex Bodies by Probabilistic Methods

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Abstract

The recognition of bounded convex bodies **D** by means of random k-flats (k-dimensional planes) intersecting **D** is one of the interesting problems of Stochastic Geometry. In particular, the problem of recognition of bounded convex domains **D** by chord length distribution function is of much interest. One can consider the case when the orientation and the length of the chords are observed. We refer this case as the orientation-dependent chord length distribution. All these problems are the problems of geometric tomography, since orientation-dependent chord length distribution function at point y is the probability that parallel **X**-ray in a fixed direction is less than or equal to y. Investigation of convex bodies by orientation-dependent chord length distribution is equivalent to the investigation of their covariograms. The present note considers some problems and recent results related to covariograms, and their applications to various problems of tomography.

Keywords: chord length distribution function, covariogam, bounded convex domain.
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Introduction

The main purpose of the stereology is to obtain information about the geometric properties of n-dimensional structures, if there is information on the forms of smaller dimensions as through the k-plane sections ($0 < k < n$), and with the help of projections on infinitesimal layers. The most popular application is the tomography (see [1] - [3]). Reconstruction of a body over its cross sections is one of the main tasks of geometric tomography, a term introduced by R. Gardner in [3]. If $D \subset R^n$ (R^n is n-dimensional Euclidean space) compact convex body, it is possible to intersect it by a random k-plane ($1 \le k \le n-1$). If the body D is intersected by k-plane, then arises a k-dimensional section that contains some information on D. A natural question arises whether it is possible to reconstruct D, if we have a subclass of k-dimensional cross-sections. Tomography is mainly engaged in the description of the subclasses for which the calculation of the geometrical characteristics is often a difficult task.

Discussion

Reconstruction of convex bodies using random sections makes it possible to simplify the calculation, since the estimates of probability characteristics can be obtained using the methods of mathematical statistics. Quantities characterizing random sections of the body D carry some information on D and if there is a connection between the geometrical characteristics of D and probabilistic characteristics of random cross-sections, then by a sample of results of experiments we can estimate the geometric characteristics of the body D (see [4] - [6]).

Let $\Pi r_{u^\perp} D$ be the orthogonal projection of D onto the hyperplane $u^\perp$ ($u^\perp$ - hyperplane passing through the origin with normal vector u). Random line parallel to the direction u and intersecting D has an intersecting point (denoted by y) with $\Pi r_{u^\perp} D$. We can identify points $\Pi r_{u^\perp} D$ with lines that intersect D and parallel to the direction u. Assuming that the intersection point y uniformly distributed in the convex body $\Pi r_{u^\perp} D$ we arrive at the following definition. Function

$$F_D(u,x) = \frac{V_{n-1}\{y \in \Pi r_{u^\perp} D : \; V_1(g(u,y) \cap D) < x\}}{b_D(u)}$$

Is called orientation dependent chord length distribution function of D in the direction u, where g (u, y) is a line parallel to the u and intersecting $\Pi r_{u^\perp} D$ at point y, $b_D(u) = V_{n-1}(\Pi r_{u^\perp} D)$, and $V_n(\cdot)$ is n-dimensional Lebesgue measure.

Let L(ω) be a random segment with length l> 0 intersecting D and parallel to a fixed direction u. Consider the random variable $|L(\omega)| = V_1(L(\omega) \cap D), \; \omega \in \Omega(u)$, which is defined as follows:

Ω (u) = {segments of length l, parallel to u and intersecting D}.

A random segment L (ω), which lies on the line g (u, y), can be set by the coordinates (g (u, y), z), where z is the one-dimensional coordinate of the center of the segment L (ω) on the line

g(u, y). The origin in the line g (u, y) is one of the intersection points g (u, y) with D. Using the above notation we can identify Ω (u) with the following set:

$$\Omega(u) = \{(y,z), y \in \Pi r_{u^\perp} D, z \in \left[-\frac{l}{2}, \; \chi(u,y) + \frac{l}{2}\right]\}.$$

where $\chi(u,y) = V_1(g(u,y) \cap D)$. Further, we denote

$$B_D^{u,x} = \{(y,z) \in \Omega(u) \, : \, |L|(y,z) < x\}, \quad x \in R^1$$

It is clear that Ω (u) and $B_D^{u,x}$ are measurable subsets of R^n. The function

$$F_{|L|}(u,x) = \frac{V_n(B_D^{u,x})}{V_n(\Omega(u))} = \frac{1}{V_n(\Omega(u))} \int_{B_D^{u,x}} dydz\,, \quad u \in S^{n-1}$$

is called the orientation dependent distribution of the length of the part of random segment which lies within the body. Can we reconstruct a bounded convex body if we know $F_{|L|}(u,x)$ by changing the value of the length l, and how to reconstruct the geometric characteristics of the body D by means of $F_{|L|}(u,x)$. Equation (1) establishes a relationship between the distribution functions $F_D(u,x)$ and $F_{|L|}(u,x)$ in the interval [0, l]:

$$F_{|L|}(u,x) = \frac{b_D(u)}{V_n(D) + l\, b_D(u)} \left[2x + F_D(u,x)(l - x) - \int_0^x F_D(u,z)dz\right]. \tag{1}$$

It is obvious that the distribution function $F_{|L|}(u,x)$ equals 0 when x≤0 and equals 1 if x≥l. Consequently, having $F_D(u,x)$ we can reconstruct $F_{|L|}(u,x)$. There is a problem of inverting the equation (1), that is to find out whether it is possible by the values of the function $F_{|L|}(u,x)$ reconstruct $F_D(u,x)$ distribution function.

Compact convex bodies from n-dimensional Euclidean space can be intersected by k-dimensional planes and we can consider distribution functions of the characteristics of cross-sections (for example, 3-dimensional space a body can be intersected by lines and by planes, and in the last case we can investigate distributions of areas and perimeters).

Let G_n be the space of all lines in R^n. The line $g \in G_n$ can be specified by direction $u \in S^{n-1}$ and the point of intersection y with the hyperplane $u^\perp$. Density $du^\perp$ is the volume element du of the unit sphere S^{n-1}), and dy - the volume element $u^\perp$ at the point y. Let μ (·) is a locally finite measure on G_n, invariant under the group of Euclidean motions of R^n. It is known that an element of this measure up to a constant factor has the following form (see [1]) μ(dg) = dg = dudy. We denote $O_{n-1} = V_{n-1}(S^{n-1})$ the Lebesgue measure of the unit sphere in R^n. For each bounded convex body D, the set of lines intersecting D we denote by $[D] = \{g \in G_n \, , g \cap D \neq \emptyset\}$. We have

$$\mu([D]) = \frac{O_{n-2} V_{n-1}(\partial D)}{2(n-1)}$$

Random line in [D] is the line with distribution proportional to the restriction of μ to [D]. Consequently, for each $x \in R^1$ we have

$$F_D(x) = \frac{\mu(\{g\epsilon[D], V_1(g \cap D) < x\})}{\mu([D])}.$$

$F_D(x)$ is called the chord length distribution function of D.

In the study of the distribution function $F_D(u,x)$ an important role plays the covariogram concept introduced by French mathematician Matheron (see [6], [7]). Let $h \in R^n$, and the translation of D on the vector h is denoted by D + h:

$$D + h = \{x + h,\ h \in R^n,\ x \in D\}.$$

The function

$C(D,h) = V_n\big(D \cap (D+h)\big),\ h \in R^n$

is called covariogram of D. Covariogram C(D, h) is invariant with respect to the group of translations and reflections.

Equation (2) establishes a connection between the covariogram and the orientation dependent chord length distribution function. Therefore, having an explicit form of the covariogram, you can investigate an interesting properties of the distribution function $F_D(u,x)$ (see [6], [7]).

$$-\frac{\partial C(D,xu)}{\partial x} = \big(1 - F_D(u,x)\big) \cdot b_D(u), \qquad (2)$$

where xu - vector in R^n having length x and direction u.

The question of the existence of a bijection between bounded convex bodies D and distribution functions of the chord length $F_D(x)$ was made by the famous German mathematician Wilhelm Blaschke. This question has received a negative response. Further mathematicians considered subclasses of bounded convex bodies for which the chord length distribution function reconstructed non-congruent elements of the subclass. Although the function of the chord length distribution $F_D(x)$ does not reconstruct the compact convex body, yet it contains information about the volume, surface area and other characteristics of the body (see [1] and [3]).

Matheron in [6] (see also [7]) formulated a hypothesis that there exists a one-to-one correspondence between $F_D(u,x)$ and bounded convex bodies. In the plane, a positive answer to the Matheron's hypothesis in the class of convex polygons received Nagel (see [11]). Matheron's hypothesis received a positive solution for any D in the planar case (see [8] - [10]). In the case of finite-dimensional spaces with n>3 Matheron's hypothesis has received a negative answer. In the case of 3-dimensional space the problem is open. Nevertheless, for the case of bounded convex polyhedron for n=3 Matheron's hypothesis received a positive answer (see [4]). Note that convexity is essential for this set of problems. The authors of [12] have given an example of two non-congruent and non-convex polygons with the same covariogram.

Distribution function $F_D(u,x)$ in the class of triangles and ellipses is a function that depends on direction through the maximal chord in this direction, and we can construct a class of parallelograms, for which such a result is not true. It is interesting to examine the cases in which the distribution function $F_D(u,x)$ depends on the direction only through the maximal chord. We can put a general problem: in any case, the maximal cross section is characterized the compact convex body up to translations and reflections. This type of problems considered Gardner in his book [3]. Gardner, in particular, proved that a centrally symmetric convex body in R^n is uniquely reconstructed in the class of all centrally symmetrical bodies (up to parallel translations and reflections), if the maximal cross-sections are the same for all directions. Since orientation dependent chord length distribution function reconstructs a compact convex body in the planar case, and in some cases, it depends on the maximal chord in a given direction, then in the planar case by means of $F_D(u,x)$ we can solve tomographic problems using maximal sections, in particular, we can prove that the maximal chords in all directions reconstruct triangle in the class of all triangles (and a similar result holds in the case of ellipses).

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Распознавание выпуклых тел вероятностными методами.

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Аннотация. Восстановление ограниченных выпуклых тел **D** с помощью случайных k-мерных плоскостей пересекающих тело **D** является одной из интересных задач Стохастической Геометрии. В частности, задача восстановления ограниченных выпуклых тел **D** по функции распределения длины хорды. Можно рассмотреть случай, когда ориентации и длины хорд наблюдаемы. В этом случае мы говорим о зависящей от ориентации распределении длины хорды. Все эти проблемы являются задачами геометрической томографии, так как зависящяя от ориентации функция распределения длины хорды в точке y совпадает с вероятностью, что параллельные **X**-лучи в фиксированном направлении меньше или равны y. Изучение выпуклых тел по зависящей от ориентации функции распределения длины хорды эквивалентно изучению тел по их ковариограмме. В этой заметке рассматриваются некоторые проблемы и недавние результаты касающиеся ковариограмм и их применения к различным задачам томографии.

Ключевые слова: функция распределения длины хорды, ковариограмма, ограниченная выпуклая область.