SECTION 2. Applied mathematics. Mathematical modeling.

Peretiatko Anastasiia Sergeevna
assistant
Specialized computer systems department
Ukrainian State University of Chemical Technology
Dnipropetrovsk, Ukraine

USING SEMIDEFINITE SIMPLEX METHOD FOR SOLVING SEMIDEFINITE PROBLEMS

Abstract: *Semidefinite optimization is relatively a new field of researches. It finds a lot of applications in combinatorial optimization, computational geometry and network theory. Over the last years applications of semidefinite optimization are continuously expanded. We can find exact or approximate solution of many NP-hard problems by using semidefinite relaxation. In this paper we use a generalization of simplex-method for solving semidefinite problems. The main idea of this method is to use the approximation of the cone of semidefinite matrices by the sum of one-rank matrices. In this way we replace the original objective function by a linear combination of one-rank matrices. A lot of numerical experiments were performed and the findings are very encouraging.*

Key words: *Semidefinite optimization, Semidefinite programming, Semidefinite simplex method.*

1. Introduction. For the last 15 years semidefinite optimization has been an important subject of theoretical and practical researches. Its advance lies in the fact that we can efficiently solve a lot of NP-hard problems. Many applications of computational geometry, quadratic, combinatorial and polynomial optimization, network and optimal control theory can be efficiently solved by semidefinite relaxation [1].

There are a lot of methods for solving semidefinite problems. Primal-dual interior point method [2, 3] is widely used. The condition of positive semidefiniteness can be obtained only algorithmically, that considerably complicates the development of efficient algorithms for semidefinite programming. Interior point methods for linear programming problems were extended for solving semidefinite programming problems. This class of methods demonstrates good results for problems of middle dimension. Other methods for solving semidefinite problems didn't obtain a wide circulation. Development of more efficient methods for solving semidefinite problems is continuing [4, 5]. In paper [6] a new semidefinite simplex method for solving this class of problems was proposed.

In semidefinite problem we search the minimum point that is a semidefinite matrix. A set of such matrices forms a convex cone in the space of all matrices. Generatrices of semidefinite cone are the infinite number of one-rank matrices, such that each two of them are neighboring (the sum of generatrices is also a generatrix). This fact we will use for building a local approximation of a semidefinite cone by a polyhedral cone.

2. SDP Formulation. Consider the following semidefinite problem:

$$\min\{C \bullet X \mid A_i \bullet X = b_i, X \succeq 0, i = 1,\ldots,m\}, \quad (1)$$

where X is semidefinite matrix $(n \times n)$, C and A_i are symmetric matrices, and $C \bullet X = \sum\sum c_{ij} x_{ij}$.

This problem has the dual

$$\max\{b^T y \mid \sum_{i=1}^{m} A_i y + Z = C, Z\succeq 0\},$$

where Z is also semidefinite matrix $(n \times n)$.

3. Semidefinite Simplex method. Let's consider primal semidefinite problem (1). It is well-known that any semidefinite matrix can be written as the sum of semidefinite matrices of rank one. [7, p.542]. They are formed by vectors $x^i = (x^1, \ldots, x^n)$, where initial vector components of x^i are -1, 0, 1. One-rank matrix equals xx^T. There are a great number of all one-rank matrices. Let X_j denote this matrices. We seek for the solution of (1) as $X = \sum \alpha_j X_j$, where the number of summands is greater than m and $\alpha \geq 0$. Then the problem (1) can be formulated as follows:

$$\min\{C \bullet \sum \alpha_j X_j \mid A_i \bullet \sum \alpha_j X_j = b_i,\ i = 1,\ldots,m,\ \alpha \geq 0\},$$

or

$$\min\{\sum \alpha_j C \bullet X_j \mid \sum \alpha_j A_i \bullet X_j = b_i, i = 1,\ldots,m, \alpha \geq 0\}. \qquad (2)$$

The problem (2) is linear programming one that can be solved by simplex-method [6]. Its solution α^* defines approximate solution for (1)

$$X = \sum \alpha_j^* X_j.$$

In order to continue the minimization process it is necessary to add a new semidefinite one-rank matrix to the basis, such that estimate in modified line of objective function is negative. If there isn't correction with negative value in objective function line, then current solution of problem (2) provides solution of problem (1).

Let B denote the matrix of basic elements of the optimal solution (2). Then elements of new k-th matrix column in (2) are equal to

$$B^{-1} A_i \bullet x_k x_k^T,$$

and the row of objective function is equal to

$$C \bullet x_k x_k^T - \sum C \bullet x_k x_k^T B^{-1} A_j \bullet x_k x_k^T$$

or

$$(C - \sum C \bullet x_k x_k^T B^{-1} A_j) \bullet x_k x_k^T.$$

Let $Q = C - \sum C \bullet x_k x_k^T B^{-1} A_j$, then

$$Q \bullet x_k x_k^T = x_k^T Q x_k$$

and this expression should be minimal. If matrix Q is positive definite, than the value of the objective function (2) can not be reduced and the current solution is optimal for the problem (1). For minimizing $x_k^T Q x_k$ we have to find the solution of the quadratic problem

$$\min\{x^T Q x \mid \|x\|^2 = q\}, \qquad (3)$$

for arbitrary $q > 0$. It is well-known that problem (3) is effectively solved [8]. Let's use the method of quadratic regularization for it's solving [9]. We can rewrite (3) as follows:

$$\min\{x_{n+1} \mid x^T Q x + s \leq x_{n+1}, \|x\|^2 = q\}, \qquad (4)$$

where s is chosen such as $x^{*T} Q x^* + s \geq \|x^*\|^2$ (x^* is solution of problem (1)). Then by using transformation $z = Px$, we rewrite (4) as follows

$$\min\{\|z\|^2 \mid z^T Q z + s \leq \|z\|^2,\ \|z\|^2 = q\}, \qquad (5)$$

where P is matrix $(n+1)\times(n+1)$ and equals to

$$\begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ z_1 & z_2 & ... & z_{n+1} \end{pmatrix}.$$

Quadratic regularization method lies in transformation of (5) to one-parameter problems

$$\min\{ \|z\|^2 \mid z^T Qz + (r-1)\|z\|^2 + s \le d,\ r\|z\|^2 = d,\ \|z\|^2 = q\}, \quad (6)$$

where parameter $r > 0$ is minimal and such that matrix $Q^* = z^T Qz + (r-1)I$ is semidefinite, d needs to be evaluated. The solution of (6) can be found from the solution of the problem

$$\max\{ \|x\|^2 \mid x^T Q^* x = 1\}, \quad (7)$$

that is equivalent to searching the eigenvector of a semidefinite matrix Q^*.

Let x^* be a solution of (7). Than Q is semidefinite if $x^{*T} Qx^* > 0$. In this case the problem (1) is solved; otherwise we add a new column $x^* x^{*T}$ in (2) and again use the simplex method to solve the updated problem (1).

If the feasible set of (2) is empty than the method of false basis is used for finding the initial basis.

So when we solve (2) and the matrix Q is negative definite on some step than the value of the objective function decreases and bounded below by the solution of the problem. That's why semidefinite simplex method converges to the solution of the problem (1).

4. Numerical experiments. We implement two methods: semidefinite simplex method and infeasible interior point method [10]. The algorithms were implemented in VBA for Excel. Semidefinite simplex method showed good results in solving semidefinite problems. Its main advantage before interior point methods is that simplex method doesn't need an equality of primal and dual objective functions.

Consider a small semidefinite optimization problem:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix},\ b = \begin{pmatrix} 0 \\ 10 \end{pmatrix}.$$

Its primal and dual objective functions are not equal. Interior point methods can't solve such problem. Simplex method found an optimal solution

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

5. Conclusions. We can conclude that semidefinite simplex-method has some advantages over interior point methods. Let's list these advantages:

1.The dimension of the problem in simplex method is equal to $(n+1)n/2$, and for interior point methods – $m+(n+1)n$.

2.Simplex method solves a wide variety of problems, because equality of primal objective function and dual objective function are not necessary.

3.Simplex method is not sensitive to the choice of the initial point, while for interior point methods the initial point must be feasible (there are modifications of this method for infeasible interior point [10]).

4.Interior point methods find the approximate solution of the problem (1).

References:

1. Vandenberghe L. Semidefinite programming / Lieven Vandenberghe, Stephan Boyd // SIAM Review. – 1996. – vol. 38. – P. 49–95.

2. Todd M. J. Semidefinite optimization / M. J. Todd // Acta Numerica. – 2001. – vol. 10. – P. 515–560.

3. Todd M. J. On the Nesterov-Todd direction semidefinite programming / M. J. Todd, K. C. Toh, R. H. Tutuncu // SIAM J. on Optim. – 1998. – vol. 8. – P. 769-796.

4. Malick J. Regularization methods for semidefinite programming / J. Malick, J. Povh, F. Rendl, A. Wiegele // SIAM J. on Optim. – 2009. – vol. 20, 1. – P. 336-356.

5. Helmberg C. A spectral bundle method for semidefinite programming / C. Helmberg, F. Rendl // SIAM J. Optim. – 2000. – vol.10. – P. 673–696.

6. Косолап А.И. Обобщение симплекс-метода для решения задач полуопределенной оптимизации / А.И. Косолап // Математичне та компютерне моделювання. – 2010. С. 99–106.

7. Horn R. Matrix analysis / R. Horn, C. Jonson. – Moscow: Mir. – 1989. – 656 p.

8. Fortin C. Computing the local minimizers of a large and sparse trust-region subproblem / C. Fortin. – Montreal : McGill University. – 2004. – 149 p.

9. Косолап А.И. Метод квадратической регуляризации для решения систем нелинейных уравнений / А.И. Косолап. – Журнал вычислительной и прикладной математики. – 2010, №4. – С. 44-50.

10. Roumili H. Infeasible Interior Point Method for Semidefinite Programs / H. Roumili, A. Keraghel, A. Yassine // Applied Mathematical Sciences. – 2007. – 10 p.