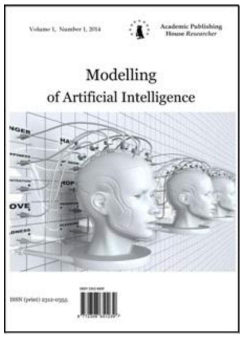
Published in the Russian Federation
Modeling of Artificial Intelligence
Has been issued since 2014.
ISSN: 2312-0355
Vol. 2, No. 2, pp. 72-76, 2014

DOI: 10.13187/mai.2014.2.72
www.ejournal11.com



UDC 519.2

Distribution of Cross-Sections of Convex Bodies

Victor K. Ohanyan

Yerevan State University, Armenia
Yerevan, 0025, 1 Alex Manoogian
Dr. (Mathematics), Professor
E-mail: victo@aua.am

Abstract. The paper continues the investigations in [1]. We consider the problem of recognition of convex bodies in n-dimensional Euclidean space by k-flats sections, in particular for n=3 by linear or planar sections. This problem is equivalent to recognition of convex bodies by orientation-dependent chord length distributions (for linear sections) and orientation-dependent area distributions (for planar sections). The main goal of the article is to enlarge the class of bodies for which the form of the orientation dependent chord length distribution function and the cross-section area distribution function are known.
Keywords: Bounded convex domain; orientation dependent chord length distribution; cross-section area distribution.

Mathematics Subject Classification 2010: 60D05; 52A22; 53C65

Introduction

Let **G** be the space of all lines in the Euclidean plane R^2, $g \in \boldsymbol{G}$, $(p, \varphi) =$ the polar coordinates of the foot of the perpendicular to g from the origin; $p \geq 0$, $\varphi \in S^1$ (S^1 is the unit circle on the plane with center at the origin).

Let $\mu(\cdot)$ stands for a locally finite measure on **G** invariant with respect to the group of all Euclidean motions (translations and rotations). It is well known that the element of the measure up to a constant factor has the following form (see [2], [4]): $\mu(dg) = dpd\varphi$, where dp is one dimensional Lebesgue measure, while $d\varphi$ is the uniform measure on the unit circle. For a closed bounded convex domain D and a point O from the interior of D we denote by $S_D(\varphi)$ the support function in direction $\varphi \in S^1$, i.e. the maximum value of p with respect to the point O for which the line $g(p, \varphi)$ intersects D:

$$S_D(\varphi) = \max\{p \in \boldsymbol{R}^+ : \ g(p, \varphi) \cap D \neq \emptyset\},$$

where $\boldsymbol{R}^+$ is the set of nonnegative real numbers. When D has a center of symmetry we usually take the point O in center of symmetry. The function

$$b_D(\varphi) = S_D(\varphi) + S_D(\varphi + \pi)$$

is known as the breadth function of D, which does not depend on the choice of the point O. For the bounded convex domain D the chord length distribution function in direction $F_D(x, \varphi)$ is defined as the probability of having chord $\chi(g) = g \cap D$ with length less than or equal to x in the bundle of lines g with coordinates (p, φ) or $(p, \varphi + \pi)$. A random line which is perpendicular to the direction φ and intersects D has an intersection point (denote that point by y) with the line which is parallel to the direction φ and passes through the origin. The intersection point y is uniformly distributed on the segment $[-S_D(\varphi+\pi),\ S_D(\varphi)]$ or $[0, b_D(\varphi)]$. Thus,

$$F_D(x, \varphi) = \frac{L_1\{y: L_1(g_y(\varphi) \cap D) \le x\}}{S_D(\varphi) + S_D(\varphi+\pi)} = \frac{L_1\left\{y: \chi\left(g_y(\varphi)\right) \le x\right\}}{b_D(\varphi)},$$

where $g_y(\varphi)$ is the line which is perpendicular to the direction φ and intersects $[-S_D(\varphi+\pi), S_D(\varphi)]$ at point y.

Since 1930s in line with the problem of finding forms of these functions for special domains the problem of determining bounded convex domain by its chord length distributions (mixed and orientation dependent) is studied. In [3] G. Matheron introduced the concept of the covariogram formulating a hypothesis that the covariogram of a bounded convex domain determines that domain in the subclass of all bounded convex domains up to translations and reflections.

Let $\boldsymbol{R}^n$ be the n dimensional Euclidean space, $D \subset \boldsymbol{R}^n$ be a bounded convex domain, S^{n-1} be $(n-1)$-dimensional unit sphere centered at the origin and $L_n(\cdot)$ be n-dimensional Lebesgue measure in $\boldsymbol{R}^n$. The function

$C_D(x) = L_n\big(D \cap (D + x)\big)$ for any $x \in \boldsymbol{R}^n$

is called the covariogram of convex body D. $(D + x) = \{y + x:\ y \in D\}$.

C_D is invariant with respect to translations and reflection. G. Matheron (see [3]) showed that for every $t > 0$ and $\varphi \in S^{n-1}$

$$\frac{\partial C_D(t\varphi)}{\partial t} = -L_{n-1}(\left\{y \in \varphi^{\perp}:\ L_1\left(D \cap \left(l_\varphi + y\right)\right) \ge t\right\}), \qquad (1)$$

where $l_\varphi + y$ denotes the line parallel to the direction φ through y and $\varphi^{\perp}$ denotes the orthogonal complement of φ.

It is not difficult to verify that for $n = 2$ (1) is equivalent to

$$-\frac{\partial C_D(t\varphi)}{\partial t} = \left(1 - F_D\left(t,\ \varphi + \frac{\pi}{2}\right)\right) b_D(\varphi + \pi/2),$$

i.e. the problem of determining bounded convex domain by its covariogram is equivalent to that of determining it by its orientation dependent chord length distribution.

In [6] G. Bianchi and G. Averkov confirmed Matheron's conjecture for $n = 2$. Bianchi has also proved that for $n \ge 4$ the hypothesis is false (see [7]). For $n = 3$ the problem is open.

All the above mentioned results and facts indicate that orientation dependent distribution function and covariogram are key concepts while studying bounded convex domains and investigation of these functions may give us results which can be generalized for all bounded convex domains or for some subclasses of such domains.

Results

In the three dimensional space two types of orientations dependent distributions can be considered. First is the probability that the random chord generated by intersection of the spatial line with the domain has length less than or equal to given number. In the second case random planes and their intersections with the domain are observed.

Denote by $\boldsymbol{E}$ the space of all planes in $\boldsymbol{R}^3$. Each $e \in \boldsymbol{E}$ can be introduced by spatial direction $\omega = (\varphi, \theta) \in S^2$ ($\varphi \in S^1$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$) and by the distance p of the plane e from the origin. Denote by $A(e)$ the area of the cross-section $D \cap e$ and by $S_D(\omega)$ the support function in direction ω with respect to a certain point O from the interior of D:

$S_D(\omega) = \max\{p: \ e(p,\omega) \cap D \neq \emptyset\}$.

A random plane with direction ω and intersecting D has an intersection point y with the line which is parallel to direction ω and passes through the origin. The intersection point y is uniformly distributed over the interval $[0, b_D(\omega)]$, where $b_D(\omega)$ is the breadth function in direction ω ($b_D(\omega) = S_D(\omega) + S_D(-\omega)$), which does not depend on the choice of the point O. We can identify the points of the interval $[0, b_D(\omega)]$ and the planes which intersect D and have direction ω. Thus,

$$F_D(x,\omega) = \frac{L_1\left\{y: A\left(e_y(\omega)\right) \leq x\right\}}{b_D(\omega)}$$

where $e_y(\omega)$ is the plane with direction ω intersection $[0, b_D(\omega)]$ at point y.

In the paper [8] is proved that for any finite subset A from S^1, there are two non-congruent domains for which orientation-dependent chord length distribution functions coincide for any direction from A. Moreover, in [8] explicit forms for covariogram and orientation-dependent chord length distribution function $F_D(y,\varphi)$ for arbitrary triangle are obtained. Finally, if we have the values of $F_D(y,\varphi)$ for everywhere dense set from S^1, then we can uniquely recognized the triangle with respect to translations and reflections (see [8] – [13]). Thus, investigating covariograms of convex bodies we investigate the geometric properties of them. Find the explicit form of covariogram for subclass of convex domains: Using the explicit form of covariogram and Materon's formula find $F_D(y,\varphi)$ for the corresponding subclass of convex bodies (see [1]). Construct algorithms to reconstruct convex body by its covariogram for finite number of directions (see [6], [7]) (the same problem for chord length distribution function $F_D(y,\varphi)$ (see [5], [9]) in finite number of directions has negative solution (see [8]). The explicit forms for Covariogram of a triangle and for $F_D(y,\varphi)$ follows that their can be written in the form (see [8] and [10]):

$$C(D, x\varphi) = L_2(D)\left(1 - \frac{x}{t_{max}(\varphi)}\right)^2$$

$$F_D(y,\varphi) = \frac{x}{t_{max}(\varphi)},$$

where $t_{max}(\varphi)$ is the maximal chord length in direction φ.

It is not difficult to note both for an ellipse and a triangle that the function through which the covariogram depends on $\frac{t}{\chi_{\max}(\varphi)}$ is an integral of the boundary of the domain up to a constant factor.

Basing on this facts we suggest the following hypothesis:

Does (or in what cases) the covariogram $C_D(t\,\varphi)$ of a bounded convex domain D depend on φ through the function $\chi_{\max}(\varphi)$, i.e

$$C_D(t\,\varphi) = C_D(t, \chi_{\max}(\varphi))?$$

Is (or in what cases) the covariogram $C_D(t\,\varphi)$ of a bounded convex domain D a function depending only on $\frac{t}{\chi_{\max}(\varphi)}$, i.e.

$$C_D(t\,\varphi) = C_D\left(\frac{t}{\chi_{\max}(\varphi)}\right)?$$

Let $y = f(x)$ is the equation of the boundary of a bounded convex domain D around the point of intersection of ∂D with the ray φ.

Is it possible (or in what cases it is possible) to introduce $C_D(t\ \varphi)$ in the form

$$C_D(t\ \varphi) = A + B\int_0^{C\frac{t}{x_{\max}(\varphi)}} f(x)dx\,,$$

where A, B and C are constants?

In the last 2 years our group has obtained important results to calculate explicit forms of chord length distribution functions $F_D(y,\varphi)$ for different convex bodies. In particular, if D is a lens (this problem has important applications in crystallography, [14]). An algorithm for calculation of values of $F_D(y)$ for any bounded convex polygon is constructed.

The program for effective implementation of this algorithm is constructed. For any triangle explicit forms of $F_D(y)$ and $F_D(y,\varphi)$ are obtained (see [8]—[13]).

We obtained the following results:

1. Chord length distribution function in direction φ for a regular polygon,
2. Chord length distribution function in direction φ for an ellipse,
3. Cross-section area distribution function in a direction for an ellipsoid,
4. Cross-section area distribution function in a direction for a cylinder,
5. The covariogram of a parallelogram and cylinder.

References:

1. Ohanyan V.K. Tomography of bounded convex domains, European Researcher. vol. 48, no. 5 (1), pp. 1092—1096, 2013.

2. L.A. Santalo, Integral Geometry and Geometric Probability (Addision-Wesley, Reading, MA, 2004).

3. G. Matheron, Random Sets and Integral Geometry (New York, Wiley, 1975).

4. R. V. Ambartzumian, Factorization Calculus and Geometric Probability (Cambridge University Press, Cambridge, 1990).

5. R. Schneider, W. Weil, Stochastic and Integral Geometry (Springer, Berlin-Heidelberg, 2008).

6. G. Bianchi, G. Averkov, ``Confirmation of Matheron's Conjecture on the covariogram of a planar convex body'', Journal of the European Mathematical Society, 11, 1187--1202, 2009.

7. G. Bianchi, ``Matheron's conjecture for the covariogram problem'', J. London Math. Soc., 71 (1), 203--220, 2005.

8. A. G. Gasparyan, V. K. Ohanyan, ``Recognition of triangles by covariogram'', Journal of Contemporary Mathematical Analysis (Armenian Academy of sciences), 48 (3), 110--122, 2013.

9. R. De-Lin, Topics in Integral Geometry. (Utopia press, Singapore, 1994).

10. A. G. Gasparyan and V. K. Ohanyan, ``Recognition of triangles by covariogram'', Journal of Contemporary Mathematical Analysis (Armenian Academy of sciences), vol. 48, no. 3, 110-122, 2013.

11. H. S. Harutyunyan and V. K. Ohanyan, Linear and planar sections of the convex domains'', Journal of Contemporary Mathematical Analysis (Armenian Academy of sciences), vol. 49, no. 3, 2014.

12. A. G. Gasparyan and V. K. Ohanyan, ``Covariogram of a parallelogram'', Journal of Contemporary Mathematical Analysis (Armenian Academy of sciences), vol. 49, no. 4, 2014.

13. N. G. Aharonyan, H. S. Haruttyunyan, and V. K. Ohanyan, ``Random copy of a segment within a convex domain'', Journal of Contemporary Mathematical Analysis (Armenian Academy of sciences), vol. 45, no. 5, 348-356, 2010.

14. W. Gille, N. G. Aharonyan and H. S. Harutyunyan, ``Chord length distribution of pentagonal and hexagonal rods: relation to small--angle scattering'', Journal of Applied Crystallography, vol. 42, pp. 326--328, 2009.

УДК 519.2

Распределения поперечных сечений выпуклых тел

Виктор Кароевич Оганян

Ереванский государственный университет, Армения
0025, Ереван, ул. Алека Манукяна, 1
Д.ф.-м.н., профессор
E-mail: victo@aua.am

Аннотация. Настоящая работа продолжает исследования начатые в [1]. Рассматривается проблема восстановления выпуклых тел в n-мерном евклидовом пространстве по сечениям k-плоскостями. В частности, для n=3 линейными и плоскими сечениями. Эта задача эквивалентна восстановлению выпуклых тел по зависящим от направления распределениям длин хорд (для линейных сечений) и распределениям площади (для плоских поперечных сечений). Основной целью работы является расширение класса тел, для которых известны формы зависящих от направления функции распределения длины хорды и функции распределения площади поперечного сечения.

Ключевые слова: Ограниченное выпуклое тело; зависящее от направления распределение длины хорды; площади поперечных сечений.