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STABILTY RESULTS FOR FUZZY DIFFERENCE EQUATIONS

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ABSTRACT

In this paper, consider fuzzy difference equations using Hukahara difference for the fuzzy elements. Using Lyapunov like functions and comparison results, we show under suitable assumptions that the stability properties for the fuzzy difference equations follows from stability properties of the trivial solutions of a scalar difference equation.

1. Introduction and preliminaries

The difference equations appear as a natural description of observed evolution phenomenon because most measurements of time evolving variables are discrete and as such these equations are, in their own right, important models. More importantly difference equations also appear in the study of discretization methods for differential equations.

There exists an extensive literature dealing with difference equations and their applications. We refer to the monograph [3] and references therein. On the other hand, when a dynamical system is modeled by deterministic ordinary differential equations we cannot usually be sure that the model is perfect because, in general, knowledge of dynamical system is often incomplete or vague. If the underlying structure of the model depends upon subjective choices, one way to incorporate these into the model, is to utilize the aspect of fuzziness, which leads to the consideration of fuzzy differential equations. The study of fuzzy differential equations has been initiated as an independent subject in conjunction with fuzzy valued analysis [1] and set valued differential equations [2]

In this paper, we consider the fuzzy difference equations

(1) $$u_n = F(n, u_n),\ u_{n_0} = u_0,$$

where $F: N_{n_0} \times E^m \to E^m$ is continuous in *u* for each $n \in N_{n_0}$ and $u \in E^n$.

Let $K_c(R^m)$ be denote the collection of all nonempty, compact convex subsets of R^m. We define the Hausdorff distance between sets A, B $\in K_c(R^m)$ by

$$d_H(A,B) = \max\{sup_{x\in A} inf_{y\in B}\|x-y\|, sup_{y\in B} inf_{x\in A}\|x-y\|\},$$

where ||.|| denotes the norm on R^m.

Denote

E^m={u:R^m→[0,1]; u satisfies (i)-(iv) below}.

(i) u is normal, that is, there exists an $x_0 \in R^m$ such that $u(x_0)=1$;

(ii) u is fuzzy convex, that is, for x, y$\in R^m$ and $0\leq\lambda\leq 1$,

$$u(\lambda x+(1-\lambda)y)\geq\min\{u(x),u(y)\};$$

(iii) u is upper semicontinuous;

(iv) $[u]^0=cl\{x\in R^m; u(x)>0\}$ is compact.

For $0<\alpha\leq 1$, denote $[u]^\alpha=\{x\in R^m; u(x)\geq\alpha\}$. Then from (i)-(iv), it follows that the α-level set $[u]^\alpha\in K_c(R^m)$ for all $0<\alpha\leq 1$. For later purposes, we define $\hat{0}\in E^m$ as $\hat{0}(x)=1$ if x=0 and $\hat{0}(x)=0$ if $x\neq 0$.

If we define

$$D[u,v]=sup_{0\leq\alpha\leq 1}d_H([u]^\alpha,[v]^\alpha),$$

then it is well known that D is a metric in E^m and that (E^m, D) is a complete metric space [2]. For any $u\in E^m$, let $||u||:=D[u,\hat{0}]$.

We list the following properties of D[u,v]:

D[u+w,v+w]=D[u,v] and D[u,v]=D[v,u],

D[λu, λv]=|λ|D[u,v],

D[u,v]≤D[u,w]+D[w,v],

for all u, v, w$\in E^m$ and λ∈R.

If there exists $w \in E^m$ such that u=v+w, then w is called the Hukuhara difference of u and v and is denoted by u-v.

Let N denote the set of natural numbers and $N_+ = N \cup \{0\}$. We denote by $N_{n_0}^+$ the set $N_{n_0}^+ = \{n_0, n_0+1, \ldots, n_0+l, \ldots\}$ with $l \in N_+$ and $n_0 \in N$. Let $u: N_{n_0}^+ \to E^m$ such that Hukuhara differences u(n+1)-u(n) exist for all $n \in N_{n_0}^+$. It is well known (see [2]) that ||u(n)|| in nondecreasing on $N_{n_0}^+$.

We need the following comparison principle for ordinary difference equations (see [3]).

Lemma 1.1. *Let $N_{n_0}^+$, $r \geq 0$ and g(n, r) be a nondecreasing function in r for each n. Suppose that for each $n \geq n_0$, the inequalities*

(2) $$y_{n+1} \leq g(n, y_n),$$

(3) $$z_{n+1} \geq g(n, z_n),$$

hold. If $y_{n_0} \leq z_{n_0}$, then $y_n \leq z_n$, for all $n \geq n_0$.

2. The main results

The following result estimates the solution of the fuzzy difference equation in terms of the is solution of the scalar difference equation

(4) $$z_{n+1} = g(n, z_n), \qquad z_{n_0} = z_0,$$

where g(n, r) is continuous in r for each n and nondecreasing in r for each n.

Theorem 2.1. *Assume that F(n, u) is continuous in u for each n and*

(5) $$D[F(n,u), \hat{0}] = \|F(n,u)\| \leq g(n, \|u\|)$$

where g(n, r) is given in (4). Then, $\|u_{n_0}\| \leq z_{n_0}$ implies that

(6) $$\|u_{n+1}\| \leq z_{n+1}, \text{ for } n \geq n_0.$$

Proof. If $y_{n+1} := \|u_{n+1}\|$, then we have

$$y_{n+1} = \|F(n, u_n)\| \leq g(n, \|u_n\|) = g(n, y_n), \quad n \geq n_0.$$

Let z_{n+1} be the solution of (4) with $z_{n_0} = y_{n_0}$. Then, by Lemma 1.1, we obtain that $y_{n+1} \leq y_{n+1}$, for $n \geq n_0$, which implies (6) completing the proof. ■

Further, let g(n, r)=r+h(n, r) and assume that g(n, r) nondecreasing in r, for each n. We have the following comparison result. This version of Theorem 2.1 is more suitable because h(n, r) need not be positive and hence the solution of (4) could have better properties. This observation is useful in extending the Lyapunov-like method for (1).

Theorem 2.2. *Let* V: $N_{n_0} \times E^m \to R_+$ *be a given function that satisfy*

$$V(n+1, u_{n+1}) \leq V(n, u_n) + h(n, u_n) \equiv g(n, V(n, u_n)), n \geq n_0.$$

Then $V(n, u_{n_0}) \leq z_{n_0}$ *implies that*

(7) $$V(n+1, u_{n+1}) \leq z_{n+1}, \ n \geq n_0,$$

where $z_{n+1}=z_{n+1}(n_0, z_{n_0})$ *is the solution of (4).*

Proof. Set $y_{n+1} = V(n+1, u_{n+1})$, so that $y_{n_0} = V(n_0, u_{n_0}) \leq z_{n_0}$ and

$$y_{n+1} \leq y_n + h(n, y_n) = g(n, y_n), n \geq n_0.$$

Hence, by Lemma 1.1, we obtain that $y_{n+1} \leq z_{n+1}$, $n \geq n_0$, where z_{n+1} is the solution of (4). This implies the stated estimate.■

Now, we can prove the stability results for the solutions of the fuzzy difference equation (1). In the following, we denote by K the class of continuous and increasing functions β defined on [0,∞) such that β(0)=0.

Theorem 2.3. *Let the assumptions of Theorem 2.2 hold. If for* a, b∈$\mathcal{K}$, n∈ N *and* $u \in E^m$ *we have that*

$$b(||u||) \leq V(n, u) \leq a(||u||),$$

Then the stability properties of the trivial solution of (4) imply the corresponding stability properties of the trivial solution of (1).

Proof. Suppose that the trivial solution of (4) is stable and let $\varepsilon > 0$, $n_0 \in N$ be given. Then we have b(ε)>0 and there exists a $\delta_1 = \delta_1(n_0, \varepsilon) > 0$ such that $0 \leq z_{n_0} < \delta_1$ implies $z_{n+1} < b(\varepsilon), n \geq n_0$. Choose $\delta = \delta(n_0, \varepsilon) > 0$ such that a(δ)<δ_1. Then, by Theorem 2.1 we obtain that $V(n+1, u_{n+1}) \leq z_{n+1}, n \geq n_0$, which shows that $b(||u_{n+1}||) \leq V(n+1, u_{n+1}) \leq z_{n+1}, n \geq n_0$. Choose $z_{n_0} = V(n_0, u_{n_0})$ so that we have $z_{n_0} \leq a(||u_{n_0}||) \leq a(\delta) < \delta_1$. We then get b(||$u_{n+1}$||) $< b(\varepsilon), n \geq n_0$, which implies the stability of the trivial solution of (4).■

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