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On the solution set of fuzzy systems

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Abstract

In this paper we investigate ordinary fuzzy differential inclusions. Using compactness type conditions we prove main results for the existence of solutions. We prove that the solution set is R_δ extending the classical Kneser's theorem.

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1 Introduction

The fuzzy differential equations have many real applications, and were studied extensively during the last decades [12, 15, 16, 22]. The fuzzy differential equations with discontinuous right-hand side and the optimal control of fuzzy differential equations require fuzzy differential inclusions [3, 18, 20]. The theory of the fuzzy differential equations and inclusions is presented in [9, 14, 17]. Among others we notice [1], where fuzzy differential equations with state constrains are studied with connection to set differential equations.

We study the following fuzzy differential inclusion

$$\dot{x}(t) \in F(t, x(t)),\ x(0) = x_0,\ t \in [0, 1]. \tag{1}$$

Here $F : [0, 1] \times \mathbb{E} \rightrightarrows \mathbb{E}$, where $\mathbb{E}$ is the space of fuzzy numbers.

Our target is to investigate the main properties of the solution set of (1) under compactness type assumptions. We first prove the variant of Kneser's theorem.

We refer to [2, 8, 10] for theory of ordinary and evolution differential inclusions with and without state constrains.

Let $\mathbb{E} = \{x : \mathbb{R}^n \to [0, 1];\ x \text{ satisfies } 1) - 4)\ \}$ be the space of fuzzy numbers. Here:

1) x is normal, i.e. there exists $y_0 \in \mathbb{R}^n$ such that $x(y_0) = 1$.

2) x is fuzzy convex, i.e. $x(\lambda y + (1 - \lambda)z) \geq \min\{x(y), x(z)\}$ whenever $y, z \in \mathbb{R}^n$ and $\lambda \in [0, 1]$.

3) x is upper semicontinuous, i.e. for any $y_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta(y_0, \varepsilon) > 0$ such that $x(y) < x(y_0) + \varepsilon$ whenever $|y - y_0| < \delta,\ y \in \mathbb{R}^n$.

4) The closure of the set $\{y \in \mathbb{R}^n;\ x(y) > 0\}$ is compact.

The set $[x]^\alpha = \{y \in \mathbb{R}^n;\ x(y) \geq \alpha\}$ is called α-level set of x.

It follows from 1) – 4) that the α-level sets $[x]^\alpha$ are convex compact subsets of $\mathbb{R}^n$ for all $\alpha \in (0, 1]$. The fuzzy zero is defined by $\hat{0}(y) = \begin{cases} 0\ if\ y \neq 0, \\ 1\ if\ y = 0. \end{cases}$

The metric in $\mathbb{E}$ is defined by $D(x, y) = \sup\limits_{\alpha \in (0,1]} D_H([x]^\alpha, [y]^\alpha)$, where $D_H(A, B) = \max\{\max\limits_{a \in A} \min\limits_{b \in B} |a - b|, \max\limits_{b \in B} \min\limits_{a \in A} |a - b|\}$ is the Hausdorff distance between the convex compact subsets of $\mathbb{R}^n$.

The distance from $a \in \mathbb{E}$ to the closed bounded set $B \subset \mathbb{E}$ is defined by $dist(a, B) = \inf\limits_{b \in B} d(a, b)$ and the Hausdorff distance between the closed and bounded subsets of $\mathbb{E}$ is defined by $d_H(A, B) = \max\{\sup\limits_{a \in A} \inf\limits_{b \in B} d(a, b), \sup\limits_{b \in B} \inf\limits_{a \in A} d(a, b)\}$.

The map $f : I \to \mathbb{E}$ is said to be strongly measurable if there exists a sequence

$\{f_m\}_{m=1}^{\infty}$ of step functions $f_m : I \to \mathbb{E}$ such that $\lim\limits_{m\to\infty} D(f_m(t), f(t)) = 0$ for a.a $t \in I$.

It is easy to see that if $f(\cdot)$ is strongly measurable then $[f]^\alpha(\cdot)$ are measurable for every $\alpha \in (0, 1]$. The converse does not necessarily holds (cf. [12]).

It is well known that every strongly measurable function is almost separably valued i.e there exists a null set $N \subset I$ such that $f(I \setminus N)$ is contained in a separable metric space. Due to Lusin's Theorem $f : I \to \mathbb{E}^n$ is strongly measurable if and only if for all $\varepsilon > 0$ there exists $I_\varepsilon \subset I$ with $meas(I \backslash I_\varepsilon) \le \varepsilon$ such that $f(\cdot)$ is continuous on I_ε. If $f : I \to \mathbb{E}$ is strongly measurable and $D(f(t), \hat{0}) \le \lambda(t)$, where $\lambda(\cdot)$ is Lebesgue integrable real valued function then f is Bochner integrable and

$$\int_0^t f(s)ds = \lim_{m\to\infty} \int_0^t f_m(s)ds.$$

where $f_m(\cdot)$ are step functions with $f_m(t) \to f(t)$ for a.a. t. We refer the reader to [21] for the theory of vector valued (Bochner) integrals.

In the fuzzy set literature starting from [19] the integral of fuzzy functions is defined level-wise, i.e. there exists $g(t) \in \mathbb{E}$ such that $[g]^\alpha(t) = \int_0^t [f]^\alpha(s)ds$. As it is shown in [12] there are level-wise integrable functions which are not almost everywhere separably valued, i.e. not Bochner integrable.

The function $g : I \to \mathbb{E}$ is called absolutely continuous (AC) if there exists a strongly measurable and continuous function $f : I \to \mathbb{E}$ such that $g(t) = \int_0^t f(s)ds$.

Denote by $\mathfrak{F}^n$, the space of all compact and convex fuzzy sets of $\mathbb{E}$. If $u \in \mathfrak{F}^1$, then u is called a fuzzy interval and the α-level set $[u]^\alpha$ is a non empty compact and convex set for all $\alpha \in [0, 1]$. The operations of sum and scalar multiplication on $\mathfrak{F}$ are defined as $(u \oplus v)(x) = \sup\limits_{y\in\mathbb{R}^n} \{u(y) \wedge v(x - y)\}$ and

$$(\lambda \cdot u)(x) = \begin{cases} u\left(\frac{x}{\lambda}\right), & \lambda \neq 0 \\ \chi_{\{0\}}(x), & \lambda = 0 \end{cases}$$

where $\chi_\{0\}$ is characteristic function of $\{0\}$. The following properties are true $[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha$ and $[\lambda \cdot u]^\alpha = \lambda[u]^\alpha$, for all $\alpha \in [0, 1]$.

Let $u, v \in \mathfrak{F}^n$. If there exists $w \in \mathfrak{F}^n$ such that $u = v \oplus w$, then w is called the $H-$difference of u and v, and it is denoted by $u \ominus v$.

Let $F : T \to \mathfrak{F}^n$ and $t_0 \in T$. we say that F is)generalized) differentiable at t_0 if (H) or (B) (seee.g. [4, 5]), where

(H) there is an element $F'(t_0) \in \mathfrak{F}^n$ such that for all $h > 0$ sufficiently near to 0, there are $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} \text{ and } \lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h}$$

are equal.

(B) there is an element $F'(t_0) \in \mathfrak{F}^n$ such that for all $h < 0$ sufficiently nearby 0, there are $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and limits

$$\lim_{h \to 0^-} \frac{F(t_0 + h) \ominus F(t_0)}{h} \text{ and } \lim_{h \to 0^-} \frac{F(t_0) \ominus F(t_0 - h)}{h}$$

which are equal.

Note that if F is (H) differentiable, then it is not (B) differentiable and vice versa.

Theorem 1. *Let $F : I \to \mathfrak{F}$ be a function. Then:*

(1) If F is (H) differentiable, then f_α and g_α are differentiable functions and

$$[F'(t)]^\alpha = [f'_\alpha, g'_\alpha]. \tag{2}$$

(2) If F is (B) differentiable, then f_α and g_α are differentiable functions and

$$[F'(t)]^\alpha = [g'_\alpha, f'_\alpha]. \tag{3}$$

Theorem 2. *Let $F : T \to \mathfrak{F}$ be a continuous function.*

(1) Let F be (H) differentiable. If F' is integrable then for all $t \in T$,

$$F(t) = F(a) \oplus \int_a^t F'(s)ds. \tag{4}$$

(2) Let F be (B) differentiable. If F' is integrable then for all $t \in T$,

$$F(t) = F(a) \ominus (-1) \int_a^t F'(s)ds. \tag{5}$$

Theorem 3. *The mapping $x : T \to \mathbb{E}$ is a solution of* (1) *w.r.t. (H) iff*

$$x(t) = x_0 \oplus \int_a^t f(s)ds, \text{ where } f(s) \in F(s, x(s)) \text{ is strongly measurable.} \tag{6}$$

The mapping $x : T \to \mathbb{E}$ is a solution of (1) *w.r.t. (B) iff*

$$x(t) = x_0 \ominus \int_a^t f(s)ds, \text{ where } f(s) \in F(s, x(s)) \text{ is strongly measurable.} \tag{7}$$

Example 1. *Let C be fuzzy and not crisp and let x_0 be crisp. Consider the fuzzy differential equation:*

$$\dot{x} = C, \ x(0) = x_0. \tag{8}$$

There exists a solution of (8) w.r.t. (H) On the other hand there is no solution of (8) w.r.t. (B).

In general, because $\int_a^t f(s)ds \not\succ \chi_{\{0\}}$ for $f(t) \neq \chi_{\{0\}}$, the fundamental fuzzy differential equation

$$\left\{ \begin{array}{l} x'(t) = f(t) \\ x(0) = \chi_{\{0\}}, \end{array} \right\} \tag{9}$$

does not have solution w.r.t. (B).

Recall that the map $F : I \times \mathbb{E} \rightrightarrows \mathbb{E}$ is said to be upper semicontinuous (USC) at (s, y) if for every $\varepsilon > 0$ there exists δ such that $F\left(I \bigcap \left[s - \frac{\delta}{2}, s + \frac{\delta}{2}\right], y + \delta\mathbb{B}\right) \subset F(s, y) + \varepsilon\mathbb{B}$. Here $\mathbb{B} = \{x \in \mathbb{E} : \ D(0, x) \leq 1\}$ is the unit ball. It is said to be continuous at (s, y) when for every $\varepsilon > 0$ there exists δ such that $d_H(F(s, y), F(t, x)) < \varepsilon$ for every $t \in I$ and $x \in \mathbb{E}$ such that $|t - s| + D(x, y) < \delta$. If δ does not depend on (s, y) then $F(\cdot, \cdot)$ is called uniformly continuous.

The multimap $F(\cdot, \cdot)$ is said to be almost USC (continuous, uniformly continuous) when for every $\delta > 0$ there exists a compact $I_\delta \subset I$ with Lebesgue measure $meas(I \backslash I_\delta) < \delta$ such that $F(\cdot, \cdot)$ is USC (continuous, uniformly continuous) on $I_\delta \times \mathbb{E}$. The latter is equivalent to say that there exists a sequence $\{I_m\}_{m=1}^\infty$ of pairwise disjoint compact sets with $\text{meas}(I_m) > 0$ and $\text{meas}\left(\bigcup_{m=1}^\infty I_m\right) = \text{meas}(I)$ such that $F(\cdot, \cdot)$ is continuous on $I_m \times \mathbb{E}$ for every m.

The set $\mathbb{E}$ is a complete semilinear metric space with respect to metric $D(\cdot, \cdot)$. This space is not locally compact and nonseparable. From Theorem 2.1 of [12] we know that

$\mathbb{E}$ can be embedded as a closed convex cone in a Banach space $\mathbb{X}$. The embedding map $j : \mathbb{E} \to \mathbb{X}$ is isometry and isomorphism and hence $f : I \to \mathbb{E}$ is continuous iff $j(f)(\cdot)$ is continuous. Furthermore, $j(\cdot)$ preserves differentiation and integration. Namely if $\dot{f}(t)$ exists then $\frac{d}{dt}j(f)(t)$ also exists and $j\left(\dot{f}\right)(t) = \frac{d}{dt}j(f)(t)$, where $\frac{d}{dt}$ is the usual differential operator.

Now if $g(\cdot) : I \to \mathbb{E}$ is strongly measurable and integrable then $j(g)(\cdot)$ is strongly measurable and Bochner integrable and

$$j\left(\int_0^t g(s)ds\right) = \int_0^t j\,(g)\,(s)ds \text{ for all } t \in I. \tag{10}$$

Consider the fuzzy differential equation:

$$\dot{x}(t) = f(t, x(t)),\ x(0) = x_0,\ t \in [0, T]. \tag{11}$$

Theorem 4. *Let there exists an integrable function $\eta(\cdot)$ such that $D(f(t,x), \hat{0}) \leq \eta(t)(1 + |x|)$. Assume that $f(\cdot, x)$ is strongly measurable, while $f(t, \cdot)$ is locally Lipschitz, i.e. for every $x \in \mathbb{E}$ there exists a neighborhood $U_x \in x$ and a Lebesgue integrable function $l_x(\cdot)$ such that $D(f(t,y), f(t,z)) \leq l_x(t)D(y,z)$ for every $y, z \in U_x$. Then the differential equation (11) admits an unique solution w.r.t. (H) on the interval I which depends continuously on x_0.*

Proof. Suppose $j : \mathbb{E} \to \mathbb{X}$ is an embedding map defined by $j(x) = y$, Also we denote $g(t, y) = j\,(f(t, x))$. Then $g(\cdot, y)$ is strongly measurable, while $g(t, \cdot)$ is locally Lipschitz and $|g(t, y)| \leq \eta(t)(1 + |y|)$. It is well known that the problem

$$\dot{y}(t) = g(t, y);\ y(0) = y_0$$

has a unique solution $y(\cdot)$, which depends continuously on y_0. Thus $x(t) = j^{-1}\,(y(t))$ is a unique solution of (11). □

In the next section we recall the main properties of the measures of noncompactness used in this paper. In the third section the differential inclusion (1) is studied.

2 Measure of non compactness and some of its properties

Let A be a bounded subset of complete metric space Y with metric $\varrho_Y(\cdot,\cdot)$. The Hausdorff measure of noncompactness $\beta : A \to \mathbb{R}^+$ is defined as

$$\beta(A) := inf\{d > 0; A \text{ can be covered by finite many balls with radius } \leq d\},$$

and "Kuratowski measure" of noncompactness $\nu : A \to \mathbb{R}^+$ is defined by

$$\nu(A) := \inf\{d > 0 : A \text{ can be covered by finite many sets with diameter } \leq d\},$$

where for any bounded set $A \subset Y$ we denote $\operatorname{diam}(A) = \sup\limits_{a,b\in A} \varrho_Y(a,b)$, and $\varrho_Y(\cdot,\cdot)$ is the distance in Y. It is easy to see that for any $\varepsilon > 0$ there exists a ball $\mathbb{B}_r$ with radius $r \leq \operatorname{diam}(A) + \varepsilon$ such that $A \subset \mathbb{B}_r$.

Further in this section $\gamma(\cdot)$ is either $\nu(\cdot)$ or $\beta(\cdot)$. Some properties of $\gamma(\cdot)$ are listed below:

(i) $\gamma(A) = 0$, iff $\bar{A}$ is compact and $\gamma(A) = \gamma(\overline{co}\ A)$.
(ii) $\gamma(aA + bB) \leq |a|\gamma(A) + |b|\gamma(B)$.
(iii) If $A \subset B$ then $\gamma(A) \leq \gamma(B)$.
(iv) $\gamma(A \bigcup B) = max(\gamma(A), \gamma(B))$.
(v) $\gamma(\cdot)$ is continuous with respect to the Hausdorff distance.

The following property of β is proved in [10] and [13].

Theorem 5. *Let $\mathbb{Y}$ be separable Banach space and let $\{g_n(\cdot)\}_{n=1}^{\infty}$ be an integrally bounded sequence of measurable functions from $[0,T]$ into $\mathbb{Y}$. Then:*

$$\beta\left(\int_t^{t+h}\left\{\bigcup_{i=1}^{\infty} g_i(s)\right\}ds\right) \leq \int_t^{t+h} \beta\left\{\bigcup_{i=1}^{\infty} g_i(s)\right\}ds,$$

where $t, t+h \in [0,T]$.

Now we prove another property of $j(\cdot)$.

Theorem 6. *If $A \subset \mathbb{E}$ then $\beta(j(A)) \leq \beta(A) \leq 2\beta(j(A))$.*

Proof. Let $A \subset \mathbb{E}$ and let $A \subset \bigcup\limits_{k=1}^{m} B_k$, where B_k are balls in $\mathbb{E}$. Then $j(A) \subset \bigcup\limits_{k=1}^{m} j(B_k)$, i.e. $\beta(A) \geq \beta(j(A))$, because $j(\cdot)$ preserves the diameters of B_k. Furthermore, if $j(A) \subset \bigcup_{i=1}^{p} U_k$, where U_k are balls in $\mathbb{X}$ with $U_k \bigcap j(A) \neq \emptyset$, then $A \subset \bigcup_{i=1}^{p} j^{-1}(U_k \bigcap j(A))$. Furthermore, the diameter $\operatorname{diam}(U_k \bigcap j(A)) \leq \operatorname{diam}(U_k)$ and $2\beta(j(A)) \geq \beta(A)$. □

Remark 1. *Since $\beta(A) < 2\beta(A)$, one has that $\beta(A) \leq \beta(j(A)) \leq 2\beta(A)$ due to Theorem 6.*

Theorem 7. *Let $\{f_n(\cdot)\}_{n=1}^{\infty}$ be a (integrally bounded) sequence of strongly measurable fuzzy functions defined from I into $\mathbb{E}$. Then $t \to \beta\{f_m(t), m \geq 1\}$ is measurable and*

$$\beta\left(\int_t^{t+h} \left\{\bigcup_{m=1}^{\infty} f_m(s)\right\} ds\right) \leq 2\int_t^{t+h} \beta\left\{\bigcup_{m=1}^{\infty} f_m(s)\right\} ds,$$

Proof. Since f_m are strongly measurable, one has that $j(f_m)(\cdot)$ are also strongly measurable and hence almost everywhere separably valued.

Thus there exists a separable Banach space $Y \subset \mathbb{Y}$ such that $j(f_m)(I \setminus N) \subset Y$, where $N \subset I$ is a null set.

Furthermore without loss of generality we assume that $j(f_m) : I \to Y$ and from Theorem 5 we have

$$\beta\left(\int_t^{t+h} \bigcup_{m=1}^{\infty} j(f_m(s))ds\right) \leq \int_t^{t+h} \beta\left(\bigcup_{m=1}^{\infty} j(f_m(s))\right) ds.$$

By (10), one has that

$$\beta\left(j\left(\int_t^{t+h} \bigcup_{m=1}^{\infty} f_m(s)ds\right)\right) \leq \int_t^{t+h} \beta\left(j\left(\bigcup_{m=1}^{\infty} f_m(s)\right)\right) ds.$$

Consequently, using Theorem 6 and Remark 1

$$\beta\left(\int_t^{t+h} \bigcup_{m=1}^{\infty} f_m(s)ds\right) \leq 2\int_t^{t+h} \beta\left(\bigcup_{m=1}^{\infty} f_m(s)\right) ds.$$

□

The multifunction $G : I \times \mathbb{E} \rightrightarrows \mathbb{E}$ is said to satisfy compactness type condition (CTC) if there exists a Perron function $v(\cdot,\cdot)$ $\lambda : I \to \mathbb{R}^+$ such that $\beta\left(G(t, A)\right) \leq \frac{1}{2}v(t, \beta(A))$ for any bounded set $A \subset \mathbb{E}$.

Recall that the Caratheodory function $v : I \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be Perron function if it is integrally bounded on the bounded sets, $v(t, 0) \equiv 0$ and the unique solution of $\dot{r}(t) = v(t, r(t))$, $r(0) = 0$ is 0.

Remark 2. *The factor* 2 *does not allow to use usual compactness type condition, i.e.* $\beta\left(F(t, A)\right) \leq v\left(t, \beta(A)\right)$

3 Structure of the solution set of fuzzy differential inclusions

In this section we will study the properties of the solution set of the fuzzy differential inclusion (1) w.r.t. (H).

Assume that

$$\max_{v \in F(t,x)} D(v, \hat{0}) \leq w(t, D(x, \hat{0})). \tag{12}$$

Here $w : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is almost continuous, integrally bounded on the bounded sets and nondecreasing on the second argument such that the maximal solution $\hat{r}(\cdot)$ of

$$\dot{r}(t) = w(t, r(t)), \ r(0) = D(x_0, \hat{0})$$

exists on $[0, 1]$.

A particular case of the following lemma is proved in [10].

Lemma 1. *Under the condition stated above the problem (1) can be transformed in an equivalent system*

$$\dot{x}(t) \in \widetilde{F}(t, x), \ x(0) = x_0 \tag{13}$$

such that $\max_{v \in \widetilde{F}(t,x)} D(v, \hat{0}) \leq 1$*, where the properties continuity and measurability of the right-hand side are preserved.*

Proof. Notice that $\hat{r}(\cdot)$ is increasing, i.e. $\max_{t \in I} \hat{r}(t) = \hat{r}(1)$. Suppose $\hat{r}(1) = M$ and define the multimap:

$$\hat{F}(t, x) = \begin{cases} F(t, x), & \text{for} \quad D(x, \hat{0}) \leq M + 1; \\ F\left(t, \dfrac{Mx}{D(x, \hat{0})}\right), & \text{otherwise.} \end{cases}$$

Evidently $\hat{F}(\cdot,\cdot)$ admits the same measurability and continuity properties as $F(\cdot,\cdot)$ and replacement of F with $\hat{F}$ in the right-hand side of (1) does not change the solution set.

Denote $\lambda(t) = w(t, M) + 1$, where $\lambda(\cdot)$ is positive Lebesgue integrable function. It is easy to see that $D(\hat{F}(t,x), \hat{0}) \le \lambda(t)$, because $w(t,\cdot)$ is nondecreasing. Define

$$\mu(t) = \int_0^t \lambda(s)ds, \text{ where } t \in [0,1].$$

Thus $\mu(\cdot)$ is AC strictly increasing and hence invertible with inverse function $\mu^{-1}(\cdot)$.

If we let $y(t) = x(\mu^{-1}(t))$, then the multifunction $\widetilde{F}(t,y) = \frac{1}{\lambda(t)}\hat{F}(\mu^{-1}(t), y)$ satisfies $\max\limits_{v\in\widetilde{F}(t,y)} D(v,\hat{0}) \le 1$ and preserves measurability and continuity properties of F. Furthermore, for every solution $y(\cdot)$ of (13) the AC function $x(t) = y(\mu(t))$ is a solution of (1) and vise versa. Since $\lambda(t) \ge 1$ one has that the function $w(t,r) = \frac{1}{\lambda(t)} v(\mu^{-1}(t), r)$ is also Kamke function.

Notice that $\widetilde{F} : [0,T] \times \mathbb{E} \rightrightarrows \mathbb{E}$, where $T = \mu(1)$. □

Due to Lemma 1 and studying (1) on $[0,T]$ one can assume that $\max\limits_{v\in F(t,y)} D(v,\hat{0}) \le 1$. Further in the paper we study the differential inclusion (1) on the interval $I = [0,T]$.

The following topological definitions are used in the next Theorem 8:

Definition 1. *Let $\mathfrak{X}$ be a complete metric space. The set $Y \subset \mathfrak{X}$ is said to be retract of $\mathfrak{X}$ if there exists a continuous function $r : \mathfrak{X} \to Y$ such that $r(x) = x$ for all $x \in Y$. The function r is called a retraction.*

a) The set $Y \subset \mathfrak{X}$ is called a deformation retract of $\mathfrak{X}$ if there exists a retract $r : \mathfrak{X} \to Y$ and a homotopy $H : \mathfrak{X} \times [0,1] \to \mathfrak{X}$ such that $H(x,0) = x$ and $H(x,1) = r(x)$ for all $x \in \mathfrak{X}$.

b) The set Y is contractible (contractible to a point) if and only if there exists a point $a \in Y$ such that a is a deformation retract of Y.

c) The set A is said to be R_δ if it is an intersection of decreasing sequence of compact contractible sets.

Notice that every R_δ set is connected and acyclic. It is well known (see Lemma 5 of [6]) that A is R_δ set iff there exists a sequence $B_{k+1} \subset B_k$ of closed sets with $\lim\limits_{k\to\infty} \beta(B_k) = 0$ such that $A = \bigcap\limits_{k=1}^{\infty} B_k$.

Theorem 8. *Let $G(\cdot,\cdot)$ be almost USC satisfying CTC and (12) then (1) (with right-hand side G) admist a nonempty R_δ solution set.*

Proof. Define $\widetilde{G}(t,x) = j(F(t,x)$. We have that $\widetilde{G} : I \times X \rightrightarrows X$ is almost USC and hence there exists a sequence of pairwise disjoint sets $\{I_k\}_{k=1}^{\infty}$ with meas $\left(\bigcup_{k=1}^{\infty} I_k\right) = T$ and $\widetilde{G}$ is USC on $I_k \times X$ for each k.

Consider the locally Lipschitz approximations

$$\widetilde{G}_k(t, j(x)) = \sum_{\lambda\in\Lambda} \varphi_\lambda(j(x)) C_\lambda(t) \text{ with } C_\lambda(t) = \overline{co}\ \widetilde{G}(t, B_{3r_k}(j(x_\lambda))). \tag{14}$$

Recall that $(\varphi_\lambda)_{\lambda\in\Lambda}$ is a locally Lipschitz partition of unity subordinate to some locally finite refinement $(U_\lambda)_{\lambda\in\Lambda}$ of $\{j(x) + r_k\mathbb{B}_j : \ j(x) \in X\}$ with $r_k = 3^{-k}$ and $j(x_\lambda)$ is such that $U_\lambda \subset j(x_\lambda) + r_k\mathbb{B}_j$ as shown in 2.4 [10].

We pick a measurable selection f_λ of $G_k(\cdot, x_\lambda)$ and define $g_\lambda^k : I \times X \to \mathbb{B}$ by

$$g_\lambda^k(t,x) = \sum_{\lambda} \varphi_\lambda(x) f_\lambda(t) \in G_k(t,x). \tag{15}$$

Since $(U_\lambda)_{\lambda\in\Lambda}$ is locally finite, one has that $g_\lambda^k(\cdot, x)$ is strongly measurable and $g_\lambda^k(t,\cdot)$ is locally Lipschitz. Thus the fuzzy equation (11) with f replaced by g_λ^k admits a unique solution $x^k(\cdot)$.

Therefore we have

$$\widetilde{G}(t, j(x)) \subset \widetilde{G}_{k+1}(t, j(x)) \subset \widetilde{G}_k(t, j(x)) \subset \overline{co}\ \widetilde{G}(t, j(x_\lambda) + 2r_k\mathbb{B}_j) \text{ on } I \times X. \tag{16}$$

Denote $M(t) = \bigcup_{m=1}^{\infty} \{x^m(t)\}$. Thus $j(M(t)) := \{j(\{x^m(t)\})\}_{m=1}^{\infty}$, and $j(\dot{x}^m(t)) \in \overline{co}\ \widetilde{G}_m(t, j(x^m(t)))$, and due to Theorem 6

$$\begin{aligned}\beta(j(M(t+h))) - \beta(j(M(t))) &\le \int_t^{t+h} \beta\left(\widetilde{G}(s, j(M(s)) + \varepsilon_k\mathbb{B}_j)\right) ds \\ &\le \int_t^{t+h} v(s, \beta(j(M(s))) + \varepsilon_k)\, ds.\end{aligned}$$

Since

$$\lim_{k\to\infty} \int_t^{t+h} v(s, \beta(j(M(s))) + \varepsilon_k) ds = \int_t^{t+h} v(s, \beta(j(M(s)))) ds$$

one has that

$$\beta\left(j\left(M(t+h)\right)\right)-\beta\left(j\left(M(t)\right)\right)\leq\int_{t}^{t+h}v\left(s,\beta\left(j\left(M(s)\right)\right)\right)ds.$$

If $\beta\left(j(M(t))\right)=g(t)$ then $g(\cdot)$ is AC, and moreover,

$$\dot{g}(t)\leq v\left(t,g(t)\right),\ g(0)=0,\ \text{i.e. } g(t)\equiv 0.$$

$$\text{Thus } \beta\left(\bigcup_{k=1}^{\infty}\left\{j\left(x^k(t)\right)\right\}\right)=0.$$

However, $j(\cdot)$ preserves the measure of non-compactness and hence $\beta\left(\bigcup_{k=1}^{\infty}\left\{x^k(t)\right\}\right)=0$. Since $\left\{x^k(\cdot)\right\}$ is bounded, and since $\left\{\dot{x}^k(\cdot)\right\}$ is integrally bounded, one has that $\left\{x^k(\cdot)\right\}$ is equicontinuous.

Due to Arzela's theorem passing to subsequences we have $x^k(t)\to x(t)$. Furthermore, $j(\dot{x}^k(\cdot))\to j(\dot{x}(\cdot))$ in $L_1(I,X)$ in weak sense. It is standard to prove that $x(\cdot)$ is a solution of (1).

If S_k is the solution set of (1) with $\tilde{G}_k$ instead of G then $S_{k+1}\subset S_k$. Moreover the set $S=\bigcap_{k=1}^{\infty}S_k\neq\emptyset$.

To see that S_k is contractible we consider the initial value problem $\dot{z}=g_\lambda^k(t,z)$ a.e. on I with $z(x)=y$. It has a unique solution $z_k(\cdot;s,y)$ which depends continuously on $(s,y)\in I\times X$. Define the map

$$h(\tau,u)(t)=\begin{cases}u(t) & \text{on } [0,\tau T],\\ z(t;\tau a,u(t)) & \text{on } [\tau T,T].\end{cases}$$

The continuous function $h:[0,1]\times S_k\to S_k$ with $h(0,u)=z(\cdot;0,x_0)$ and $h(1,u)=u$ is the contraction. Due to CTC we have $D_H\left(S_k,S\right)\to 0$ as $k\to\infty$. Since $\lim_{k\to\infty}\beta\left(S_k\right)=0$ one has that S is R_δ set. □

Now we will prove a version of Filippov–Pliss lemma which has many applications in optimal control and different variants of it are studied by many authors. We refer to [11] for some recent results in that topic.

Definition 2. *The almost continuous function $g:I\times\mathbb{R}^+\to\mathbb{R}^+$ which is is said to be modulus of the multifunction $F:I\times\mathbb{E}\rightrightarrows\mathbb{E}$ integrally bounded on the bounded sets,*

$g(t,0) \equiv 0$ and $d_H(F(t,x),F(t,y)) \le g(t,D(x,y))$ for every $x,y \in \mathbb{E}$.

Theorem 9. *(Lemma or Filippov–Pliss) Let $F(\cdot,\cdot)$ be almost continuous with nonempty closed convex values, satisfying CTC and (12). Assume that $y(\cdot)$ is AC function such that $dist\,(\dot{y}(t),F(t,y(t))) \le \lambda(t)$, where $\lambda(\cdot)$ is integrable (on $[0,T]$) function. If there exists modulus $g(\cdot,\cdot)$ of $F(\cdot,\cdot)$, then for every $\varepsilon > 0$ there exists a solution $x_\varepsilon(\cdot)$ of (1) such that $D(x_\varepsilon(t),y(t)) \le r(t)$, where $r(\cdot)$ is the maximal solution of*

$$\dot{r}(t) = g(t,r(t)) + \lambda(t),\ \ r(0) = 0.$$

Proof. Define the multimap

$$G(t,u) = \{v \in F(t,u):\ \ D(v,\dot{y}(t)) \le g(t,D(u,y(t))) + \lambda(t)\}.$$

Since $dist(\dot{y}(t),F(t,u)) \le dist(\dot{y}(t),F(t,y(t)) + d_H(F(t,u),F(t,y(t))) \le g(t,D(u,y(t))) + \lambda(t)$, one has that $H_\varepsilon(\cdot,\cdot)$ is with nonempty values. Consequently $G(\cdot,\cdot)$ has nonempty closed values. Obviously it satisfies CTC and (12). Since $D(a,\lambda b + (1-\lambda)c) \le \lambda D(a,b) + (1-\lambda)D(a,c)$, one has that $G(\cdot.\cdot)$, and also convex values.

We claim that $G(\cdot,\cdot)$ is almost USC. For our next purpose fix $\delta > 0$. Let $\dot{y}(\cdot)$ and $\lambda(\cdot)$ be continuous on I_δ, and $g(\cdot,\cdot)$ and $F(\cdot,\cdot)$ be continuous on $I_\delta \times \mathbb{E}$, where $I_\delta \subset I$ is compact with $meas(I \setminus I_\delta) < \delta$. Let $t_m \to t \in I_\delta$, $u_m \to u$ and let $v_m \in G(t_m,u_m)$. If $v_m \to v$, then $v \in F(t,u)$ and $D(v,\dot{y}(t)) = \lim_{m\to\infty} D(v_m,\dot{y}(t_m))$. Thus $v \in G(t,u)$, i.e. the graph of $G(\cdot,\cdot)$ restricted on $I_\delta \times \mathbb{E}$ is closed. The latter together with CTC implies that $G(\cdot,\cdot)$ is almost USC.

From Theorem 8 we know that there exists a solution $x(\cdot)$ of

$$\dot{x}(t) \in G(t,x(t)),\ \ x(0) = x_0.$$

Thus $D(\dot{x}(t),\dot{y}(t)) \le g(t,D(x(t),y(t))) + \lambda(t)$, and hence $D(x(t),y(t)) \le r(t)$, where $r(\cdot)$ is the maximal solution of $\dot{r}(t) = g(t,r(t)) + \lambda(t)$, $r(0) = D(x_0,y_0)$. ☐

3.1 Fuzzy differential equations w.r.t. (B)

The derivative (H) has many bad properties. For example the diameter of $A+B$ is commonly greater than the maximal diameter of any one of the two A and B. Consequently it is almost impossible to study asymptotic stability of fuzzy differential equations w.r.t. (H). The advantage is that for every locally Lipschitz differential equation (inclusion) the solution w.r.t. (H) always exists. From Example 1 the differential equation in general does not have solutions w.r.t. (B) even in very simple cases.

There is a connection between both derivatives.

If (11) admits a solution w.r.t. (H) say $x(t)$ on $[0,T]$ then $x(T-t)$ is a backward solution of (11) w.r.t. (B). Namely define new variable $\tau = T - t$ and

$$\dot{y}_g(\tau) = f(\tau, y), \quad y(0) = x(T),$$

then $y(\cdot)$ exists on $[0,T]$ and $y(t) = x(T-t)$ and vise versa.

Let $K \subset I \times \mathbb{E}$ be domain such that for any $(t,x) \in K$ there exist $\tau = \tau(t,x) > 0$, and $f(t,x) \in F(t,x)$ with $x(t) \ominus \displaystyle\int_t^{t+h} f(\tau,x)d\tau \in \mathbb{E}$ for any $0 < h < \tau$.

Theorem 10. *If* $(t_0, x_0) \in int(K)$*, then under CTC there exists* $\delta > 0$ *such that the differential inclusion* (1) *has a solution on* $[t_0, t_0 + a)$ *w.r.t. (B).*

Proof. The proof can be accomplished using for this purpose the classical Euler Cauchy approach. Almost all details are standard. We will sketch the proof for the reader convenience.

Fix $\varepsilon > 0$, $t_0 = 0$, and let $f_0(t) \in F(t_0, t)$ be strongly measurable. Define $x^0(t) = x_0 \ominus \displaystyle\int_{t_0}^t f(\tau)d\tau$. Clearly there exists $t_1 > t_0 = 0$ such that $(t, x^0(t)) \in \int K$ and $f_0(t) \in F(t, x_0 + \varepsilon\mathbb{B}$. Now we can define $x^0(t)$ on $]t_1, t_2]$ with $x^0(t) = x(t_1) \ominus \displaystyle\int_{t_1}^t f_0(\tau)d\tau$.

Clearly there exists $T \in (0,1]$ such that $x^0(\cdot)$ can be extended on $[0,T]$. Now we consider a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ and the corresponding sequence of absolute continuous $\left\{x^k(\cdot)\right\}$ such that $(t, x^k(t)) \in K$ for every k and every $t \in [0,T]$. Moreover, $x^k(t) = x_0 \ominus \displaystyle\int_0^t f_k(\tau)d\tau$ with $f_k(t) \in F)t, x^k(t) + \varepsilon_k\mathbb{B})$.

Now using obvious modifications of the proof of Theorem 8 one can show that passing to subsequences $x^k(t) \to x(t)$ uniformly on $[0,T]$. Clearly $x(\cdot)$ is a solution of (1) on the interval $[0,T]$ w.r.t. (B). ☐

We point out that the existence of solution of (11) w.r.t. (B) in the case of Lipscjitz $f(t,\cdot)$ can not be proved by the classical Pickar successive approximations, because in general $x_0 \ominus \displaystyle\int_0^t f(\tau, x^k(\tau))d\tau \notin \mathbb{E}$.

Notice that if CTC holds true and $(t_0, x_0) \in int(K)$, then due to Theorem 10 there exists w.r.t. (H) and at least one w.r.t. (B) stating from (t_0, x_0). Consequently even for

Lipschitz single valued differential equation the solution w,r,t, generalized derivative is not unique. Also it is almost impossible to prove good properties of the solution set.

Notice that it is impossible to prove that the solution set of (1) is R_δ w.r.t. (B), because the locally Lipschitz selections $g_k(t,x) \in \tilde{G}_k(\cdot,\cdot)$ do not satisfy $x \ominus \int_0^h f_k(\tau,x)d\tau \in \mathbb{E}$.

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References

[1] R. Agarwal, D. O'Regan, V. Lakshmikantham, Viability theory and fuzzy differential equations, *Fuzzy Sets and Systems* **151** (2005), 563-580.

[2] J.-P. Aubin, *Viability Theory*, Birghäuser, Boston, 1991.

[3] V. Baidosov, Fuzzy differential inclusions, *J. of Appl. Math. and Mechan.* **54** (1) (1990), 8-13.

[4] B. Bede, L. Stefanini, Generalized differentiability of fuzzy-valued functions. *Fuzzy Sets and Systems* **230** (2013), 119-141.

[5] B. Bede, L. Stefanini, Numerical Solution of Interval Differential Equations with Generalized Hukuhara Differentiability, *IFSA/EUSFLAT Conf.* (2009), 730-735.

[6] D. Bothe, Multivalued perturbations of m-accretive differential inclusions, *Israel J. Math.* **108** (1998), 109–138.

[7] A. Bressan, G. Colombo, Selections and Representations of Multifunctions in Paracompact Spaces, *Studia Math.* **102** (1992), 209-216.

[8] O. Carja, M. Necula, I. Vrabie, *Viability, Invariance and applications*, Nort Holland, New York, 2007.

[9] R. Choudary, V. Lupulescu, *Fuzzy Sets and Fuzzy Differential Equations*, Abdus Salam SMS-Lahore, 2010.

[10] K. Deimling, *Multivalued Differential Equations*, De Grujter, Berlin, 1992.

[11] T. Donchev, E. Farkhi, On the theorem of Filippov–Pliss and some applications, *Control and Cybernetics* **38** (2009), 1251–1271.

[12] O. Kaleva, Fuzzy differential equations, *Fuzzy Sets and Systems* **35** (1990), 389–396.

[13] M. Kisielewicz, Multivalued differential equations in separable Banach spaces, *J. Opt. Theory Appl.* **37** (1982), 231–249.

[14] V. Lakshmikantham, R. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor - Francis, London, 2003.

[15] V. Lupulescu, Initial value problem for fuzzy differential equations under dissipative conditions, *Information Sciences* **178** (2008), 4523–4533.

[16] J. Park, H. Han, Existence and uniquiness theorem for a solution of fuzzy differential equation, *Intern. J. Math. Math. Sci.* **22** (1999), 271–279.

[17] A. Plotnikov, N. Skripnik, *Differential equations with "clear" and fuzzy multivalued right-hand side. Asymptotic methods*, Astroprint, Odessa, 2009 (in Russian)

[18] A. Plotnikov, N. Skripnik, The generalized solutions of the fuzzy differential inclusions, *Int. J. Pure Appl. Math.* **56** (2) (2009), 165-172.

[19] M. Puri, D. Ralescu, Fuzzy random variables, *J. Math. Anal. Appl.* **114** (1986), 409–422.

[20] N. Skripnik, Existence of classic solutions of the fuzzy differential inclusions, *Ukr. Math. Bull.* **5** (2008), 244-257 (in Russian).

[21] S. Schwabik, Y. Guoju, Topics in Banach space integration, Chapter 1, World Scientific, Singapore, 2005.

[22] C. Wu, S. Song, Existence theorem to Cauchy problem for fuzzy differential equations under compactness type conditions, *Information Sciences* **108** (1998), 123-134.