"Constantin Brâncuşi" University of Târgu Jiu ENGINEERING FACULTY
SCIENTIFIC CONFERENCE – 13 th edition – with international participation
November 07-08, 2008 – Târgu Jiu

OPTIMAL CONTROL UNDER PARTIAL OBSERVATIONS AND STOCHASTIC UNIFORM OBSERVABILITY

Ungureanu Viorica Mariela, *dr., Constantin Brancuşi University*

ABSTRACT. This paper discuss linear quadratic control problem for stochastic differential systems with partial observations. Following [4] and using the results in [8] concerning the Riccati equation, we apply the well known separation principle to obtain a solution of the optimal control problem considered.

1. Optimal control under partial observations

Let $H, U, V, \dim(V) < \infty$ be separable real Hilbert spaces. Throughout this section we will assume that the next hypothesis are satisfied:
Throughout this chapter we assume the following hypotheses:
(G1) *U(t,s)* is a strong evolution operator generated by the family $A(t), t \in [0,\infty)$; there exists a sequence $\{A_n\}_{n\in \mathbf{N}} \subset C_s([0,\infty), L(H))$ such that for every $n \in \mathbf{N}$, the family $\{A_n(t)\}_{t\geq 0}$ generates the strong evolution operator $U_n(t,s)$ and $U_n(t,s)x \underset{n\to\infty}{\to} U(t,s)x, x \in H$ uniformly on bounded subsets of Δ. The families $A^*(t), A_n^*(t), n \in \mathbf{N}, t \in [0,\infty)$ satisfy the hypothesis (G1) and we will say that G1 (A^*) holds.
(G2): $B \in C_b(\mathbf{R}_+, L(U,H)), B^* \in C_b(\mathbf{R}_+, L(H,U)),\ C \in C_b(\mathbf{R}_+, L(H,V))$,$C^* \in\ C_b(\mathbf{R}_+, L(V,H))$,$N, G, G^* \in C_b(\mathbf{R}_+, L(H)),\ \overline{V} \in C([0,\infty), L(V)), V(t)$ is nonsingular for all $t \in [0,\infty)$ and $K \in C_b(\mathbf{R}_+, L(U)$ is uniformly positive, that is there exists $\delta > 0$ such that $K(t) \geq \delta I$, for all $t \in \mathbf{R}_+$.
We consider the following stochastic system with control and partial observations

(1) $dy(t) = A(t)y(t)dt + B(t)u(t)dt + G(t)dw(t),$

$$y(0) = y_0 \in L_s^2(H);$$

(2) $dz(t) = C(t)y(t)dt + \overline{V}(t)dv(t), z(0) = 0,$

where $w(t),v(t),t \geq 0$ are real valued Wiener processes and y_0 is Gaussian with mean m_0 and covariance Q_0 and y_0, $w(t),v(t),t \geq 0$ are mutually independent. We also introduce the cost functional

(3) $J(u) = \overline{\lim}_{t\to\infty} \frac{1}{t} E\int_0^t \|N(s)y(s)\|^2 + \langle K(s)u(s), u(s)\rangle ds$

Our problem is to minimize *J(u)* over a class of admissible controls *u(t)*, which are adapted to the σ-algebra generated by the observations $z(s), s \in [0,t]$. We will define the set of admissible controls later. Now, we recall some basic results on filtering theory. Let Z_t be the σ-algebra generated by $z(s), s \in [0,t]$ and let us consider the system

(4) $dy(t) = A(t)y(t)dt + G(t)dw(t), y(0) = y_0$

(5) $dz(t) = Cy(t)dt + V(t)dv(t), z(0) = 0$

The filtering problem consists in estimating the state *y(t)* based on the observation process $z(s), 0 \le s \le t$. The most popular, optimal in the mean square sense, filtering estimator of $y(t)$ is its projection onto $L^2(\Omega, Z_t, P, H)$. This estimator is called *the best global estimator* of y given $z(s), 0 \le s \le t$ and is equal to the conditional expectation $E[y(t) | Z_t]$ [6]. The conditional expectation, as a function of $z(s), s \in [0,t]$, may have too complicated structure. Therefore, especially in engineering applications, it makes sense to use a linear function $\hat{y}(t)$ of $z(s), s \in [0,t]$, minimizing the mean square error $E\|y(t) - \hat{y}(t)\|^2$ in a class of linear filtering estimates. Then we define below the best affine estimator of *y(t)*.

Let $T > 0$ be fixed and let $z \in C([0,T], L^2(\Omega, \mathbf{F}, P, V)), y \in L^2(\Omega, P, H)$. We denote by H_z^t the closure in $L^2(\Omega, P, H)$ of the linear space generated by the set $\{x + L(z(s_j)), x \in H, L \in L(V,H), s_j \in [0,t]\}$. Obviously H_z^t is the analogue of the Hilbert space H_y defined in the time invariant case and it is a Hilbert subspace of $L^2(\Omega, P, H)$. If P_z^t denotes the projection on $L^2(\Omega, P, H)$ upon H_z^t then $P_z^t(y)$ is called *the best affine estimator* of $y \in L^2(\Omega, P, H)$ given $\{z(s), 0 \le s \le t\}$. As in the discrete-time case we note that if we set $\bar{z}(t) = z(t) - E(z(t))$, then $H_z^t = H_{\bar{z}}^t$.

If $H_0 = \{y \in L^2(\Omega, P, H), E(y) = 0\}$ is the Hilbert subspace of $L^2(\Omega, P, H)$ and $H_{0,\bar{z}}^t$ is the closure in $L^2(\Omega, P, H)$ of the linear space generated by the set $\{L(z(s_j)), L \in L(V,H), s_j \in [0,t]\}$ then $P_{0,\bar{z}}^t(y)$ will denote the projection on $L^2(\Omega, P, H)$ upon $H_{0,\bar{z}}^t$. Obviously $H_{0,\bar{z}}^t \subset H_z^t$. Then $P_{0,\bar{z}}^t(y)$ is called the best linear estimator of $L^2(\Omega, P, H)$ *y* given $\{\bar{z}(s), 0 \le s \le t\}$. It is clear that $P_z^t(y) = E(y) + P_{0,\bar{z}}^t(y)$ and consequently the problem of finding the best affine estimator reduces to that of finding the best linear estimator [7]. We note here that the linear estimates are easier to calculate, but unlike the best global estimates they need not always exist.

It is known that if $y \in L^2(\Omega, P, H), z \in L^2(\Omega, P, V)$ are Gaussian random variables and the best linear estimate of *y* given *z* exists, then it coincides with the best global estimate [2].

As in the discrete-time case, the filtering problem for (4), (5) is to find the best affine estimate $\widehat{y}(t)=P_z^t(y), t\geq 0$. It is well known [2], [1], [7] that, because of the Gaussian property of the processes, the best affine estimate of the solution *y(t)* of (4), (5) coincides with the best global estimator $E\left[y(t)\,|\,Z_t\right]$. Moreover, it is the unique mild solution of

$$(6)\, d\hat{y}(t)=A(t)\hat{y}(t)dt+P(t)C^*(t)\left[\overline{V}(t)\overline{V}^*(t)\right]^{-1}d\eta(t),$$

$$(7)\, \hat{y}(0)=E(y_0),$$

where η is the innovation process defined by

$$(8)\, d\eta(t)=dz(t)-C(t)\hat{y}(t)dt$$

and *P(t)* is the covariance of the error process $e(t)=y(t)-\hat{y}(t)$. A rather surprising properties are the following: the innovation process $\eta(t)$ is a finite dimensional Wiener process with the covariance $t\overline{V}(t)\overline{V}(t)^*$ relative to the σ -algebra Z_t; $H^t_{0,\bar{z}}=H^t_{0,\bar{\eta}}$ [2], [7]. Also it is known that the covariance of the error process is the unique mild solution of the following Riccati equation [2], [5]

$$(9)\quad P'(t)=A(t)P(t)+P(t)A^*(t)+G(t)G^*(t)-P(t)C^*(t)\left[\overline{V}(t)\overline{V}^*(t)\right]^{-1}C(t)P(t)$$

$$(10)\quad P(0)=P_0=\mathrm{cov}(e_0,e_0), e_0=y_0-\overline{y}_0.$$

The next result shows that, under stabilizability conditions, the above Riccati equation has a bounded on **R**+ solution. We recall here that the pair $\{A^*\ :\ C^*\}$ is *stabilizable* iff there exists $F\in C_b([0,\infty),L(H,V))$ such that $\{A^*+C^*F\}$ is uniformly exponentially stable.

Theorem 1 *[4] Assume* $\{A^*\ :\ C^*\}$ *is stabilizable. The solution P(t) of (9)-(10) is bounded on* **R**+.

Returning to the quadratic control problem, we now define the class of admissible controls $\mathcal{U}$. Let Γ_t be the σ - algebra generated by $\eta(s), s\in[0,t]$, where η is the innovation process given by (8).

Following [4], [5], [7] we that $\mathcal{U}$ is the set of all controls $u\in L^2_{loc}([0,\infty)\times\Omega,U)$ that are $\mathbf{F}_t$ - adapted and satisfy the conditions $u(t)\in L^2(\Omega,Z_t,P,H)\cap L^2(\Omega,\Gamma_t,P,H)$ for almost all *t,*

$$\overline{\lim}_{t\to\infty}\tfrac{1}{t}E\int_0^t\|u(s)\|^2ds<\infty,\quad \sup_{t\geq 0}E\|y(s)\|^2<\infty\ .$$

If $u\in\mathcal{U}$, we associate to the system (6), (7) the following control system

$$(11)\, d\hat{y}(t)=A(t)\hat{y}(t)dt+B(t)u(t)dt+P(t)C^*(t)\left[\overline{V}(t)\overline{V}^*(t)\right]^{-1}d\eta(t),$$

$$(12)\, \hat{y}(0)=E(y_0)$$

Denoting by $\hat{y}_u$ the solution of the above system, then $\hat{y}-\hat{y}_u$ is the unique solution of the system

(13) $dx(t) = A(t)x(t)dt + B(t)u(t)dt, x(0) = 0.$

Choosing the same control u, we consider the unique mild solutions y_u and y of (4) and (1) respectively, and we see that y- y_u is also the unique mild solution (13). Therefore $y - y_u = \hat{y} - \hat{y}_u$ and $y_u(t) - \hat{y}_u(t) = e(t)$. An easy computation shows that

$$E\|N(t)y_u(t)\|^2 = TrN(t)\operatorname{cov}(e(t),e(t))N^*(t) + 2E\langle N(t)\hat{y}_u(t), e(t)\rangle + E\|N(t)\hat{y}_u(t)\|^2 .$$

If $U(t,s)$ is the evolution operator generated by A then

$$\hat{y}_u(t) = U(t,r)\bar{y}_0 + \int_0^t U(t,r)B(r)u(r)dr +$$

$$\int_0^t U(t,r)P(r)C^*(r)\left[V(r)V^*(r)\right]^{-1} d\eta(r) = T_1 + T_2 + T_3.$$

We note that $T_3 \in H^t_{0,\bar{\eta}}$ and $H^t_{0,\bar{\eta}} = H^t_{0,\bar{z}} \subset H^t_z$. Since $e(t) \in \left(H^t_z\right)^{\perp}$ we deduce $E\langle T_3, e(t)\rangle = 0$. Also $T_1 \in H^t_z$ and $E\langle T_1, e(t)\rangle = 0.$ Further $\int_0^t U(t,r)B(r)u(r)dr$ is Z_t measurable and therefore $E\langle T_2, e(t)\rangle = E\left(E\left[\langle T_2, e(t)\rangle |_{Z_t}\right]\right) = E\left[\left\langle T_2, E\left[e(t)|_{Z_t}\right]\right\rangle\right] = 0$. We just have proved that $E\langle \hat{y}_u(t), e(t)\rangle = 0$. Hence $2E\langle N(t)\hat{y}_u(t), e(t)\rangle = 0$ and

(14) $E\|N(t)y_u(t)\|^2 = TrN(t)P(t)N^*(t) + E\|N(t)\hat{y}_u(t)\|^2,$

where $P(t)$ is the solution of (9)-(10). Let now introduce the cost functional

(15) $\hat{J}(u) = \overline{\lim}_{t\to\infty} \frac{1}{t} E\int_0^t \|N(s)\hat{y}_u(s)\|^2 + \langle K(s)u(s), u(s)\rangle ds.$

If $P(t)$ is bounded on $\mathbf{R}_+$ we deduce by (14)

(16) $J(u) = \hat{J}(u) + \overline{\lim}_{t\to\infty} \frac{1}{t}\int_0^t TrN(s)P(s)N^*(s)ds.$

Now it is clear that the optimal control in the class $\mathcal{U}$, which minimize (15) subject to (11)-(12) is also optimal for our control problem with partial observations (separation principle).
Therefore, we can solve the linear quadratic control problem with complete observations (11)-(12), (15). Let us consider the Riccati equation

(17) $R'(t) + A^*(t)R(t) + R(t)A(t)$

$+N(t)N^*(t) - R(t)B(t)\left[K(t)\right]^{-1} B^*(t)R(t) = 0.$

We recall here that a global solution $R(t)$ of *(17) is stabilizing for* $\{A : B\}$ iff $\{A - B\left[K(t)\right]^{-1} B^*R\}$ is uniformly exponentially stable. The following result is known:

Proposition 2 [4,8] *Assume* $\{A : B\}$ *is stabilizable and* $\{A; N\}$ *is either uniformly* *observable or detectable. Then the Riccati equation (17) has a unique nonnegative, bounded on* $\boldsymbol{R}_+$ *and stabilizing solution* R(t).

Before to prove the main result of this section it is useful to introduce the following approximating systems

(18) $R_n'(t)+A_n^*(t)R_n(t)+R_n(t)A_n(t)+N(t)N^*(t)-R_n(t)B(t)\left[K(t)\right]^{-1}B^*(t)R_n(t)=0,$

(19) $d\hat{y}_n(t)=A_n(t)\hat{y}_n(t)dt+B(t)u(t)dt+P(t)C^*(t)\left[\overline{V}(t)\overline{V}^*(t)\right]^{-1}d\eta(t),\hat{y}_n(0)=\overline{y}_0,n\in\mathbf{N}.$

Let $Q \in L^+(H)$. It is known [5] that (18) (respectively *(17))* with the final condition $R_n(T) = Q$ (respectively $R(T) = Q$ *)* has a unique classical (respectively mild) solution $R_n(T,s;Q)$. Moreover, $Q_n(T,s;R)x \underset{n\to\infty}{\to} Q(T,s;R)x$, uniformly on *[0,T]*. Analogously, the system (19)-(20) has unique strong solution $\hat{y}_u^{(n)}(t)$, which converges to the unique mild solution $\hat{y}_u(t)$ of (11)-(12) uniformly on *[0,T]*.

Theorem 3 *Assume that* $\{A^* : C^*\}$ *is stabilizable and the hypotheses of the above proposition are satisfied. Then the optimal control for the problem (11)-(15) is given by the feedback law* $\tilde{u}(t)=-\left[K(t)\right]^{-1}B^*(t)R(t)\hat{y}_{\tilde{u}}(t),$ *where R is the unique bounded and nonnegative solution of (17). The optimal cost is*

(20) $\hat{J}(\tilde{u})=\overline{\lim}_{t\to\infty}\frac{1}{t}\int_0^t TrR(s)P(s)C^*(s)\left[\overline{V}(s)\overline{V}^*(s)\right]^{-1}C(s)P(s)ds.$

Moreover, the control $\tilde{u}$ *is also optimal for the control problem with partial observations (1)-(2), (3) and*

$$\min_{u\in\mathbf{U}} J(u)=J(\tilde{u})=\hat{J}(\tilde{u})+\overline{\lim}_{t\to\infty}\frac{1}{t}\int_0^t TrN(s)P(s)N^*(s)ds.$$

Proof. Let *R(t)* be the unique global solution of *(17).* If $R_n(s)=R_n(t,s,R(t))$ is the solution of (18), then we consider the function $F_n(s,x)=\langle R_n(s)x,x\rangle$ which is continuous together its partial derivatives $F_{n,s}$, $F_{n,x}$, $F_{n,xx}$ on $[0,\infty)\times H$. Let $u\in\mathcal{U}$ and $\hat{y}_u(s)$ be its response. Using Ito's formula for $F_n(s,x)$ and the strong solution $\hat{y}_u^{(n)}(s)$ of (19) and taking expectations we get

$$E\left\langle R_n(t)\hat{y}_u^{(n)}(t),\hat{y}_u^{(n)}(t)\right\rangle-\left\langle R_n(0)\overline{y}_0,\overline{y}_0\right\rangle=-E\int_0^t\left\|N(s)\hat{y}_u^{(n)}(s)\right\|^2+\left\langle K(s)u(s),u(s)\right\rangle ds$$

$$+E\int_0^t\left\|K(s)^{1/2}\left[u(s)+\left[K(s)\right]^{-1}B^*(s)R_n(s)\hat{y}_u^{(n)}(u)\right]\right\|^2 ds$$

$$+\int_0^t TrR_n(s)P(s)C^*(s)\left[\overline{V}(s)\overline{V}^*(s)\right]^{-1}C(s)P(s)ds.$$

We first note that $R_n(s) \underset{n\to\infty}{\to} R(s)$ uniformly on $[0,t]$. Letting $n\to\infty$ in the above relation and using the Dominated Convergence Theorem of Lebesgue we get

$$E\left\langle R(t)\hat{y}_u(t),\hat{y}_u(t)\right\rangle - \left\langle R(0)\bar{y}_0,\bar{y}_0\right\rangle = -E\int_0^t \left\|N(s)\hat{y}_u(s)\right\|^2 + \left\langle K(s)u(s),u(s)\right\rangle ds$$

$$+E\int_0^t \left\|K(s)^{1/2}\left[u(s)+[K(s)]^{-1}B^*(s)R(s)\hat{y}_u(u)\right]\right\|^2 ds$$

$$+\int_0^t TrR(s)P(s)C^*(s)\left[\bar{V}(s)\bar{V}^*(s)\right]^{-1}C(s)P(s)ds.$$

We make use of the boundedness of $R(t)$ and we easily deduce that

$$\hat{J}(u) = \overline{\lim}_{t\to\infty}\frac{1}{t}\int_0^t TrR(s)P(s)C^*(s)\left[\bar{V}(s)\bar{V}^*(s)\right]^{-1}C(s)P(s)ds$$

$$+\overline{\lim}_{t\to\infty}\frac{1}{t}E\int_0^t \left\|K(s)^{1/2}\left[u(s)+[K(s)]^{-1}B^*(s)R(s)\hat{y}_u(u)\right]\right\|^2 ds.$$

It is not difficult to see that $\tilde{u}(s) = -[K(s)]^{-1}B^*(s)R(s)\hat{y}_u(u), s\geq 0$ belongs to $\mathcal{U}$ and minimize $\hat{J}(u)$ in the class of admissible controls. Obviously, (20) follows by (16). The proof is complete.

BIBLIOGRAPHY

[1] Bensoussan A., Filtrage Optimal des Systemes Lineaires, Dunod, Paris, 1971.
[2] Curtain R., Pritchard J., Infinite dimensional linear systems theory, Springer Verlag, Berlin, 1978.
[3] Da Prato G.,, Quelques resultats d'existence, unicite et regularite pour une probleme de la theorie du controle, J. Math. pures et appl. 52, 1973, pp. 353-375.
[4] Da Prato G., Ichikawa A., Quadratic Control for Linear Time-Varying Systems, SIAM.J.Control and Optimization, vol 28, No 2, 1990, 359-381
[5] Da Prato G., Ichikawa A., Quadratic control for linear periodic systems, Appl. Math. Optim., 18(1988), pp. 39-66.
[6] Davis M. H. A., Linear Estimation and Stochastic Control, Chapman and Hall, London, 1977.
[7] Ichikawa A., Filtering and control of stochastic differential equations with unbounded coefficients, Stochastic analysis and applications 4(2)(1986), pp. 187-211
[8] Ungureanu, V. M, Optimal control of linear stochastic evolution equations in Hilbert spaces and uniform observability, submitted.