ON POPULARIZATION OF *SCIENTIFIC EDUCATION* IN ITALY BETWEEN 12TH AND 16TH CENTURY

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Abstract

Mathematics education is also a social phenomenon because it is influenced both by the needs of the labour market and by the basic knowledge of mathematics necessary for every person to be able to face some operations indispensable in the social and economic daily life. Therefore the way in which mathematics education is framed changes according to modifications of the social environment and know–how. For example, until the end of the 20th century, in the Italian faculties of engineering the teaching of mathematical analysis was profound: there were two complex examinations in which the theory was as important as the ability in solving exercises. Now the situation is different. In some universities there is only a proof of mathematical analysis; in others there are two proves, but they are sixth–month and not annual proves. The theoretical requirements have been drastically reduced and the exercises themselves are often far easier than those proposed in the recent past. With some modifications, the situation is similar for the teaching of other modern mathematical disciplines: many operations needing of calculations and mathematical reasoning are developed by the computers or other intelligent machines and hence an engineer needs less theoretical mathematics than in the past. The problem has historical roots. In this research an analysis of the phenomenon of "scientific education" (teaching geometry, arithmetic, mathematics only) with respect the methods used from the late Middle Ages by "maestri d'abaco" to the Renaissance humanists, and with respect to mathematics education nowadays is discussed. Particularly the ways through which mathematical knowledge was spread in Italy between late Middle ages and early Modern age is shown. At that time, the term "scientific education" corresponded to "teaching of mathematics, physics"; hence something different from what nowadays is called science education, NoS, etc. Moreover, the relationships between mathematics education and civilization in Italy between the 12th and the 16th century is also popularized within the Abacus schools and Niccolò Tartaglia. These are significant cases because the events connected to them are strictly interrelated. The knowledge of such significant relationships between society, mathematics education, advanced mathematics and scientific knowledge can be useful for the scholars who are nowadays engaged in mathematics education research.
Key words: *Abacus schools, mathematics education, science & society, scientific education, Tartaglia*

Introduction

During a period in between late Middle Age and Renaissance a conspicuous amount of literature/commentaries/copies were managed by particular groups of scholars. The Renaissance cultural movement took place basically within intellectual groups, as Greek and Roman teachings, undertaken by scholars, writers, and civic leaders who nowadays are known as *Ren-*

aissance humanists[1]. Among the humanists there were also scientists who dealt with a subject called *philososphia naturalis*. Nowadays this subject is simply called *physics*. Generally speaking no particular difference existed between *scientific* and *humanistic* Renaissance scholars. A scholar who worked on science but with humanistic background was appreciated for his results not for his background.

In the late Middle Ages until early modern age, another category of professionals – having contact with the humanist scholars but who cannot be identified with them – proposed a kind of education which was substantially different from the medieval scholastic one, emphasizing practical, pre–professional and scientific studies: these professionals were the so called *maestri d'abaco*. They taught mathematics in public or private schools. Most of these schools took place in Italy, but the phenomenon rapidly crossed the whole Europe introducing several concepts and methods both in advanced mathematics and, later one, in the teaching practice. The role of the *maestri d'abaco* was soon significant for the improvements/development of scientific method, still though this remains, nowadays, a matter of controversy.

Until the 13th century, the decimal positional system and the Indo–Arabic numerals were almost unknown in Europe. From the 11th to the 13th century, the four arithmetical operations were carried out using the abacus. This instrument provides the possibility to develop all the four operations through standard procedures and its basis is a positional idea of the number because every column or line (it depends on the kind of abacus) can represent a power of the number assumed as the basis of the system. However, the possibility to operate in the lines does not exist with the abacus and the step from the representation of a number on an abacus in a positional form to the possibility to write a number in a positional form is not trivial at all because this step implies the full comprehension of the role of the digit zero inside the positional system. This is not necessary for calculations with the abacus, but it is to write the positional form of a number. The multiplication and, even more, the division become complicated with the abacus. The division can be carried out only by successive subtractions and this implies, in general, a long series of operations. The situation is even more complicated for other operations, as the calculation of the square roots (Pisano, 2013, p. 4–10). The history of the calculations with the abacus is an interesting chapter of history of mathematics. The results obtained using this instrument are amazing. Of course they cannot be compared with the ease of the calculations in the lines which is possible by writing the numbers in a positional form and operating directly on the written numbers. It is necessary to underline that the abacus, in various forms, was used in the Greek and Roman society, but it was almost disappeared in the medieval Europe. It was reintroduced by Gerbert d'Aurillac (955–1003) – the Pope Sylvester II – who had spent a period of his life in the Catalan abbeys where he had contact with Arabic civilization and where he learnt the decimal positional system. It is still uncertain if he fully understood the role of the zero, but certainly he caught the importance of the calculus with abacus. He invented a particular kind of abacus that was largely spread in Europe from the 11th to the 13th century[2]. Before the reintroduction of the abacus, the calculations in Europe during the middle ages were developed with the Roman digits and, despite a certain amount of studies has shown that more operations than one could think are possible with the Roman digits these calculations are incomparably more limited the ones permitted by the abacus. Leonardo Pisano (Fibonacci) was the great mathematician who fully realized that the phase of the calculations with the abacus had to be overcome, too. He understood the role of the zero and the great progress that the numerical Indo–Arabic system would have brought into mathematics and – more in general – to civilization. The book in which the Arabic digits and all the operations were presented is Fibonacci's *Liber Abaci*[3]. The problem is not easy and presents many aspects.

1   Humanism is not the study of humans. The term *umanista* was used, in 15th century Italian academic slang to describe a teacher or student of classical literature and the arts associated with it, including that of rhetoric.

2   Pekonen, 2000; Atkinson, 2005; Vallicrosa, 1931; Baird, 1994; Seife, 2000; Flusche, 2006.

3   1202, second version 1228. The first version is lost. Only this later version in available.

On Arithmetic Popularization and the *Abacus Schools*

A certain amount of studies on Fibonacci's influence on eventual mathematics and mathematics education in Italy exists from the work of Guglielmo Libri. Boncompagni[4] was fundamental for the rediscovery of Fibonacci and to reconstruct the life and the influence of the great Pisan mathematician. In the 20th century, thanks to the work of scholars as ver Eecke (Fibonacci, 1952), Gino Arrighi[5] and Ettore Picutti (Picutti 1979, 1983); many manuscripts of late Medieval and early modern age Italian mathematicians were discovered. In this sense Arrighi's work is fundamental. However, it is mainly starting from the half of the 70ies in the 20th century that a huge and systematic series of works and discoveries about mathematics and mathematics education in Italy in the late middle ages and early modern period have been carried out (for example, in the last twenty years of the 20th century the university of Siena created a Centre for the study of Medieval mathematics that, under the direction of Franci and Toti Rigatelli has published a series of works of Medieval mathematicians). Hence nowadays the picture is relatively clear and it reveals interesting aspect and relations between advanced mathematics, didactics and economic–social history: the first *maestro d'abaco* (abacus teacher, namely a person who taught the calculation with the Indo–Arabic decimal system) was likely Fibonacci himself because a document (1241) exists in which the Pisan Municipality assigned an annual salary to Leonardo probably for a series of services inside which the teaching of the Indo–Arabic system was included, too (Ulivi, 2002, Chap I).

Even if the decimal system was not adopted in a systematic way in the second half of the 13th century (let us think that at the beginning of the 14th century the *Statuti dell'Arte del Cambio di Firenze* – the rules of the persons employed in the money exchange – prohibited the use of the decimal system to avoid the risk of frauds in the calculations towards people who ignored the decimal system (Giusti, s.d., p. 1), the new way of calculation became progressively more popular and replaced almost completely the old ones. Basically, this was due to the fact that the business became rapidly more extended than in the past and the structure itself of the production and trade companies became more complex: a precise way to keep the books got important; the book entries were systematically introduced, the trade–nets became more extended and inter-related. This brought to the need of calculating quickly and with precision (Giusti, 2002, Chap. 15). Therefore, the *scuole d'abaco* (like above *Abacus schools*) were introduced in many cities. Between the second half of the 13th century and the beginning of the 16th one the abacus schools became a very mass phenomenon in many Northern and Central Italian cities. The Toscana was the region in which their number was more conspicuous. Private and public abacus schools existed. The public schools were organized and financed by the municipalities. In general, the salaries of the teachers were different if they taught in the public or in the private schools: the teachers of the private schools had far superior salaries than those of the public schools (Ulivi, 2002, Chap. 3). The economic and social *status* of the abacus teachers varied from that of little artisans with economic problems to that of relatively rich persons. They can be considered the first professional mathematicians because their maintenance derived from their mathematical capabilities and from their abilities in the teaching (Franci, 1988, p. 183). Firenze is the city for which the research has been more profound and likely the one in which the abacus schools were more common. Between the second half of the 13th century and the first half of the 16th century, about 20 abacus schools and about 70 abacus masters operated in Firenze. The schools were private. The most famous one was the *Bottega di Santa Trinita*, active between the first half of the 14th century and the half of the 15th century. The famous abacus master, Paolo dell'Abaco likely taught in the Bottega di Santa Trinita.

Another famous school was the *Scuola di Santa Maria della Scala*, second half of the 15th century. Here Benedetto da Firenze worked. Likely, he was the most important mathema-

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4 Boncompagni 1852, 1854. He is the editor of all Fibonacci's works (Fibonacci 1228, 1857; 1862a; 1862b; 1862c).

5 Arrighi's publications on Medieval mathematics until 1983 (Pancanti & Santini, 1983).

ticians among the abacus masters and was the teacher of Leonardo da Vinci (Franci, 1988, p. 190). Some families, as the Calandri, were abacus masters for generations (Ulivi, 2002, chapter 2). Pier Maria Calandri was the teacher of Machiavelli and Filippo Calandri the teacher of Giuliano de' Medici (Franci, 1988, p. 190).

Smaller Tuscan cities had public abacus schools: San Gimignano since 1279, Siena, at least from the beginning of the 14ᵗʰ century, Lucca and Pistoia at least from the half of the 14ᵗʰ century, Arezzo at least from the last ten years of the 14ᵗʰ century, Volterra from the beginning of the 16ᵗʰ century, as well as Prato. In Pisa, after Fibonacci, the first certain document proving the presence of abacus schools dates back to the second half of the 14ᵗʰ century. Out Toscana, private abacus schools existed in Bologna at least since 1265, public schools in Verona since 1285, in Savona since 1345, in Perugia from the last twenty years of the 14ᵗʰ century. Modena was an important centre for the public abacus schools from the beginning of the 15ᵗʰ century, as well as Brescia. In Venezia the work of the abacus schools was almost as important as in Firenze and had similar characteristics: the schools were, in general, private.

One of the most important was the *Scuola di Rialto* where Luca Pacioli studied as a student with the master Domenico Bragadin (Ulivi, 2002, Chap. I). This to give an idea of the importance and the diffusion of the abacus schools. The age of the students that frequented them was not constant. In general the pupils began their arithmetical instruction at 10–11 and this education prosecuted for two years. However, this rule was not general. There are evidences of young who began to frequent the abacus schools at 6 and others at 17 (Ulivi, 2002, Chap. III). With regard to the social *status* of the pupils, it was rather variegated. For, the abacus schools were, if not a popular phenomenon, at least a phenomenon that concerned most part of the bourgeoisie: hence, the sons of the artisans, of the men involved – in any manner – with the trades, of the public officials and of almost all learned or reach families frequented these schools. At the end of the 15th century it is calculated that more than 25% of all young frequented the abacus schools in Firenze, while, in a little city as Arezzo, the percentage was only the 5%.

In Venezia, one century later, it was about 40% (Ulivi, 2002, Chap. III). The teaching was organized in *mute*. In general, the *mute* were divided like this: 1. first *muta* colled *librettine* containing an introduction to the Indo–Arabic digits, addition, subtraction and multiplications among integers; 2. more than one *muta* – in general three – dedicated to the division; 3. fractions and their operations; 4. three rule; 5. calculations for money exchange (Ulivi, 2002, Chap. III).

An interesting aspect connected to mathematics education in the middle ages is given by the Italian abacus treatises. These texts can be considered as handbooks in which the subjects explained in the abacus schools were dealt with. The first abacus treatises date back to the end of the 13ᵗʰ century and this kind of literature was flourishing until the 16ᵗʰ century. In 1980 van Egmond carried out a profound research in the Italian libraries and archives and catalogued about 300 abacus treatises written (manuscript or printed) in Italy between the late middle ages and the Renaissance (Van Egmond, 1980; Bagni, 1998). The structure of the abacus treatises varies according to the epochs and the authors. There are some sections that are, more or less, the same in every treatise: they concern the introduction of the Indo–Arabic digits, the four arithmetical operations, the solution of the equations of the second degree, some problems connected to trades and money–exchange, the square roots and some notions of geometry. However, for other aspects the abacus treatises present conspicuous differences because the level of the problems proposed, their explanations and solutions vary from author to author. Some authors present a long series of formulas to solve the equations of second and (in their wrong opinion) of the third degree. Sometimes long parts dedicated to astronomy and astrology exist[6]. So that, this is a variegated literature in which the bonds between didactics of mathematics and mathematical research are not always clear, in which good intuitions and mistakes handed down from an author to another one are present. In the described context, the most remarkable aspect concern algebra because almost all treatises present solutions of equations of second

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6   Paolo dell'Abaco, 1985; Arrighi, 1980; Arrighi, 1981; Piochi, 1984.

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and third degree, often of superior degree, too[7]. As to the equation of second degree, the abacus treatises follow the classical treatment of Al–Khwarizmi and Fibonacci, but the question becomes interesting from a historical point of view when we consider the equation of the third degree. The authors who deal with this subject[8] refer the same wrong solutions and precisely, for the equations

1) $\begin{aligned} ax^3 &= bx + c \\ ax^3 &= bx^2 + c \end{aligned}$ The solution: $x = \sqrt{\left(\dfrac{b}{2a}\right)^2 + \dfrac{c}{a}} + \dfrac{b}{2a}$

And for the equation

2) $ax^3 = bx^2 + cx + d$ The solution: $x = \sqrt{\left(\dfrac{b}{2a}\right)^2 + \dfrac{c+d}{a}} + \dfrac{b}{2a}$

(They do not use our algebraic symbolism). Clearly these authors, not only did not try to prove the general validity of their formulas, but they neither tried to empirically verify them. An interesting case is the one of the abacus treatise of the famous artist Piero della Francesca (Piero della Francesca, 1970), to which Giusti has dedicated an interesting paper (Giusti, 1991): Piero della Francesco reported the wrong formulas 1) and 2) and used them to solve a series of problems of compound interest. In the specific cases considered by Piero these formulas work, even if, at all appearances, he had the wrong opinion they were valid in general. Giusti shows that the formulas work for the compound interest because they are equivalent – only in this specific case – to the right formulas for the compound interest given by Fibonacci in the *Liber Abaci* and that are independent of the general solution of the equation of the third degree. This consideration puts in evidence a series of remarkable questions connected to the way in which scientific ideas are spread and to the relation between science and society: the *Liber Abaci* was the model for almost all the abacus treatises written in Italy between the 13th and the 15th century (even if some scholars tried to identify a mathematical tradition different from Leonardo's). The level of these treatises is inferior to that of the *Liber Abaci* because no mathematicians as well as Fibonacci existed in those centuries. At the same time these treatises were written in vernacular Italian since they had to be read by the students and by the merchants and business men who often were not confident with Latin. They were not foundational either learned books, but practical texts, even if theoretical parts exist in some cases. Despite these limitations, at the beginning of the 16th century an original mathematical interest began in the abacus masters: it concerned the solution of the equation of a degree greater than two, in particular those of the third degree. At the beginning the difficulty of the question was underestimated and the results were often superficial, but these attempts induced an interest in algebra that was superior than Fibonacci's, because he was a great mathematician, but not exactly and algebraist, even if his contribution was fundamental, rather a geometer and a number theoretician. The Italian mathematical environment was hence – so to speak – ready to see the birth of the great algebraic Italian mathematicians of the 16th century whose contributions to the solution of third and fourth degree equations was decisive. On the other hand, in the second half of the 16th century, the calculation with and the use of the Indo–Arabic digits were consolidated, there was an abundant literature and the need to refer directly to Fibonacci became less urgent. The mathematical research began to follow its own *iter*, in the first phase of which algebra had the fundamental role. The consequence of this situation was that the works and the name itself of Fibonacci progressively

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7   Paolo Dell'Abaco, 1964; Della Francesca, 1970; Gilio, 1983; Canacci, 1983; Franci & Pancanti, 1983; Gori, 1984; Bastiano da Pisa detto il Bevilacqua (sec. XVI), 1987; Catellani, Degani & Mantovani, 2000. Important literature: Arrighi, 1966–67; Van Egmond, 1978; Franci & Toti Rigatelli, 1982, 1983, 1985.

8   Gherardi, in Arrighi, 1966–1967; Gilio, 1983.

disappeared and the mathematician who was the most famous and known until the half of the 15th century, one century later was almost unknown[9] and it was necessary to wait for the half of the 19th century and Baldassarre Boncompagni for him were rediscovered.

Mathematics Popularization: Niccolò Tartaglia's Equations

The events of the life and of the scientific activity of Niccolò Tartaglia (Pisano and Capecchi 2014–*forthcoming*) complete the picture described in the previous sections because Tartaglia taught mathematics in different contexts during his whole life, belonged to the environment to which the abacus masters themselves belonged, but, due to his conflicts with important personalities as Gerolamo Cardano, reached never a social position coherent with his great scientific value. From the point of view of mathematics education, Tartaglia represents the classical example of an intelligent autodidact who overcame the average level of the advanced mathematics of his time. Nowadays a case as the one of Tartaglia looks difficult to be reproposed because advanced mathematics has become so specific that an autodidact giving original contributions to advanced mathematics can hardly be conceived, even if this possibility cannot be a priori excluded. However, despite Tartaglia was an autodidact, the environment in which he lived – the one of the Italian Renaissance with its interest in algebraic questions – had a decisive influence on his mathematical interests and researches.

Tartaglia produced crucial contributions to mathematics, physics, and fortifications: solution of the third degree equations, scientific foundations of ballistics, criticism to Aristotle's lever, statics, the measurement of calibers and land surveying and fortifications. He discussed them principally in *General trattato di numeri et misure* (Venice, 1556–1560), *Nova scientia* (Venice, 1537) and in *Quesiti*. Thanks to his mathematical studies at an early age, Tartaglia went to Verona (Masotti reported the existence of some documents present in the *Archivio di Stato di Verona* that declared his stay in Verona to be around 1529–1533 (Masotti, 1976, Vol. 13, p. 259) where (fl. 1516–1518) he had a job as a *teacher of the abacus* at a school in Palazzo Mazzanti. In 1534 he moved once again to Venice for giving public lectures in mathematics, e.g., at the Church of San Zanipolo. Venice would be the most important setting for his main scientific works. In fact, all of his studies were published in this city where he essentially spent all of his life. Tartaglia stayed in Brescia for a period of time (1548–1549), teaching at S. Afra, S. Barnaba, S. Lorenzo and at the Academy of Rezzato. In the last years of his life he was thriving scientific activities in Venice.

The resolution of the third degree equations (Tartaglia 1554, Book IX, Q. 14, *folio* 101rv and Q. 25, *folio* 106rv) and his subsequent controversy with Girolamo Cardano (1501–1576) surely represent one of the most significant subjects related to Tartaglia's name in history. Cardano knew of the innovations directly through Tartaglia himself (1539); then he published them in his *Ars magna* (1545). Generally speaking, the resolution (which at the end of the 15th century Luca Pacioli considered impossible through using known calculations of the time) was studied and separately proved by both Scipione del Ferro and Tartaglia. Cardano and Ferrari improved the method. The Book IX of *Quesiti et invention diverse* (Tartaglia 1554, Book IX) explains this procedure. It is known that the solution of third–degree equations was acknowledged in one of Tartaglia's poems:

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9   Bussotti 2009, p. 48–50.

Quando chel cubo con le cose appresso $\qquad$ $x^3 + px$

Se agguaglia à qualche numero discreto $\qquad$ $= q$

Trouan dul altri differenti in esso. $\qquad$ $u - v = q$

Dapoi terrai questo per consueto

Che'l lor produtto sempre sia eguale $\qquad$ $uv = (p/3)^3$

Al terzo cubo delle cose neto,

El residuo poi suo generale

Delli lor lati cubi ben sottratti $\qquad$ $\sqrt[3]{u} - \sqrt[3]{v} = x$

Varra la tua cosa principale.

In el secondo de cotesti atti

Quando che'l cubo restasse lui solo $\qquad$ $x^3 = px + q$

Tu osseruarai quest'aitri contratti,

Del numer farai due tal part'à uolo

Che l'una in l'altra si produca schietto $\qquad$ $uv = (p/3)^3$

El terzo cubo delle cose in stolo

Delle qual poi, per commun precetto

Torrai li lati cubi insieme gionti $\qquad$ $\sqrt[3]{u} + \sqrt[3]{v} = x$

Et cotal somma sara il tuo concetto.

El terzo poi de questi nostri conti

Se solue col secondo se ben guardi $\qquad$ $x^3 + q = px$

Che per natura son quasi congionti.

Questi trouai, & non con passi tardi

Nel mille cinquecente, quatroe trenta

Con fondamenti ben sald'è gagliardi

Nella citta dal mar'intorno centa.

When the cube and its things [that is the unknown] near. Add a new, discrete number. Determine two new, different numbers. By that one; this feat will be kept as a rule. Their product always equals, the same, to the cube of a third. Of the number of things named. Then, generally speaking. The remaining amount. Of the cube roots subtracted will be our desired count. When a cube and its things near. Add to a new, discrete number. Determine two new, different numbers. By that one, then, generally speaking, the remaining amount of the cube roots subtracted will be our desired count". This is the solution in the poem, not the demonstration Tartaglia sent to Cardano. The last verse could allude to the fact that Tartaglia found the formula while he was in Venice.

Figure 1: Plates from Tartaglia's poem solution of the third–degree equation.

Tartaglia studied the following equations (modern expression in which all the terms are positive):

$$x^3 + px^2 = q \qquad\qquad x^3 + q = px^2$$

The events and reasons surrounding the origin of the *disfida matematica* (mathematical challenge) which arose between 1547 and 1548 between Lodovico Ferrari (1522–1565), who sought to defend his mentor Cardano, and Tartaglia are well known. The scientific dispute began with *cartelli* and six *controcartelli* in which 62 mathematical problems referring to Euclidean geometry were put forth and partially solved (Pisano and Casolaro, 2012). Nevertheless, a concise timeline with presumably historical discoveries related to the evidence of the rule for solving third–degree equations is here below:

- Scipione del Ferro (1465–1526) in the 1510s (fl. 1520s) but never published.
- Tartaglia's solution (Tartaglia 1535) since his mathematical debate with Anton Maria del Fiore, a disciple of del Ferro who claimed to be able to solve the cubic equations.
- Tartaglia did not publish his solution, but only Tartaglia succeeded in solving the thirty problems proposed in the challenge organized on 12 February 1535.
- Lodovico Ferrari (1522–1565) and his six "Cartelli" (Pamphlets) (1547–48) against Tartaglia.
- Tartgalia's *Risposte* (*Replies*) *to Lodovico Ferrari*, Venice 1547 (1–4) and Brescia 1548 (5–6).

Let us see the main details (Tartaglia 1554, Qs. 20, 25, 26, 28, 29, 31–41).

Tartaglia, after great insistence, relayed the solution to Girolamo Cardano (25 March 1539) who, in addition to being a very famous doctor, was also an excellent mathematician (Bolletti 1958, p. 93–111). Fortunately for Cardano, despite the fact that Tartaglia's solution was expressed in coded verses, his skills helped him to decipher the solution and publish it before Tartaglia. There is an interesting exchange between Cardano and Tartaglia (4 August 1539) (Tartaglia 1554, Book IX, Q 38), in *Quesiti et invention diverse* (Tartaglia 1554) not only regarding the solution of the superior degree equations but also geometrical topics. In this exchange, Cardano put forth a specific request concerning a geometrical problem. Tartaglia begins by proposing 17 problems for Ferrari which involve using a compass with fixed opening (Tartaglia, *Seconda risposta*, Venice, 21 April 1547), [p. 15–18], p. 53–56). Among the 31 inquiries which Ferrari sent to Tartaglia in *Terzo Cartello di disfida matematica* (1547–1548), there are two inherent to the inscription and reciprocal circumscription of regular polygons, which can also be found in the Parisian manuscript *Elementa* by Cardano inserted in *Commentaria* (Masotti 1974, pp 66–68). Ferrari responds by solving these problems, adding that not only Tartaglia's problems but all of the Euclidean propositions can be solved by using the compass with fixed opening (Ferrari, *Quinto cartello* (Milan, October 1547), [pp 25–39], p. 141–155). The subject was translated into Latin and published by Cardano in *De subtilitate* (Cardano 1550, Book XV, p. 296–302). To Ferrari's solution and Cardano's solution, Tartaglia introduces one of his main topic (*Sesta Risposta*, Brescia, 24 July 1548). In fact, Tartaglia goes back to the 17 problems and solves them in *General Trattato* (Tartaglia 1556–1560, Part V, *folios* 63v–83v). Today both Ferrari and Tartaglia's merits in their conclusive and demonstrative procedures are recognized (Bortolotti 1935, p. 75–76). Most importantly, Tartaglia and Ferrari are recognized, thanks to the *Cartelli matematica disfida* for creating a conclusive approach using a straightedge and compass with fixed opening (assigned at will) which became a public use.

After a long written diatribe, the two rivals faced each other in Milan on 10 August 1548. The outcome of this encounter was subject to opposing judgements (Masotti 1974, p. XXXIV–XL). The fact that some problems discussed in Tartaglia and Ferrari's dispute concerned Euclidean geometry is noteworthy. These problems concerning plane geometry were quite significant (Masotti 1974, p. XXI–XXIII and footnotes 104–107) since they were always solved by using a straightedge and compass, the latter using the fixed opening technique (*Ivi*).

In *Ars magna* (1545) Cardano also published the solution to the fourth degree equation. It must be noted, however, that Cardano cites Tartaglia as author of the solution of the cubic equation and Ludovico Ferrari as the person who discovered the solution to the fourth degree equation. Therefore, Cardano's error in regard to Tartaglia (which he avoided mentioning) was not keeping his promise not to divulge the secret of the solution. On can image that Tartaglia – with such a discovery – could have acquired a certain visibility in the academic and professional panorama. This occurrence engendered a series of disputes between the two mathematicians that lasted two years and a ferocious dispute between Tartaglia and Cardano's student, Lodovico Ferrari. Obviously, whatever the historical truth about such misdeeds is not what interests us. We note only that on the first page of *Ars Magna* (*The Great Art*) Cardano attributes (Baldi and Canziani) the solution of the cubic equation to Scipione dal Ferro (ca. 1465–1526), instructor of mathematics at the Medieval University of Bologna –a solution it seems he had already studied in 1515. Although he didn't publish his discovery, before his death, Scipione dal Ferro revealed it to one of his students, the Venetian Anton Maria Florido (*Floridus*). As to the formula that gave the solution to the cubic equation, both Tartaglia's version and Scipione dal Ferro's previous version were not immediately reducible since both contained a quadratic term that neither mathematicians initially knew how to eliminate. It seems that Tartaglia was not able to overcome this obstacle before Cardano's publication of *Ars Magna*. Some maintain that this publication was justified both because six years had passed since Cardano's promise to Tartaglia and because Cardano was not expected to respect a promise based on a discovery belonging to Dal Ferro and not to Tartaglia. Tartaglia responded to such claims by *Quesiti,*

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where – in in addition to the disputes with Cardano – he lists some others. Ferrari did the same in a pamphlet entitled *matematica disfida*.

In the end the historical legend concerning an eventual plagiarism and other accusations directed to Tartaglia made his ascent into the academic world difficult even though his works, today, are impartially seen as a milestone in the history of mathematics and an important contribution to statics.

Conclusion

The problem of mathematics education in the high school is a huge question and many scholars and equips of scholars have dedicated profound studies to this problem at least from the half of the 19th century. However, it looks legitimate to wonder why the study of the Euclidean geometry has been progressively reduced in all European countries, despite we are convinced that the Euclidean geometry is one of the most educative disciplines. Perhaps this is also connected to the fact that in our society, mathematics education in the high schools is oriented to the development of the algorithmic abilities of the students (algebra, analytic geometry, calculus) rather than to the development of the demonstrative and rigorous reasoning. This kind of mathematics education is considered more useful for the students when they will face the university and the labour market.

If the social needs are a consistent part of mathematics education, the other aspect is the connection between mathematics education and advanced mathematics research. Namely: does advanced mathematics influence mathematics education? The problem is complex: between the end of the 19th and the beginning of the 20th century important mathematicians as Felix Klein (Klein, 1872) and Federigo Enriques[10] thought this should be the case. Sixty years later, in a different context (in the 60ies and 70ies of the 20th century), an important movement of mathematics education (of which Piaget himself was a supporter) promoted by outstanding mathematicians belonging to the Bourbaki movement, theorized the need to organize mathematics education relying upon to the concepts of abstract algebra[11]. Hence the relation between didactics and advanced mathematics is anyway dialectic.

In this research, the following discussions were proposed:

1) The dialectic between mathematics education and advanced research because the environment of the abacus masters was the one who produced the most famous and learned teachers of mathematics in that period and also the one around which some subjects of advanced mathematics – in particular algebra – were begun to be dealt with, even if not always solved. Tartaglia, who – as we have seen – gave fundamental contributions to the solution of the third degree equation, was a mathematician and teacher of mathematics, not a professor.

2) The relationship between mathematics education and social structures: at the end of the 12th-begin of the 13th century Fibonacci succeeded in introducing the operations with the Indo-Arabic numerals because the Italian maritime republics and, more in general, the Italian cities state developed an economy based on trades and hence a quick and precise way of calculation was necessary. Because of this same reason a plurality of *Abacus schools* were born between the 13th and the 16th century.

Finally, in order to propos the general lines of a series of interconnected phenomena an a) analysis of the relationships between advanced research and education, and an b) analysis between scientific discoveries and social structure should be provided. In fact it is a subject that can open new perspectives for scholars engaged in educational research. For example,

10   Enriques, 1921, 2003; Bussotti, 2012.
11   Bussotti, 2013 also for bibliographical indications.

in the period we analysed the reception and popularization of Leonardo da Vinci's work in Western Europe and, in particular, in Italy between the end of the 15th and the first half of the 16th century is an important subject in mathematics, mathematics education and the relation between history of mathematics and civilization (Pisano 2013b; Pisano, 2009). Particularly, the knowledge, as a part of society at that time, is a complex account since a social integration of knowledge is essential and scientific knowledge is important; but it seems that the latter is not indispensable condition.

The knowledge of important historical cases can also be useful nowadays to have a vast material on which to work and to reason in order to deal with the complex problems of science education.

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