The First Derivative of an Exponential Function with the "White Box/Black Box" Didactical Principle and Observations with GeoGebra

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Abstract. This paper shows how GeoGebra—a dynamic mathematics software—can be used to experiment, visualize and connect various concepts such as function, first derivative, slope, and tangent line. Students were given an assignment to determine the first derivative of the exponential function that they solved while experimenting with GeoGebra. GeoGebra enables students to experiment, model, and research their ideas in order to get desired results for mathematical problems. The software package, GeoGebra, enriches the educational process and opens up new questions and possibilities. Learning and teaching mathematics in contemporary, technology-rich surroundings requires new didactical approaches. The findings in this paper are based on the "White Box/Black Box" didactical principle. Recommendations for mathematics teachers are shared.
Keywords: first derivative; exponential function; dynamic software; GeoGebra; the white box/black box principle

1. Introduction

In traditional mathematical courses, derivatives are introduced through definition. Most of the related classroom time is spent proving and practicing derivation rules. Therefore, there is often not much time for students to actually learn about situations in which derivatives are necessary, or the basic supporting principles. Using GeoGebra, students and teachers are able to explore various mathematical notions in a diverse and profound way. Computer Algebra Systems (CAS), such as GeoGebra, allows the teaching process to concentrate on the modeling process, while complicated calculations are handled by the CAS technology. The use of CAS requires new pedagogical approaches. One such example mentioned in this paper is the didactical principle commonly referred to as the "White Box/Black Box" principle, as developed by Buchberger (Buchberger, 1989) and described in detail in (Heugl, Klinger & Lechner, 1996). The White Box/Black Box principle bridges the traditional teaching of mathematics with the use of available computer technology.

The principle consists of several interlocked stages. The stage when a new problem is solved by hand, or by using already known functions, is called a "White Box" phase. In this phase students distinguish all the basic operations and functions that are necessary to solve the problem. Having the problem carefully analyzed, the students then define a new function that solves the problem in a single step. The problem is therefore solved by a simple function recall (i.e., the press of a button, or entry of a function code) and the related definition of the function is not seen again. This phase is called a "Black Box" phase. It is presumed that the students understand the meaning and the definition of the functions. This process is dynamic. After a certain number of hand calculations or solved examples ("White Box" phase), a function could be used as a "Black Box". One advantage of continuous, by-hand repetition of the function is that students who perhaps did not understand the meaning of a function, or how a certain problem should be solved, still would have a chance to grasp the meaning of the function during repeated practice.

2. The "White Box/Black Box" principle in GeoGebra

In this paper we suggest how students could use educational software such as GeoGebra to visualize and experiment with mathematical facts. We used GeoGebra to carry out the pedagogical observations relating to the "White Box/Black Box" principle within the classroom. GeoGebra was used to handle the computing problems. We followed the didactical pattern of the principle by introducing new concepts with elementary and illustrative methods. For example, we solved some problems of calculus using the limit of the quotient of differences. Therefore our students got a better understanding of the concept of a differential quotient, and of derivatives. The problem of computing the limit was carried out using the computer software.

In this example, it is shown how the properties of tangent line are used to bring students to certain conclusions about the first derivative of the function. Furthermore, this example expounds a distinctive way of performing mathematical tasks. This pedagogical experiment was conducted with 17 senior gymnasium students. This kind of work requires a computer laboratory with Internet access. During the class students worked in pairs, individually or, if it was required, in larger groups.

For a successful classroom computer investigation, a specific pedagogical approach and preparation is required (Tall, 2003). This approach implies technical support such as a well- equipped computer laboratory, and a familiarity with the basics of the software as demonstrated by the teacher and the students involved in the lesson. GeoGebra is one of the leading mathematical educational software packages due to its intuitive and user-friendly interface. GeoGebra is a dynamic mathematics software for all levels of education that joins arithmetic, geometry, algebra, and calculus (Hohenwarter & Jones, 2007). There is a wide range of papers and Internet materials that can help teachers in organizing

GeoGeobra-supported mathematical lessons (www.GeoGebra.org; www.GeoGebra.org/en/wiki). A well-prepared plan of the lesson, including the prediction of common as well as unexpected student questions, improves the teaching process. It is a great advantage if the files containing the lessons are prepared ahead of time with lots of tasks, assignments, and questions for the students to explore. Furthermore, GeoGebra is advantageous due to its visual features that can motivate students to explore, and can make a lesson more interesting and dynamic (Hohenwarter, Hohenwarter, Kreis, & Lavicza, 2008).

3. The Task and GeoGebra Solution

As we have noted above, the "White Box" phase requires a theoretical knowledge. For example, all of our students were familiar with the following definitions:

$$tg\,\varphi = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{\Delta y}{\Delta x} = f'(x_0) \text{ , slope of the tangent line} \qquad (1)$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \text{, definition of derivative} \qquad (2)$$

This theoretical knowledge was considered as part of the "White box".

The "Black Box" phase requires an upgrading of students' existing theoretical knowledge through the use of modeling and application. The students were set the task to find the derivative of the function $y = a^x$. It was considered as a "Black Box". The first derivative is an equation for the slope of a tangent line to a curve at an indicated point. The first derivative may be found using a derivative definition (1). With the purpose of making the task of finding the first derivative easier and more obvious, students observed specific function $y = 2^x$. Using the definition (1) students obtained the following:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h} = \lim_{h \to 0} \frac{2^x(2^h - 1)}{h} = 2^x \lim_{h \to 0} \frac{2^h - 1}{h} = f'(x) \qquad (3)$$

Using the textbook, the task would be solved in the previous steps, only it would be concluded that the value of the limit $\lim_{h \to 0} \frac{2^h - 1}{h}$ was known, as this limit is considered as an important limit. Instead of this more traditional procedure, the use of GeoGebra offers an approach that includes a wide range of experimentation, independent reasoning, and the use of relevant technology.

When using GeoGebra, after analyzing result (3), students concluded that the result of differentiation is a function, but the question at that point was, Which particular function should be expected? With the aim of obtaining the resulting function, students used GeoGebra software to explore various possibilities and experiment with the initial conditions. Plotting a graph in GeoGebra was helpful when trying to visualize the tangent line. The steps described in the text were then used to attain the correct solution.

3.1 The First GeoGebra Worksheet

This part of the task was the same for all students and they worked in pairs. The function $y = 2^x$ was keyed into the input field by the students. Subsequently, the slider marked with *h* was added to the GeoGebra sheet. The slider *h* is considered as a very small value, almost zero, being set between 0 and 0.1. These facts were adjusted to the GeoGebra features and entered as the function $g(x) = \dfrac{2^h - 1}{h}$. Further on, these steps were animated using the GeoGebra features *Navigation Bar for Constructing Steps* and the *Play Button*. Playing the steps, students noticed that $\lim\limits_{h \to 0} \dfrac{2^h - 1}{h}$ was constant and valued around 0.7. They marked that belt in colour, in order to make this fact more evident (see Figure 1).
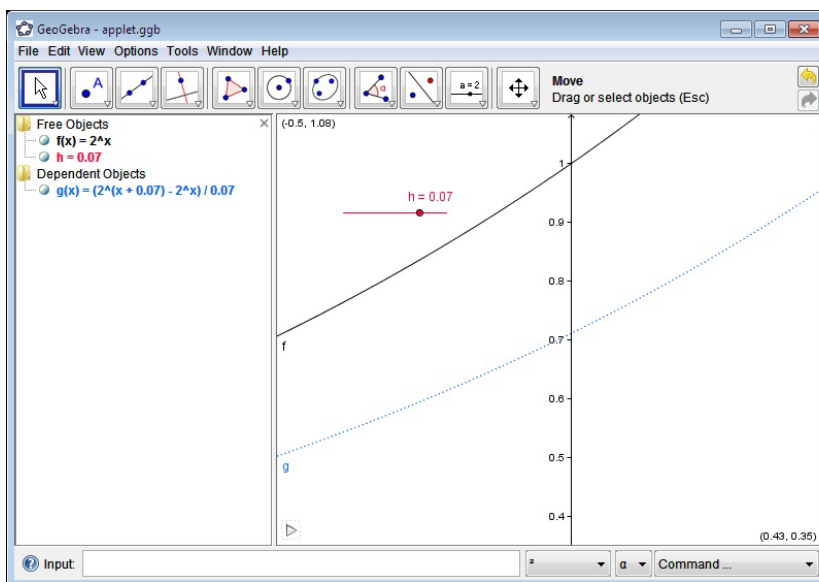

Figure 1. Visualization of the first derivative of the function $y = 2^x$

The students' task was to determine the specific constant. Note that this part could also be presented as a previously-prepared GeoGebra applet which could gradually introduce students to the topic, and thereby save class time. With the applet application, students could analyze its features with the teacher.

3.2 The Second GeoGebra Worksheet

This part of solving the task was done strictly on an individual basis. Each student was assigned a different function. For example, one took the function $y = 3^x$, another $y = 5^x$, and another $y = \left(\dfrac{1}{2}\right)^x$ etc. The idea was that each student chose a different base for the exponential function $y = a^x$ in order to experiment with lots of different functions and to ultimately arrive at the correct solution. The steps that every student followed are described in the following text. A student's solution is shown in Figure 2.
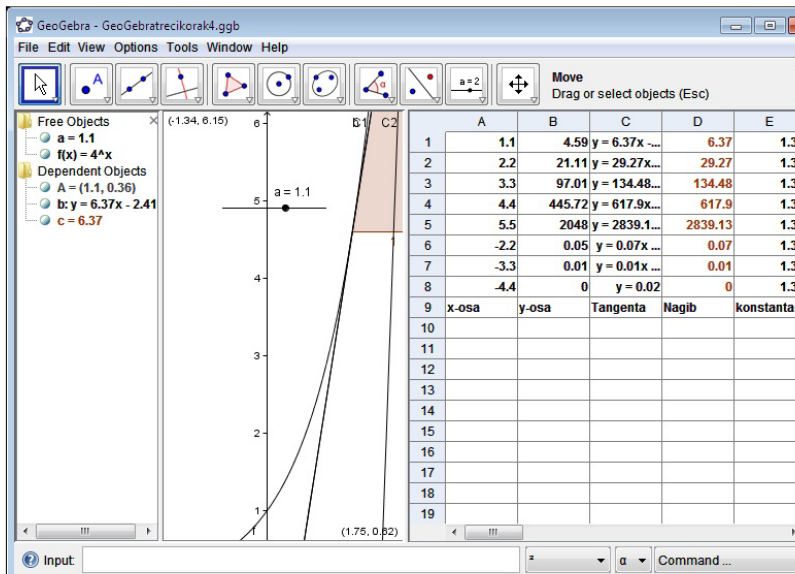
Figure 2. GeoGebra worksheet with function $y = a^x$, regarding its base, typed by student

The introduction of point A on the graph of the function $y = a^x$ followed. It was defined with a slider. The slider option was chosen in order to increase the number of different points that should provide opportunities for students to reason inductively. The next step was introducing line *b,* that represented the tangent line of the function y=a^x at the point *A.* A *tangent line* is a line that locally touches a curve at one, and only one, point and this is an embedded function within the GeoGebra software. Furthermore, students used another embedded GeoGebra feature known as the slope of the function. The students' theoretical knowledge was limited to the definition of the derivative of a function and its connection to the tangent line at the point of the function graph. After revising the theoretical framework students concluded the following:

$$a^x \lim_{h \to 0} \frac{a^h - 1}{h} = f'(x)$$

(4)

$$\lim_{h \to 0} \frac{a^h - 1}{h} = \frac{f'(x)}{a^x}$$

(5)

Students noted these results in a table, which is another very useful feature in GeoGebra. In the table they noted the following results:

In column *A*, students inputted approximately ten x-axis values for points of the graph of the function. Their x- axis values were defined as x$_A$, 2x$_A$, 3x$_A$, etc., where the value x$_A$ was defined by the slider. In column *B*, students defined y-axis values of the point. Column *C* was reserved for the equation of the tangent line at the determined point, and column *D* contained the appropriate slope's values. Finally, column F contained the results of the ratio,

$$\frac{column \ D}{column \ B}$$

(6)

corresponding to the previously analyzed and obtained

$$\frac{f'(x)}{a^x}$$

(7)

The final result was the constant number in each field of column F.

3.3 The Third GeoGebra Worksheet

This part required collaborative group work during which time students discussed and analyzed the obtained data. Students noticed that the limit $\lim_{h\to 0}\frac{a^h-1}{h}$ was constant and also observed that even though the points of the graphed function changed, the constant remained the same. Every student had a different function, and so they all obtained different constants. They collected all constants, overall 17 constants, in one GeoGebra file (see Figure 3).
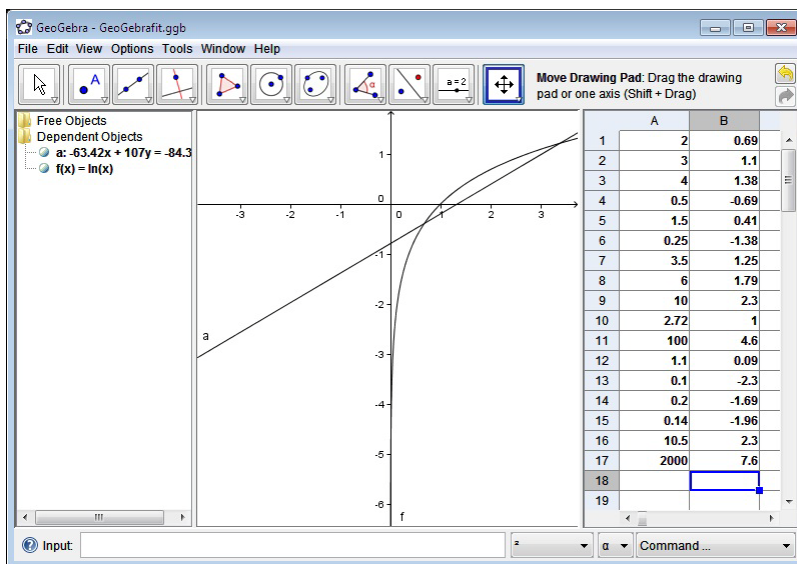

Figure 3. The students' final conclusion

They used the *Table Option* and then another GeoGebra feature for fitting the data. The first column in the table represented values of the base of the function, and the second column represented the obtained constant. They experimented with the features *FitLin, FitExp, FitLog* and came to the conclusion that the best fitting curve was $y = \ln a$. Finally, the conclusion was that the derivative of the function was $f'(x) = (a^x)' = a^x \ln a$.

4. Conclusion

While learning calculus, many students might have difficulties grasping the concept of the derivative as a function that outputs the value of the slope of a tangent. Having this in mind, we tried to get students more actively and visually involved in the development of the derivative. That goal was largely accomplished by using GeoGebra and the "White Box/Black Box" principle during the teaching process. Accordingly, using GeoGebra in the classroom provides a higher quality

of teaching, helps the students with tedious computation, and allows visualization of the problems and pattern recognition for students.

Also, the use of GeoGebra supports students in making new conjectures and in tackling experimentation. In this example, students not only proved and solved the given task, but they also revised and connected several very important mathematical facts. The possibility of recovering mathematical content experimentally is very motivating for many students. The use of a computer gives many opportunities for experiments. The usage of computers in the classroom should be followed by adequate didactical principles, for example the "White Box/Black Box" principle. Although investigations like the one described herein can be quite time-consuming, leading some students and teachers to prefer more traditional methods, its long-term benefits are nonetheless quite remarkable.

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