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A COMMON FIXED POINT THEOREM OF COMPATIBLE MAPPING OF TYPE (A-1) IN COMPLETE FUZZY METRIC SPACE

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ABSTRACT

In this paper we prove some common fixed point theorems for compatible mappings of type (A-1) in complete fuzzy metric space our result improves the result of Khan, M.S. [8].

KEYWORDS: Compatible Mappings, Compatible Mappings of Type (A), Compatible Mappings of Type (A-1), Common Fixed Point, Complete Fuzzy Metric Space, Fuzzy Metric Space

INTRODUCTION

The first important result in the theory of fixed point of compatible mapping was obtained by Gerald Jungck in 1986[6] as a generalization of commuting mapping. In 1993 Jungck and Cho [7] introduced the concept of, Compatible mappings of type (A) by generalizing the definition of weakly uniformly contraction maps. Pathak and Khan [12] introduced the concept of type A-compatible and S-compatible by splitting the definition of compatible mapping of type (A).Pathak et.al. [8] renamed A-compatible and S-compatible as compatible mappings of and type(A-1) and compatible mappings of type(A-2) respectively and introduced it in fuzzy metric space.

Zadeh [16] introduced the concept of fuzzy sets. The idea of fuzzy metric space was introduced by Kramosil and Michalek [11] which was modified by George and Veernmani [2, 3]. Singh B. and M.S. Chauhan [14] introduced the concept of compatibility in fuzzy metic space and proved some common fixed point theorems in fuzzy metric spaces in the sense of George and Veermani with continuous t-norm $*$ defined by $a*b = \min\{a, b\}$ for all $a, b \in [0,1]$.

The aim of the paper is to prove some common fixed point theorems of compatible mappings of type (A-1). These results modify and extend the result in [8, 12, 15].

PRELIMINARIES

Definition 2.1[13] A Binary Operation*: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t-norm if, it satisfies the following conditions:

- *is associative and commutative
- *is continuous

- $a * 1 = a$, for all $a \in [0, 1]$
- $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all a, b, c, d in [0, 1]

Definition 2.2[2]: 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary (non-empty), * is continuous t-norm, and M is a Fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

- $M(x, y, t) > 0$.
- $M(x, y, t) = 1$ if and only if $x = y$.
- $M(x, y, t) = M(y, x, t)$.
- $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
- $M(x, y,): (0, \infty) \rightarrow [0,1]$ is continuous.
- For all $x, y, z \in X$ and $s, t > 0$.

Let (X, d) be a metric space, and let $a * b = \min\{a, b\}$. Let $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all $x, y \in X$ and $t > 0$. Then (X, M, *) is a fuzzy metric M induced by d is called standard fuzzy metric space [3].

Definition 2.3[4]: A sequence $[x_n]$ in a fuzzy metric space (X, M, *) is said to be convergent to a point x in X (denoted by $\lim_{n\to\infty} x_n = x$), if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in N$ such that

$M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

The completeness and non completeness of fuzzy metric space was discussed in George and Veeramani [3] and M. Grabiec [5].

Definition 2.4[2]: A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is called Cauchy sequence if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

Definition 2.5[8]: Two self mapping A and S of a fuzzy metric space (X, M, *) are said to be compatible, if $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ whenever $\{x_n\}$ is a sequence such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some zin X.

Definition 2.6[7]: Self mappings A and S of a fuzzy metric space (X, M, *) are said to be compatible of type (A) if $\lim_{n\to\infty} M(ASx_n, SSx_n, t) = \lim_{n\to\infty} M(SAx_n, AAx_n, t) = 1$ for all $t > 0$,whenever $\{x_n\}$is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$.

Definition 2.7[8]: Self mappings A and S of a fuzzy metric space (X, M, *) are said to be compatible of type (A-1) if $\lim_{n\to\infty} M(SAx_n, AAx_n, t) = 1$ for all $t > 0$ whenever $\{x_n\}$is a sequence in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$.

Lemma 2.8[4]: Let (*X*, *M*, *) be a fuzzy metric space. Then for all *x*, *y* in *X*, *M* (*x*, *y*, *) is non-decreasing.

Lemma 2.9[4]: Let (*X*, *M*, *) be a fuzzy metric space. If there exists $q \in (0, 1)$ such that

$M(x, y, qt)\ M(x, y, t/q)$ for positive integer n. Taking limit as $n \to \infty M(x, y, t) \geq 1$ and hence $x = y$.

Lemma 2.10[10]: The only t-norm * satisfying $s*s \geq s$ for all $s \in [0, 1]$, is the minimum t-norm, that is,

$a * b = \min\{a, b\}$ for all a, b [0,1].

Proposition 2.11[7]: Let $(X, M, *)$ be a fuzzy metric space and let A and S be continuous mappings of X then A and S are compatible if and only if they are compatible of type (A).

Proposition 2.12[8]: Let $(X, M, *)$ be a fuzzy metric space and let A and S be compatible mappings of type (A-1) and $Az= Sz$ for some $z \in X$, then $SAz=AA z$.

Proposition 2.13[8]: Let $(X, M, *)$ be a fuzzy metric space and let A and S be compatible mappings of type (A-1) and $Az = Sz$ for some $z \in X$, then $ASz = SSz$.

Proposition 2.14[8]: Let $(X, M, *)$ be a fuzzy metric space and let A and S be compatible mappings of type (A-1) and let $Ax_n, Sx_n \to z$ as $n \to \infty$ for some $x \in X$ then $AA x_n \to Sz$ if S is continuous at z.

MAIN RESULTS

We prove the following theorem.

Theorem 3.1: Let (X, M, *) be a complete fuzzy metric space and let P, Q, S and T be a self mappings of X satisfying the following conditions:

- $P(X) \subset T(X)$, $Q(X) \subset S(X)$,
- S and T are continuous.
- The pairs {P, S} and {Q, T} are compatible mapping of type (A-1) on X.
- There exists $k \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$,

  $M(Px, Qy, kt) \geq M(Sx, Ty, t) * M(Px, Sx, t) * M(Qy, Ty, t) * M(Px, Ty, t)$

  Then P, Q, S and T have a unique common fixed point in X.

Proof: Since $P(X) \subset T(X)$ and $Q(X) \subset S(X)$ for any $x_0 \in X$, there exists $x_1 \in X$ such that $P x_0 = T x_1$

And for this $x_1 \in X$, $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$, for all n = 0, 1, 2,…….

From (iv), $M(y_{2n+1}, y_{2n+2}, kt) = M(Px_{2n}, Qx_{2n+1}, kt)$.

$\geq M(Sx2n, Tx_{2n+1}, t) * M(Px_{2n}, Sx_{2n}, t) * M(Qx_{2n+1}, Tx_{2n+1}, t) * M(Px_{2n}, Tx_{2n+1}, t)$

$= M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+1}, t)$

$\geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)$

From lemma 2.9 and 2.10, We have

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) \quad (1)$$

Similarly, we have

$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t)$ (2)

From (1) and (2), we have

$M(y_{n+1}, y_{n+2}, kt) \geq M(y_n, y_{n+1}, t)$ (3)

$M(y_n, y_{n+1}, t) \geq M(y_n, y_{n-1}, t/k)$

$\geq M(y_{n+2}, y_{n-1}, t/k^2)$

$\geq \ldots\ldots\ldots\ldots \geq M(y_1, y_2, t/k^n) \to 1$ as $n \to \infty$.

So $M(y_n, y_{n+1}, t) \to 1$ as $n \to \infty$ for any $t > 0$. For each $\varepsilon > 0$ and each $t > 0$, we can choose $n_0 \in N$ such that $M(y_n, y_{n+1}, t) > 1-\varepsilon$ for all $n > n_0$.

For m, n ∈ N we suppose $m \geq n$. Then we have that

$$M(y_n, y_m, t) \geq M\left(y_n, y_{n+1}, \frac{t}{m-n}\right) * M\left(y_{n+1}, y_{n+2}, \frac{t}{m-n}\right) * \ldots\ldots M\left(y_{m-1}, y_m, \frac{t}{m-n}\right)$$

$\geq (1-\varepsilon) * (1-\varepsilon) * \ldots\ldots\ldots\ldots$ (m-n) times.

$\geq (1-\varepsilon)$

And hence $\{y_n\}$ is a Cauchy sequence in X.

Since (X, M, *) is complete, $\{y_n\}$ converges to some point $z \in X$, and so

$\{Px_{2n-2}\}$, $\{Sx_{2n}\}$, $\{Qx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converges to z.

From proposition 2.15 and (iii), we have

$PPx_{2n-2} \to Sz$ (4)

and $QQx_{2n-1} \to Tz$ (5)

Now, from (iv), we get

$M(PPx_{2n-2}, QQx_{2n-1}, kt) \geq M(SPx_{2n-2}, TQx_{2n-1}, t) * M(PPx_{2n-2}, SPx_{2n-2}, t) * M(QQx_{2n-1}, TQx_{2n-1}, t) * M(PPx_{2n-2}, TQx_{2n-1}, t)$

Taking limit as $n \to \infty$ and using (4) and (5) we have

$M(Sz, Tz, kt) \geq M(Sz, Tz, t) * M(Sz, Sz, t) * M(Tz, Tz, t) * M(Sz, Tz, t)$

$\geq M(Sz, Tz, t) * 1 * M(Sz, Tz, t)$

$\geq M(Sz, Tz, t)$

It follows that $Sz = Tz$ (6)

Now from (iv)

$M(Pz, QQ_{2n-1}, kt) \geq M(Sz, TQx_{2n-1}, t) * M(Pz, Sz, t) * M(QQx_{2n-1}, TQx_{2n-1}, t) * M(PPx_{2n-2}, TQx_{2n-1}, t)$

Again taking limit $n \to \infty$ and using (5) and (6), we have

M (Pz, Tz, kt) ≥M (Sz, Sz, t) *M (Pz, Tz, t) *M (Pz, Tz, t) *M (Pz, Tz, t)

≥M (Pz, Tz, t)

And hence Pz =Tz (3.1.7)

From (iv), (6) and (3.1.7)

M (Pz, Qz, kt) ≥M (Sz, Tz, t) *M (Pz, Sz, t) *M (Qz, Tz, t) *M (Pz, Tz, t)

= M (Pz, Pz, t) *M (Pz, Pz, t) *M (Qz, Pz, t) *M (Pz, Pz, t)

≥ M (Pz, Qz, t).

And hence Pz =Qz. (3.1.8)

From (6), (3.1.7) and (3.1.8), we have

Pz = Qz = Tz = Sz. (3.1.9)

Now, we show that Qz = z.

From (iv),

M (Px_{2n}, Qz, kt) ≥M (Sx_{2n}, Tz, t) *M (Px_{2n}, Sx_{2n}, t) *M (Qz, Tz, t) *M (Px_{2n}, Tz, t)

And, taking limit as n →∞ and using (6) and (3.1.7), we have

M (z, Qz, kt) ≥M (z, Tz, t) *M (z, z, t) *M (Qz, Tz, t) *M (z, Tz, t)

= M (z, Qz, t) *1 *M (Qz, Qz, t) *M (z, Qz, t)

≥ M (z, Bz, t).

And hence Qz = z. Thus from (3.1.9), z = Pz = Q z = Tz = Sz and z is a common fixed point of P, Q, S and T.

In order to prove the uniqueness of fixed point, let w be another common fixed point of P, Q, S and T. Then

M (z, w, kt) = M (Pz, Qw, kt)

≥M (Sz, Tw, t) *M (Pz, Sz, t) *M (Qw, Tw, t) *M (Pz, Tw, t)

≥M (z, w, t).

From lemma 2.10, z = w. This completes the proof of theorem.

Corollary 3.2: Let (X, M, *) be a complete fuzzy metric space and let P, Q, S and T be a self mappings of X satisfying (i) - (iii) of theorem 3.1 and there exists k ∈ (0,1) such that

M (Px, Qy, kt) ≥M (Sx, Ty, t) *M (Px, Sx, t) *M (Qy, Ty, t) *M (Qy, Sx, 2t) *M (Px, Ty, t)

for every x, y ∈ X and t >0. Then P, Q, S and T have a unique common point in X.

Corollary 3.3: Let (X, M, *) be a complete fuzzy metric space and let P, Q, S and T be a self mappings of X satisfying (i) - (iii) of theorem 3.1 and there exists k ∈ (0, 1) such that M (Px, Qy, kt) ≥M (Sx, Ty, t) for every x, y ∈ X and t >0. Then P, Q, S and T have a unique common point in X.

Corrolary 3.4: Let (X, M, *) be a complete fuzzy metric space and let A, B, S and T be a self mappings of X satisfying (i) - (iii) of theorem 3.1 and there exists k ∈ (0,1) such that

M (Px, Qy, kt) ≥M (Sx, Ty, t) *M (Sx, Px, t) *M (Px, Ty, t),

for every x, y ∈ X and t >0. Then P, Q, S and T have a unique common point in X.

Corollary 3.5: Let (X, M, *) be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping P of X such that the following condition are satisfied :

- P(X) ⊂T(X) ∩ S(X),
- The pair {P, S} and {P, T} are compatible mapping of type (A-1) on X.
- There exists k ∈ (0, 1) such that for every x, y ∈ X and t >0

M (Px, Py, kt) ≥M (Sx, Ty, t) *M (Px, Sx, t) *M (Qy, Ty, t) *M (P x, Ty, t).

In fact, P, S and T have a unique common fixed point in X.

Proof: We shown that the necessity of the conditions (i) - (iii). Suppose that S and T have a common fixed point in X, say z. Then Sz = z = Tz.

Let Px = z for all x ∈ X. Then we have P(X) ⊂T(X) ∩ S(X), and we know that [P, S] and [P, T] are compatible mapping of type (A-1), in fact PoS = SoP and PoT = ToP, and hence the conditions (i) and (ii) are satisfied.

For some k ∈ (0, 1), we get M (Px, Py, kt) = 1 ≥ M (Sx, Ty, t) *M (Px, Sx, t) *M (Py, Ty, t) *M (Px, Ty, t).

for every x, y ∈ X and t >0 and hence the condition (iii) is satisfied.

Now, for the sufficiency of the conditions, let $P = Q$ in theorem 3.1. Then *P*, *S* and *T* have a unique common fixed point in X.

In fact, P, S and T have a unique common fixed point in X.

Corollary 3.6: Let (X, M, *) be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping P of X satisfying (i) – (ii) of theorem 3.5 and there exists a self mapping of X satisfying (i) - (iii) of theorem 3.5 and there exists k ∈ (0, 1) such that for every x, y ∈ X and t >0

M (Px, Py, kt) ≥M (Sx, Ty, t) *M (Px, Sx, t) *M (Py, Ty, t) *M (Px, Sx, t) *M (Px, Ty, t).

Corollary 3.7: Let (X, M, *) be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping P of X satisfying (i) - (ii) of theorem 3.5 and there exists a self mapping of X satisfying (i) - (iii) of theorem 3.5 and there exists k ∈ (0, 1) such that for every x, y ∈ X and t >0

M (Px, Py, kt) ≥M (Sx, Ty, t).

In fact, P, S and T have a unique common fixed point in X.

Corollary 3.8: Let (X, M, *) be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping P of X satisfying (i) - (ii) of theorem 3.5 and there exists a self mapping of X satisfying (i) - (iii) of theorem 3.5 and there exists $k \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$

$$M(Px, Py, kt) \geq M(Sx, Ty, t) * M(Sx, Px, t) * M(Px, Ty, t).$$

In fact, P, S and T have a unique common fixed point in X.

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