

Canonical Cosine Transform Novel Stools in Signal processing

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Abstract: In this paper a theory of distributional two-dimensional (2-D) canonical cosine is developed using Gelfand-Shilov technique and defined some operators on these spaces also the topological structure of some of the S-type spaces of distributional two dimensional canonical cosine transform.

Keywords: 2-D canonical transforms, generalized function, testing function space, s-type spaces, canonical cosine transform.

1. INTRODUCTION:

Linear canonical transform is useful tools for optical analysis and signal processing .The Fourier

Analysis is undoubtedly the one of the most valuable and powerful tools in signal processing, image processing and many other branches of engineering .The fractional Fourier transform, a special case of linear canonical transform is studied through different angles. Almeida [1], [2] Had introduced it and proved many of its properties Namias [5].Opened the way of defining the fractional transform through the Eigen value as in case of fractional Fourier transform. The conversional canonical cosine transform is defined as .

$$\{CTf(t)\}(s) = \sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i(d)}{2(b)}s^2} \int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right) \cdot e^{\frac{i(a)}{2(b)}t^2} f(t) dt,$$

It is easily seen that for each $s \in R^n$ and the function $K_c(t, s)$ belongs to $E(R^n)$ as a function of t , where

$$K_c(t, s) = \sqrt{\frac{1}{2\pi ib}} e^{\frac{i(d)}{2(b)}s^2} e^{\frac{i(a)}{2(b)}t^2} \cos\left(\frac{s}{b}t\right)$$

Hence the canonical cosine transform of $f \in E(R^n)$ can be defined by

$$\{CTf(t)\}(s) = \langle f(t), K_c(t, s) \rangle,$$

where right hand side has a meaning as the application of $f \in E^1$ to $K_c(t, s) \in E$.

As compared to one dimensional, canonical cosine transform has a considerably richer structure in two dimensional.

The definition of distributional two dimensional canonical cosine transform is given in section 2. S-type spaces using Gelfand-shilov technique are developed in section 3. Section 4 is devoted for the operators on the above spaces. In section

5, discuss the result on the topological structures of some spaces. The notation and terminology as per Zemanian[6],[7]. Gelfand-Shilov[3],[4].

2. DEFINITION OF TWO DIMENSIONAL (2D) CANONICAL COSINES TRANSFORMS:

Let $E^1(R \times R)$ denote the dual of $E(R \times R)$. Therefore the generalized canonical cosine-cosine transform of $f(t, x) \in E^1(R \times R)$ is defined as

$$\{2DCCCT f(t, x)\}(s, w) = \langle f(t, x), K_{C_1}(t, s) K_{C_2}(x, w) \rangle$$

$$\begin{aligned} & \{2DCCCT f(t, x)\}(s, w) \\ &= \frac{1}{\sqrt{2\pi i b}} \frac{1}{\sqrt{2\pi i b}} e^{\frac{i(d)}{2(b)}s^2} e^{\frac{i(d)}{2(b)}w^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right) \cos\left(\frac{w}{b}x\right) e^{\frac{i(a)}{2(b)}t^2} e^{\frac{i(a)}{2(b)}x^2} f(t, x) dx dt \end{aligned}$$

$$\text{where, } K_{C_1}(t, x) = \frac{1}{\sqrt{2\pi i b}} e^{\frac{i(d)}{2(b)}s^2} e^{\frac{i(a)}{2(b)}t^2} \cdot \cos\left(\frac{s}{b}t\right) \quad \text{when } b \neq 0$$

$$= \sqrt{d} e^{\frac{i}{2}(cds^2)} \delta(t-ds) \quad \text{when } b=0$$

$$\& K_{C_2}(x, w) = \frac{1}{\sqrt{2\pi i b}} e^{\frac{i(d)}{2(b)}w^2} e^{\frac{i(a)}{2(b)}x^2} \cdot \cos\left(\frac{w}{b}x\right) \quad \text{when } b \neq 0$$

$$= \sqrt{d} e^{\frac{i}{2}(cdw^2)} \delta(x-dw) \quad \text{where } b=0$$

where $\gamma_{E,k} \{K_{C_1}(t, s) K_{C_2}(x, w)\} = \sup_{\substack{-\infty < t < \infty \\ -\infty < x < \infty}} |D_t^k D_x^l K_{C_1}(t, s) K_{C_2}(x, w)| < \infty$.

3. VARIOUS TESTING FUNCTION SPACES:

In this section several spaces consisting of infinitely differentiable function are defined on the first and second quadrants of coordinate plane.

3.1 The space $CC_\gamma^{a,b}$: It is given by

$$CC_\gamma^{a,b} = \left\{ \phi : \phi \in E_+ / \sigma_{l,k,q} \phi(t, x) = \sup_{I_1} |t^l \cdot D_t^k D_x^q \phi(t, x)| \leq C_{k,q} A^l \cdot l^\gamma \right\} \quad (3.1)$$

The constant $C_{k,q}$ and A depend on ϕ .

3.2 The space $CC_{\beta}^{a,b}$:

$$CC_{\beta}^{a,b} = \left\{ \phi : \phi \in E_+ / \rho_{l,k,q} \phi(t,x) = \sup \left| t^l D_t^k D_x^q \phi(t,x) \right| \leq C_{l,q} B^k k^{k\beta} \right\} \quad (3.2)$$

The constants $C_{l,q}$ and B depend on ϕ .

3.3 The space $CC_{\gamma}^{a,b,\beta}$:

This space is formed by combining the condition (3.1) and (3.2)

$$CC_{\gamma}^{a,b,\beta} = \left\{ \phi : \phi \in E_+ / \xi_{a,b,q,l,k} \phi(t,x) = \sup \left| t^l D_x^q D_t^k \phi(t,x) \right| \leq C A^l l^{\gamma} B^k k^{k,\beta} \right\} \quad (3.3)$$

$l, k, q = 0, 1, 2, \dots$ Where A,B,C depend on ϕ .

In next we have introduced subspaces of each of the above space that are used in defining the inductive limits of these spaces.

3.4 The space $CC_{\gamma,m}^{a,b}$:It is defined as,

$$CC_{\gamma,m}^{a,b} = \left\{ \phi : \phi \in E_+ / \sigma_{a,b,q,l,k} \phi(t,x) = \sup \left| t^l D_t^k D_x^q \phi(t,x) \right| \leq C_{k,q} (m + \mu)^l l^{\gamma} \right\} \quad (3.4)$$

For any $\mu > 0$ where m is the constant, depending on the function ϕ .

3.5 The space $CC_{\beta,n}^{a,b}$:This space is given by

$$CC_{\beta,n}^{a,b} = \left\{ \phi : \phi \in E_+ / \rho_{a,b,q,l,k} \phi(t,x) = \sup \left| t^l D_t^k D_x^q \phi(t,x) \right| \leq C_{l,\delta,q} (n + \delta)^k k^{k\beta} \right\} \quad (3.5)$$

For any $\delta > 0$ where n the constant is depends on the function ϕ .

3.6 The space $CC_{\gamma,m}^{a,b,\beta,n}$:

This space is defined by combining the conditions in (3.4) and (3.5).

$$C_{\gamma,m}^{a,b,\beta,n} = \left\{ \phi : \phi \in E_+ / \xi_{l,k,q} \phi(t,x) = \sup_{I_1} |t^l D_t^k D_x^q \phi(t,x)| \right. \\ \left. \leq C_{\mu\delta} (m + \mu)^l (n + \delta)^k . l^{l\gamma} k^{k\beta} \right\} \quad (3.6) \text{ For any } \mu > 0, \delta > 0 \text{ and for given } m$$

$> 0, n > 0$ unless specified otherwise the space introduced in (3.1) through (3.6) will henceforth be consider equipped with their natural, Hausdoff, locally convex topologies to be denoted respectively by,

$$T_{\gamma}^{a,b}, T_{\beta}^{a,b}, T_{\gamma}^{a,b,\beta}, T_{\gamma,m}^{a,b}, T_{\beta,n}^{a,b}, T_{\gamma,m}^{a,b,\beta,n}$$

These topologies are respectively generalized by the total families of seminorms.

$$\left\{ \sigma_{a,b,q,l,k} \right\}, \left\{ \rho_{a,b,q,l,k} \right\}, \left\{ \xi_{a,b,q,l,k} \right\}, \left\{ \sigma_{a,b,q,l,k} \right\}, \left\{ \rho_{a,b,q,l,k} \right\} \text{ and } \left\{ \xi_{a,b,q,l,k} \right\}$$

4 SOME BOUNDED OPERATORS IN S-TYPE SPACES:

This section is devoted to the study of different types of linear operators, namely, shifting operator, differentiation operator, scaling operator, in the $CC_{\gamma}^{a,b,\beta}$ space. These operators are found to be bounded (continuous also) in the $CC_{\gamma}^{a,b,\beta}$.

Proposition 4.1: If $\phi(t,x) \in CC_{\gamma}^{a,b,\beta}$ and λ is fixed real number then $\phi(t + \lambda, x) \in C_{\gamma}^{a,b,\beta}, t + \lambda > 0$

Proof: Consider, $\xi_{lk} \phi(t + \lambda, x) = \sup_{I_1} |t^l D_x^q D_t^k \phi(t + \lambda, x)|$

$$\xi_{lk} \phi(t + \lambda, x) = \sup_{I_1} \left| (t' - \lambda)^l D_t^k D_x^q \phi(t', x) \right| \quad \text{where } t' = t + \lambda$$

$$\leq CA^{l\gamma} B^k k^{k\beta}$$

thus $\phi(t + \lambda, x) \in C_{\gamma}^{a,b,\beta}$, for $t + \lambda > 0$.

Proposition 4.2: The translation (shifting) operator $T : \phi(t,x) \rightarrow \phi(t + \lambda, x)$ is a topological automorphism on $C_{\gamma}^{a,b,\beta}$ for $t + \lambda > 0$.

Proposition 4.3: If $\phi(t,x) \in C_{\gamma}^{a,b,\beta}$ and $\rho > 0$ strictly positive number then $\phi(\rho t, x) \in C_{\gamma}^{a,b,\beta}$

Proof: Consider $\xi_{l,k} \phi(\rho t, x) = \sup_{I_1} \left| t^l D_t^k D_x^q \phi(\rho t, x) \right|$

$$= \sup_{I_1} \left| \left(\frac{T}{\rho} \right)^l D_t^k D_x^q \phi(T, x) \right|$$

$$= C_l \sup_{I_1} \left| T^l D_T^k D_x^q \phi(T, x) \right|$$

Where C_l is constant depending on ‘ ρ ’

$$\leq C_1 C_2 A^l l^\gamma B^k k^\beta$$

$$\leq C A^l l^\gamma B^k K^\beta, \quad \text{where } C = C_1 C_2$$

Thus $\phi(\rho t, x) \in C_\gamma^{a,b,\beta}$ for $\rho > 0$

Proposition 4.4: If $\rho > 0$ and $\phi(t, x) \in C_\gamma^{a,b,\beta}$ then the scaling operator.

$$R: C_\gamma^{a,b,\beta} \rightarrow C_\gamma^{a,b,\beta} \text{ defined } R\phi = \psi$$

Where $\psi(t, x) = \phi(\rho t, x)$ is a topological automorphism.

Proposition 4.5: The operator $\phi(t, x) \rightarrow D_t \phi(t, x)$ is defined on the space $C_\gamma^{a,b,\beta}$ and transform this space into itself

Proof: Let $\phi(t, x) \in C_\gamma^{a,b,\beta}$, if $D_t \phi(t, x) = \psi(t, x)$ we have,

$$\xi_{l,k}(\psi) = \sup_{I_1} \left| t^l D_t^k D_x^q \psi(t, x) \right| = \sup_{I_1} \left| t^l D_t^k D_x^q D_t \phi(t, x) \right|$$

$$= \sup_{I_1} \left| t^l D_x^q D_t^{k+1} \phi(t, x) \right|$$

$$\leq C A^l l^\gamma B^{(k+1)} (k+1)^{(k+1)\beta}$$

$$\therefore \psi(t, x) \in C_{\gamma}^{a,b,\beta}$$

5 TOPOLOGICAL PROPERTIES OF $CC_{\gamma}^{a,b}$ - SPACE:

This section is devoted to discuss the result on the topological structures of some of the spaces and the results exhibiting their relationship. Then attention is also paid to be strict inductive limits of some of these spaces.

Theorem 5.1: $(CC_{\gamma}^{a,b}, T_{\gamma}^{a,b})$ is a Frechet space

Proof: As the family $A_{\gamma}^{a,b}$ of seminorms $\{\delta_{l,k,q,a,b}\}_{l,k,q=0}^{\infty}$ generating $T_{\gamma}^{a,b}$ is countable it, suffices to prove the completeness of the space $(CC_{\gamma}^{a,b}, T_{\gamma}^{a,b})$.

Let us consider a Cauchy sequence $\{\phi_n\}$ in $CC_{\gamma}^{a,b}$. Hence for a given $\epsilon > 0$ there exist

an $N = N_{l,k,q}$ such that for $m, n \geq N$

$$\delta_{a,b,l,k,q}(\phi_m - \phi_n) = \sup_{I_1} |t^l D_x^q D_t^k (\phi_m - \phi_n)| < \epsilon \quad (5.1)$$

In particular for $l = k = q = 0, m, n \geq N$

$$\sup_{I_1} |\phi_m(t, x) - \phi_n(t, x)| < \epsilon \quad (5.2)$$

Consequently for fixed t in I_1 $\{\phi(t, x)\}$ is a numerical Cauchy sequence.

let $\phi(t, x)$ be the point wise limit of $\{\phi_m(t, x)\}$ using (5.2) we can easily deduce that $\{\phi_m(t, x)\}$ converges to ϕ uniformly on I_1 . Thus ϕ is continuous moreover, repeated use of (5.1) for different values l, k, q , yields that ϕ is smooth i.e. $\phi \in E_+$ further from, (5.1)

We get, $\delta_{a,b,q,l,k}(\phi_m) \leq \delta_{a,b,l,k,q}(\phi_N) + \epsilon \quad \forall m \geq n$

$$\leq C_{k,q} A^l l^{l,y} + E$$

taking $m \rightarrow \infty$ and ϵ is arbitrary we get,

$$\delta_{a,b,l,k}(\phi) = \sup_{I_1} |t^l D_x^q D_t^k \phi(t, x)|$$

$$\leq C_k A^l l^{\gamma}$$

Hence $\phi \in CC_\gamma^{a,b}$ and it is the $T_\gamma^{a,b}$ limit of ϕ_m by (5.1).

This proves the completeness of $CC_\gamma^{a,b}$ and $(CC_\gamma^{a,b}, T_\gamma^{a,b})$ is a Frechet space.

Proposition 5.2: If $m_1 < m_2$ then $C_{\gamma, m_1}^{a,b} \subset C_{\gamma, m_2}^{a,b}$. The topology of $C_{\gamma, m_1}^{a,b}$ is equivalent to the topology induced on $C_{\gamma, m_1}^{a,b}$ by $C_{\gamma, m_2}^{a,b}$

i.e $T_{\gamma, m_1}^{a,b} \sim T_{\gamma, m_2}^{a,b} / C_{\gamma, m_1}^{a,b}$

Proof: For $\phi \in C_{\gamma, m_1}^{a,b}$ and

$$\delta_{a,b,l,k}(\phi, x) \leq C_{k,\mu} (m_1 + \mu)^l l^{\gamma}$$

$$\leq C_{k,\mu} (m_2 + \mu)^l l^{\gamma} \quad \text{thus, } C_{\gamma, m_1}^{a,b} \subset C_{\gamma, m_2}^{a,b}$$

The second part is clearly from the definition of topologies of these spaces. The space

$C_\gamma^{a,b}$ can be expressed as union of countably normed spaces.

6. CONCLUSION:

In this paper two-dimensional canonical cosine is generalized in the form the distributional sense, and proved some operators on these spaces also discussed the topological structure of some of the S-type spaces.

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