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Numerical simulation of a flow past an elliptic cylinder at moderately high Reynolds numbers

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Abstract

In this paper we analyze a two-dimensional laminar flow past an elliptic cylinder in case the major axis is parallel to the flow. Attention is limited in high speed regions under a subcritical condition, using a potential function of series type comparable with separation as an outside boundary condition. Numerical analysis is based on a spectral finite difference scheme to give velocity profile near the surface and drag characteristics in comparison with traditional experimental data.

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1 Introduction

Aerodynamic drag and lift is concerned with performance characteristics of aviation and data were accumulated broadly in Ref.[4] and [5]. In a very slow flow regime, under a suitable situation analytical approximation of fluid flow is possible as in Ref.[6] and [3]. For a flow past an elliptic cylinder at moderate Reynolds numbers, experimental treatment was given in Ref.[9], and numerical one in Ref.[8]. In an intermediate regime, application of a boundary-layer theory [10] is possible. For a subcritical flow regime over a simple shape, Zahm [12] got experimental evaluation on pressure distribution and velocity profiles on the body surface, whereas Freeman [2] and Schubauer [11] got experimentally velocity profiles in the vicinity of the body surface (elliptic cylinder), and Lindsey [7] and Delany and Sorensen [1] obtained experimental data of drag coefficients vs. a Reynolds number for various aspect ratios of an elliptic section, covering data for a subcritical region.

In this paper, a moderately high but subcritical speed flow over a single two-dimensional elliptic cylinder is analyzed. Steady-state incompressible viscous laminar sublayer model is assumed. The target cylinder is assumed to be placed normal to a uniform flow, the direction of which is that of the major axis of the section. Outside of the surface sublayer, the flow is assumed to be governed by a potential flow, which can be determined by the wake configuration (i.e. sensitive to the outer barrier), resulting in any deviation but restricted from far away field condition irrespective of possessing separation or not.

2 Analysis

2.1 Basic equations

Under a subcritical regime, fluid properties such as density and viscosity are assumed to be constant, so that the fluid is regarded to be incompressible. Hereafter it is assumed that length, velocity, and time are made dimensionless with respect to the semi-major axis a (of the elliptic section), free stream velocity U_∞, and a/U_∞ respectively. Let (x, y) be a Cartesian coordinate system such that the direction of the positive x-axis is that of the uniform flow. Let (α, β) be an elliptic coordinate system (in the xy-plane) such that $\alpha = 0$ corresponds to the surface of the cylinder. Thus

$$z \equiv x + iy = \cosh(\alpha_0 + \alpha + i\beta)/\cosh\alpha_0,$$

$$(\alpha \geq 0), |\beta| \leq \pi, \tag{1}$$

where $\alpha_0 \equiv \tanh^{-1}$(semi-minor axis/semi-major axis). Then the equation of vorticity transport (in a two-dimensional flow of Newtonian fluid) can be expressed as

$$J\frac{\partial\zeta}{\partial t} + \frac{\partial(\zeta,\psi)}{\partial(\alpha,\beta)} = \frac{1}{Re}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right)\zeta, \tag{2}$$

$$J\zeta + \left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right)\psi = 0, \tag{3}$$

$$J \equiv \left|\frac{dz}{d(\alpha+i\beta)}\right|^2, \tag{4}$$

where ψ, ζ are dimensionless stream function, and vorticity respectively. Re : Reynolds number $\equiv aU_\infty/\nu$, ν : kinematic viscosity.

2.2 Fluid flow behaviour outside of the laminar sublayer

Complex flow potential function $F(\equiv \phi + i\psi)$ over a cylinder surface (ϕ: potential function) is given by

$$F = \frac{e^{\alpha_0}}{2\cosh\alpha_0}\left\{2\cosh\left(\alpha+i\beta\right) + \frac{1}{1-\sum_{n\geq 2}\epsilon_n\cosh n\left(\alpha+i\beta\right)} - 1\right\} \tag{5}$$

provided $\sum \epsilon_n^2 \ll 1$, ϵ_n : real, and for pole(s), α, of F, $|\alpha| \not\ll 1$. The latter condition for poles is automatically satisfied if nonzero coefficients are finite. If $|\alpha|$ for the argument of F is moderate,

$$F \sim \frac{e^{\alpha_0}}{2\cosh\alpha_0}\left\{2\cosh\left(\alpha+i\beta\right) + \sum_{n\geq 2}\epsilon_n\cosh n\left(\alpha+i\beta\right)\right\}, \tag{6}$$

2.3 Boundary conditions

At the surface of the cylinder, no-slip conditions are specified. That is,

$$\psi(\alpha = 0, \beta) = 0, \tag{7}$$

$$\frac{\partial \psi}{\partial \alpha}(\alpha = 0, \beta) = 0. \tag{8}$$

At the location outside of the viscous sublayer ($\alpha = \alpha_\infty (> 0)$)

$$\psi(\alpha_\infty, \beta) = \Im \left\{ F\left(\alpha_\infty + i\beta\right) \right\}, \tag{9}$$

$$\zeta(\alpha_\infty, \beta) = 0. \tag{10}$$

3 Numerical solution procedure

3.1 Primary variables

Since flow is assumed to be symmetric with respect to the x-axis, the stream function ψ and vorticity ζ can be expanded into the following Fourier series of β:

$$\psi = \sum_{n=1}^{\infty} \psi_n\left(\alpha, t\right) \sin n\beta, \tag{11}$$

$$\zeta = \sum_{n=1}^{\infty} \zeta_n\left(\alpha, t\right) \sin n\beta. \tag{12}$$

Then, using the addition formulae of trigonometric functions, Eqs.(2) and (3) can be separated into each Fourier component of β, which constitutes a system of of differential equations with respect to ψ_n's and ζ_n's.

3.2 Numerical integration scheme

By truncating the series (11) and (12) up to a certain order the governing Eqs.(2)-(4) can be decomposed into Fourier components, and by discretizing the system of equations in space and time by a finite difference method incorporated with boundary conditions, the system of equations can be integrated with respect to time by a semi-implicit method to give a steady-state solution. Special attention should be placed on boundary conditions. Eq.(8) is replaced by

$$\frac{2}{h^2}\psi\left(h, \beta\right) + J\zeta\left(h, \beta\right) = 0, \tag{13}$$

where h is the value of the coordinate α at the discretized point nearest to the cylinder surface. Among many possibilities, $\frac{\partial \zeta}{\partial \alpha}\left(\alpha_\infty, \beta\right) = 0$ (which is originally a necessary condition) is introduced for discretized equations, resulting in reducing one discretized point of application for Eq.(2) or (3). For α_∞, to cover the viscous laminar sublayer fully inside

$$\alpha_\infty = \frac{5\ c}{\sqrt{2\ Re}}, \tag{14}$$

$$c > 1,$$

where c is a suitably chosen constant.

4 Results

Velocity component parallel to the surface, u, is given by

$$u = \frac{1}{\sqrt{J}} \frac{\partial \psi}{\partial \alpha}. \tag{15}$$

In the following analysis c for Eq.(14) = 1.4.

Figure 1 shows a velocity profile on the surface of a cylinder, the ratio of a major axis to a minor axis is 3.01 at $Re = 2.09 \times 10^5$ (experimental [11]), 2×10^5 (numerical, current paper).

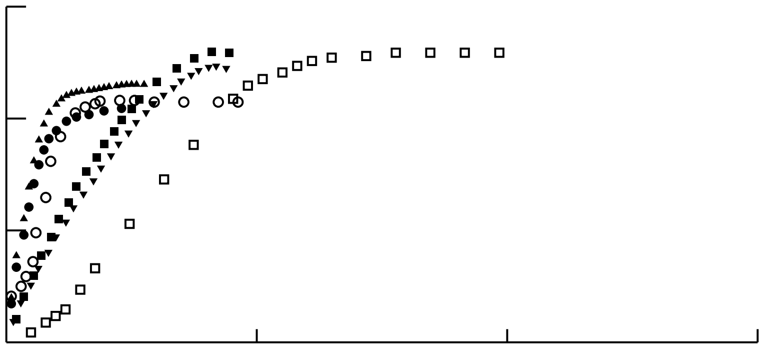

Fig. 1: Velocity profile

s : dimensionless arc length based on the minor axis from the forward stagnation point, y_n: distance (based on a) measured from the elliptic surface ($\alpha = 0$) at a given s.
○ : $s = 0.251$, [11]; ● : $s = 0.251$, $\epsilon_n = 0$, $n \geq 2$, current ($C_D = 0.322$); ▲ : $s = 0.251$, $\epsilon_2 = -0.05$, $\epsilon_n = 0$, $n \geq 3$, current ($C_D = 0.241$); □ : $s = 2.52$, [11]; ■ : $s = 2.52$, $\epsilon_n = 0$, $n \geq 2$, current, ▼ : $\epsilon_2 = -0.05$, $\epsilon_n = 0$, $n \geq 3$, current.

Drag coefficient C_D (based on $a \rho U_\infty^2$, ρ: density) is given by

$$C_D = \Re \left[\frac{i}{Re} \left\{ \oint \frac{dz}{d(\alpha + i\beta)}\ \zeta\ d\beta - \oint z \frac{\partial \zeta}{\partial \alpha}\ d\beta \right\} \right], \tag{16}$$

where integration is carried at $\alpha = 0$. Figure 2 shows characteristics of C_D in case of the ratio of the length of the major axis to that of the minor = 2. For the current analysis $\epsilon_n = 0$, $n \geq 3$.

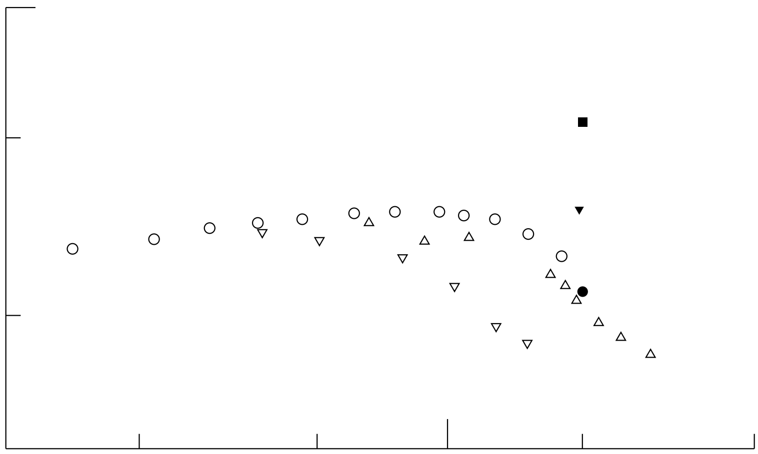

Fig. 2: Drag coefficients

$\circ$, $\triangle$: experimental [1], $\bigtriangledown$: experimental [7]; $\bullet$: $\epsilon_2 = -0.2$, current; $\blacktriangledown$: $\epsilon_2 = -0.1$, current; $\blacksquare$: $\epsilon_2 = 0$, current.

5 Discussion

Equation (5) has a possibility to give a separation, where at a point β_s, $u(\alpha_\infty, \beta_s) = 0$. For example, if $\epsilon_2 = -0.2$ and $\epsilon_3 = -0.2$, $\epsilon_n = 0$, $n \geq 4$, then $\beta_s \approx 0.47$, $x(\alpha_\infty + i\beta_s) \approx 0.76$.

6 Conclusions

Introducing a potential flow function in series type comparable with separation gives reasonable velocity profiles in the viscous layer near the cylinder surface and reasonable drag coefficients, using a spectral finite difference scheme. The assumption is just satisfied in case of a slender elliptic cylinder, the major axis of which is parallel to the flow.

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