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Domination Cover Pebbling Number for Even Cycle Lollipop

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Abstract : Given a configuration of pebbles on the vertices of a connected graph G, a pebbling move (or pebbling step) is defined as the removal of two pebbles from a vertex and placing one pebble on an adjacent vertex. The domination cover pebbling number, $\psi(G)$, of a graph G is the minimum number of pebbles that are placed on V(G) such that after a sequence of pebbling moves, the set of vertices with pebbles forms a dominating set of G, regardless of the initial configuration. In this paper, we determine $\psi(G)$ for even cycle lollipop.

Key words : Pebbling, Cover pebbling, Domination cover pebbling, Lollipop.
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1. Introduction

One recent development in graph theory, suggested by Lagarias and Saks, called pebbling, has been the subject of much research. It was first introduced into the literature by Chung [1], and has been developed by many others including Hulbert, who published a survey of graph pebbling [5]. There have been many developments since Hulbert's survey appeared.

Given a graph G, distribute k pebbles (indistinguishable markers) on its vertices in some configuration C. Specifically, a configuration on a graph G is a function from V(G) to $N \cup \{0\}$ representing an arrangement of pebbles on G. For our purposes, we will always assume that G is connected. A pebbling move (or pebbling step) is defined as the removal of two pebbles from some vertex and the placement of one of these pebbles on an adjacent vertex. Define the pebbling number, $\pi(G)$, to be the minimum number of pebbles such that regardless of their initial configuration, it is possible to move to any root vertex v a pebble by a sequence of pebbling moves. Implicit in this definition is the fact that if after moving to vertex v one desires to move to another root vertex, the pebbles reset to their original configuration.

The domination cover pebbling [3] is the combination of two ideas cover pebbling [2] and domination [4]. This introduces a new graph invariant called the domination cover pebbling number, $\psi(G)$. Recall that, a set of vertices D in G is a dominating set if every vertex in G is either in D or adjacent to a vertex of D. The cover pebbling number, $\lambda(G)$, is defined as the minimum number of pebbles required such that given any initial configuration of at least $\lambda(G)$ pebbles, it is possible to make a series of pebbling moves to place at least one pebble on every vertex of G. The domination cover pebbling number of a graph G, proposed by A. Teguia, is the minimum number $\psi(G)$ of pebbles required such that any initial configuration of at least $\psi(G)$ pebbles can be transformed so that the set of vertices that contain pebbles form a dominating set of G. In [3], Gardner et.al. have computed domination cover pebbling number for complete r-partite graph, path, wheel graph, cycle, and binary tree. We have determined the domination cover pebbling number for the odd cycle lollipop graph [6] and the square of a path [7]. In section 2, we determine the domination cover pebbling number for even cycle lollipop. We use the following theorems from [3] for further discussion :

Theorem 1.1[3] For n≥3, $\psi(P_n) = 2^{n+1}\left(\frac{1-8^{-(\beta_n+1)}}{7}\right) + \left\lfloor \frac{\alpha_n}{2} \right\rfloor$, where

$n-2 = \alpha_n + 3\beta_n \equiv \alpha_n \pmod 3$. ■

From Theorem1.1, we can derive the following:

$$\psi(P_n) = \begin{cases} \frac{2^{n+1}-1}{7}, \text{ if } \alpha_n = 0 \\ \frac{2^{n+1}-2}{7}, \text{ if } \alpha_n = 1 \\ \frac{2^{n+1}+3}{7}, \text{ if } \alpha_n = 2 \end{cases}$$

Also, from this we have,

$$\frac{2^{n+1}-2}{7} \le \psi(P_n) \le \frac{2^{n+1}+3}{7}.$$

Theorem 1.2[3] Let C_m be a cycle on m vertices. Then the domination cover pebbling number of C_m is given by,

$$\psi(C_m) = \begin{cases} \psi(P_k) + \psi(P_{k-1}) - |\alpha_k - 1||\alpha_{k-1} - 1|, \text{if } m = 2k-2 (k \ge 3) \\ 2\psi(P_k) - |\alpha_k - 1|, \text{if } m = 2k-1 (k \ge 2) \end{cases}$$

where, $k-2 \equiv \alpha_k \pmod 3$ *and* $(k-1)-2 \equiv \alpha_{k-1} \pmod 3$. ■

2 Domination cover pebbling number for even cycle lollipop

Definition2.1 [6] For a pair of integers m≥3 and n≥2, let L(m,n) be the lollipop graph of order n+m-1 obtained from a cycle C_m by attaching a path of length n-1 to a vertex of the cycle.

If the cycle C_m in L(m,n) is even, then we call L(m,n) an even cycle lollipop.
We will use the following labeling for the graphs C_m and P_n:

C_m: $v_0\ v_1\ v_2 \ldots v_{m-1}\ v_0$ (m≥3) and P_n: $v_0 v_{p_1} v_{p_2} \ldots v_{p_{n-1}}$ (n≥2)

Now, we proceed to find the domination cover pebbling number for L(m,2), where m≥4.

Here after we use the following notations: consider the paths P_A: $v_0\ v_1\ v_2 \ldots v_{k-2}$ and P_B: $v_k\ v_{k+1}\ v_{k+2} \ldots v_{m-1} v_0$ belonging to the cycle C_m, where m = 2k-2(k≥3).
Let P_C: $v_{p_1} v_{p_2} \ldots v_{p_{n-1}}$.

Let $\hat{f}(v_i)$ be the number of pebbles at the vertex v_i and $\hat{f}(P_A)$ be the number of pebbles on the path P_A.

Theorem2.2 Let L(m,2) be a lollipop graph where m=2k-2(k≥3) and $k-2 \equiv \alpha_k \pmod 3$. Then $\psi(L(m,2)) = \begin{cases} 2\psi(C_m), if\ \alpha_k = 0\ or\ 1 \\ 2\psi(C_m)+1, if\ \alpha_k = 2 \end{cases}$.

Proof: Consider the lollipop graph L(m,2), where m=2k-2(k≥3) and $k-2 \equiv \alpha_k \pmod 3$.

Case1: Let α_k=1. Then k?3.

Consider the distribution of $2\psi(C_m)$-1 pebbles on v_{p_1}, then clearly we cannot cover dominate at least one of the vertices of L(m,2). Thus, $\psi(L(m,2)) \geq 2\psi(C_m)$.

Now, consider the distribution of $2\psi(C_m)$ pebbles on the vertices of L(m,2), where $\alpha_k=1$.

Case1.1: C_m contains at least $\psi(C_m)$ pebbles.

If v_{p_1} contains one or more pebbles then we are done (by our assumption). So, assume that v_{p_1} contains zero pebbles. This implies that C_m contains $2\psi(C_m)$ pebbles. We have to send a pebble to v_0, to cover dominate the vertex v_{p_1}. Suppose we cannot send a pebble to v_0. Then, we must have

$$\hat{f}(P_A)+\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor \leq 2^{k-2}-1 \text{ and } \hat{f}(P_B)+\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor \leq 2^{k-2}-1.$$

Adding the above inequalities, we get

$$\hat{f}(P_A)+\hat{f}(P_B)+2\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor \leq 2^{k-1}-2. \qquad \text{---- (1)}$$

To minimize the L.H.S of (1), it is sufficient to assume that $\hat{f}(P_A)=0=\hat{f}(P_B)$. That is, we may assume that all the pebbles are placed at the vertex v_{k-1}. From (1), we get

$$2\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor \leq 2^{k-1}-2.$$

$$\textit{That is, } \hat{f}(v_{k-1}) \leq 2^{k-1}-1. \qquad \text{---- (2)}$$

But, we have, $\hat{f}(v_{k-1}) \geq 2\psi(C_m)$

$$=2\left[\psi(P_k)+\psi(P_{k-1})-|\alpha_k-1||\alpha_{k-1}-1|\right]$$

$$=2\left(\frac{2^{k+1}+2^k-3}{7}\right)$$

$$=\frac{6(2^k-1)}{7} \geq 2^{k-1},$$

where the second inequality follows since m=2k-2, the third inequality follows since α_k=1 and α_{k-1}=0, and the fifth inequality follows since k ≥3.

$$That\ is, \hat{f}(v_{k-1}) \geq 2^{k-1}, k \geq 3. \qquad \text{---- (3)}$$

The inequality in (2) contradicts the inequality in (3). So, we can send a pebble to v_0 using at most 2^{k-1} pebbles. If we use at most 2^{k-2} pebbles from C_m then the minimum number of pebbles that C_m contains is

$$2\psi(C_m)-2^{k-2}$$
$$=\psi(C_m)+\psi(C_m)-2^{k-2}$$
$$=\psi(C_m)+\left(\frac{5.2^{k-2}-3}{7}\right)$$
$$\geq\psi(C_m),$$

where the second equality follows since m=2k-2 and $\alpha_k = 1$, and the third inequality follows since k≥3.

If we use exactly 2^{k-1} pebbles to pebble v_0 then clearly all the pebbles are at v_{k-1}. Note that we have also cover dominated the vertices v_1 and v_{m-1} by putting a pebble at v_0 using 2^{k-1} pebbles from v_{k-1}. Now, we have to cover dominate the remaining vertices of C_m. For that we need $2\psi(P_{k-2})$ -1 pebbles at v_{k-1}, since the paths $v_{k-1}\ v_{k-2} \ldots v_2$ and $v_{k-1}\ v_k \ldots v_{m-2}$ are of length k-3. That is, we need $\frac{2^k+6}{7}-1$ pebbles (since α_{k-2} = 2) from v_{k-1}. But

$$2\psi(C_m)-2^{k-1} = 2\left(\frac{3.2^k-3}{7}\right)-2^{k-1} = \frac{5.2^{k-1}-6}{7} = \frac{2^k+3.2^{k-1}-6}{7} \geq \frac{2^k+6}{7},.$$

where the first equality follows since m = 2k-2, α_k = 1 and the fourth inequality follows since k ≥ 3.

Thus, we have enough pebbles to cover dominate the remaining vertices of C_m and we are done.

Case1.2: C_m contains $x<\psi(C_m)$ pebbles.

This implies that, v_{p_1} contains at least $2\psi(C_m)-x$ pebbles. We can send $\psi(C_m)-\left\lfloor \frac{x}{2} \right\rfloor$ pebbles to v_0. So, C_m contains at least $x+\psi(C_m)-\left\lfloor \frac{x}{2} \right\rfloor \geq \psi(C_m)$ pebbles and we are done.

Case2: Let $\alpha_k=0$. Then $k\geq 5$.

Consider the distribution of $2\psi(C_m)-1$ pebbles on v_{p_1}, then clearly we cannot cover dominate at least one of the vertices of L(m,2). Thus, $\psi(L(m,2)) \geq 2\psi(C_m)$.

Now, consider the distribution of $2\psi(C_m)$ pebbles on the vertices of L(m,2), where $\alpha_k=0$.

Case2.1: C_m contains at least $\psi(C_m)$ pebbles.

If v_{p_1} contains one or more pebbles then we are done (by our assumption). So, assume that v_{p_1} contains zero pebbles. This implies that C_m contains $2\psi(C_m)$ pebbles. We have to send a pebble to v_0, to cover dominate the vertex v_{p_1}. Suppose we cannot send a pebble to v_0. Then, we must have,

$$\hat{f}(P_A)+\hat{f}(P_B)+2\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor \leq 2^{k-1}-2. \quad ---- (4)$$

To minimize the L.H.S of (4), it is sufficient to assume that $\hat{f}(P_A)=0=\hat{f}(P_B)$. That is, we may assume that all the pebbles are at the vertex v_{k-1}. From (4), we get

$$\hat{f}(v_{k-1}) \leq 2^{k-1}-1. \quad ---- (5)$$

But, we have, $\hat{f}(v_{k-1}) \geq 2\psi(C_m)$

$$=\frac{3.2^{k+1}-10}{7} \geq 2^{k-1}, \quad ---- (6)$$

where the second equality follows since $m=2k-2$, $\alpha_k=0$ and third inequality follows since $k\geq 5$.

The inequality in (5) contradicts the inequality in (6). So, we can send a pebble to v_0 using at most 2^{k-1} pebbles. If we use at most 2^{k-2} pebbles from C_m then the minimum number of pebbles that C_m contains is

$$2\psi(C_m)-2^{k-2}$$

$$=\psi(C_m)+\left(\frac{5.2^{k-2}-5}{7}\right)$$

$$\geq\psi(C_m),$$

where the first equality follows since $m = 2k-2$ and $\alpha_k = 0$, and the second inequality follows since $k \geq 5$.

If we use exactly 2^{k-1} pebbles to pebble v_0 then clearly all the pebbles are at v_{k-1}. Note that we have also cover dominated the vertices v_1 and v_{m-1} by putting a pebble at v_0 using 2^{k-1} pebbles from v_{k-1}. Now, we have to cover dominate the remaining vertices of C_m. For that we need $2\psi(p_{k-2})$ pebbles at v_{k-1}, since the paths $v_{k-1}\ v_{k-2} \ldots v_2$ and $v_{k-1}\ v_k \ldots v_{m-2}$ are of length k-3. That is, we need $\frac{2^k-4}{7}$ pebbles (since $\alpha_{k-2}=1$) from v_{k-1}. But

$$2\psi(C_m)-2^{k-1}=2\left(\frac{3.2^k-5}{7}\right)-2^{k-1}=\frac{5.2^{k-1}-5}{7}=\frac{2^k+3.2^{k-1}-5}{7}\geq\frac{2^k-4}{7},$$

where the first equality follows since $m = 2k-2$, $\alpha_k = 0$ and the fourth inequality follows since $k \geq 5$.

Thus, we have enough pebbles to cover dominate the remaining vertices of C_m and we are done.

Case2.2: C_m contains $x<\psi(C_m)$ pebbles.

This implies that, v_{p_1} contains at least 2ψ(C_m) –x pebbles. We can send ψ(C_m)- $\left\lfloor \frac{x}{2} \right\rfloor$ pebbles to v_0. So, C_m contains at least x+ ψ(C_m)- $\left\lfloor \frac{x}{2} \right\rfloor$ ≥ψ(C_m) pebbles and we are done.

Case3: Let α_k =2. Then k≥4.

Consider the distribution of 2ψ(C_m) pebbles on v_{p_1} , then clearly we cannot cover dominate at least one of the vertices of L(m,2). Thus, $\psi\,(L(m,2)) \geq 2\psi\,(C_m)+1$.

Now, consider the distribution of 2ψ(C_m)+1 pebbles on the vertices of L(m,2), where α_k=2.

Case3.1: C_m contains at least ψ(C_m) pebbles.

If v_{p_1} contains one or more pebbles then we are done (by our assumption). So, assume that v_{p_1} contains zero pebbles. This implies that C_m contains 2ψ(C_m)+1 pebbles. We have to send a pebble to v_0, to cover dominate the vertex v_{p_1} . Suppose we cannot send a pebble to v_0. Then, we must have,

$$\hat{f}(P_A) + \hat{f}(P_B) + 2\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor \leq 2^{k-1} - 2. \qquad \text{---- (7)}$$

To minimize the L.H.S of (7), it is sufficient to assume that $\hat{f}(P_A) = 0 = \hat{f}(P_B)$. That is, we may assume that all the pebbles are at the vertex v_{k-1}. From (7), we get

$$\hat{f}(v_{k-1}) \leq 2^{k-1} - 1. \qquad \text{---- (8)}$$

But, we have, $\hat{f}(v_{k-1}) \geq 2\psi\,(C_m)+1$

$$= \frac{6(2^k)+2}{7} \geq 2^{k-1}, \qquad \text{---- (9)}$$

where the second equality follows since m=2k-2, and α_k=2 , and the third inequality follows since k≥4.

The inequality in (8) contradicts the inequality in (9). So, we can send a pebble to v_0 using at most 2^{k-1} pebbles. If we use at most 2^{k-2} pebbles from C_m then the minimum number of pebbles that C_m contains is

$2\psi(C_m)-2^{k-2} \geq \psi(C_m)$, since m = 2k-2, $\alpha_k = 0$, and k ≥ 4.

If we use exactly 2^{k-1} pebbles to pebble v_0 then clearly all the pebbles are at v_{k-1}. Note that we have also cover dominated the vertices v_1 and v_{m-1} by putting a pebble at v_0 using 2^{k-1} pebbles from v_{k-1}. Now, we have to cover dominate the remaining vertices of C_m. For that we need $2\psi(p_{k-2})-1 = \frac{2^k-2}{7}-1$ pebbles at v_{k-1}, since the paths v_{k-1} v_{k-2} …v_2 and v_{k-1} v_k …v_{m-2} are of length k-3 and α_{k-2} = 0. But

$$2\psi(C_m)-2^{k-1} = 2\left(\frac{3.2^k+1}{7}\right)-2^{k-1} = \frac{5.2^{k-1}+2}{7} = \frac{2^k+3.2^{k-1}+2}{7} \geq \frac{2^k-2}{7},$$

where the first equality follows since m = 2k-2, α_k = 2 and the fourth inequality follows since k ≥ 4.

Thus, we have enough pebbles to cover dominate the remaining vertices of C_m and we are done.

Case3.2: C_m contains $x<\psi(C_m)$ pebbles.

This implies that, v_{p_1} contains at least $2\psi(C_m)-x$ pebbles. We can send $\psi(C_m)-\left\lfloor \frac{x}{2} \right\rfloor$ pebbles to v_0. So, C_m contains at least $x+\psi(C_m)-\left\lfloor \frac{x}{2} \right\rfloor \geq \psi(C_m)$ pebbles and we are done.

Thus, $\psi(L(m,2)) \leq \begin{cases} 2\psi(C_m), if\ \alpha_k = 0\ or\ 1 \\ 2\psi(C_m)+1, if\ \alpha_k = 2 \end{cases}$, where m=2k-2(k≥3).

Therefore, $\psi(L(m,2)) = \begin{cases} 2\psi(C_m), if\ \alpha_k = 0\ or\ 1 \\ 2\psi(C_m)+1, if\ \alpha_k = 2 \end{cases}$

where m=2k-2(k≥3) and k-2=α_k(mod 3). ■

Next, we proceed to find the domination cover pebbling number of L(m,n), where m=2k-2(k≥3) and n≥3.

Theorem 2.3 Let L(m,n) be a lollipop graph with m=2k-2≥4 and n≥3. Then, the domination cover pebbling number for L(m,n) is

$$\psi(L(m,n)) = \begin{cases} 2^{n-1}\psi(C_m)+\psi(P_{n-2}), if\ \alpha_k = 0\ or\ 1 \\ 2^{n-1}\psi(C_m)+\psi(P_{n-1}), if\ \alpha_k = 2 \end{cases}$$, where k-2 ≡ α_k(mod 3).

Proof: Consider the lollipop graph L(m,n) where m=2k-2≥4 and n≥3.

Case1: Let α_k=0. Then k≥5.

Consider the distribution of ψ(L(m,n))-1 pebbles at $v_{p_{n-1}}$. Clearly, we cannot cover dominate at least one of the vertices of L(m,n). Thus, $\psi(L(m,n)) \geq 2^{n-1}\psi(C_m)+\psi(P_{n-2})$.

Now, consider the distribution of ψ(L(m,n)) pebbles on the vertices of L(m,n).

Case1.1: C_m contains at least $\psi(C_m)$ pebbles.

If P_C contains $\psi(P_{n-1})$ pebbles or more, then clearly we are done(by our assumption). So assume that P_C contains x< $\psi(P_{n-1})$ pebbles. This implies that, C_m contains $2^{n-1}\psi(C_m)+\psi(P_{n-2})-x$ pebbles. Suppose, we cannot move $\psi(P_n)$-x pebbles to v_0, then we must have,

$$\hat{f}(P_A)+\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor \leq 2^{k-2}[\psi(P_n)-x]-1$$

$$and\ \hat{f}(P_B)+\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor \leq 2^{k-2}[\psi(P_n)-x]-1.$$

Adding the above inequalities, we get

$$\hat{f}(P_A)+\hat{f}(P_B)+2\left\lfloor\frac{\hat{f}(v_{k-1})}{2}\right\rfloor \le 2^{k-1}\left[\psi(P_n)-x\right]-2. \qquad \text{---- (10)}$$

To minimize the L.H.S of (10), it is sufficient to assume that $\hat{f}(P_A)=0=\hat{f}(P_B)$. That is, we may assume that all the pebbles are at v_{k-1}.

From (10), we get

$$\hat{f}(v_{k-1}) \le 2^{k-1}\left[\psi(P_n)-x\right]-1. \qquad \text{---- (11)}$$

But, we have $\hat{f}(v_{k-1}) \ge 2^{n-1}\psi(C_m)+\psi(P_{n-2})-x$

$$=2^{n-1}\left[\psi(P_k)+\psi(P_{k-1})-|\alpha_k-1||\alpha_{k-1}-1|\right]+\psi(P_{n-2})-x$$

$$\ge 2^{n-1}\left[\frac{2^{k+1}-1}{7}+\frac{2^k+3}{7}-1\right]+\frac{2^{n-1}-2}{7}-x$$

$$=2^{n-1}\left(\frac{3(2^k)-5}{7}\right)+\frac{2^{n-1}-2}{7}-x$$

$$=\frac{2^{n+1}(2^k)-5(2^{n-1})-2^k(2^{n-1})}{7}+\frac{2^{n-1}-2}{7}-x$$

$$\ge 2^k\psi(P_n)-\left(\frac{4(2^{n-1})+2^k(2^{n-1}+3)}{7}\right)-\frac{2}{7}-x$$

$$=2^{k-1}\psi(P_n)+\left[2^{k-1}\psi(P_n)-\left(\frac{4(2^{n-1})+2^k(2^{n-1}+3)}{7}\right)\right]-\frac{2}{7}-x$$

$$=2^{k-1}\psi(P_n)+2^{k-1}\left[\frac{7(2^{n-3})-8}{7}\right]-\frac{2}{7}-x$$

$$\ge 2^{k-1}\psi(P_n)-x,$$

where the second inequality follows since m=2k-2,the third inequality follows since $\alpha_k = 0, \alpha_{k-1} = 2 \; and \; \psi(P_n) \geq \frac{2^{n+1}-2}{7}$,the eighth equality follows since $k \geq 5 \; and \; \psi(P_n) \geq \frac{2^{n+1}-2}{7}$,and the ninth inequality follows since $n \geq 4, k \geq 5 \; and \; for \; n = 3, it \; is \; obvious$.

That is, $\hat{f}(v_{k-1}) \geq 2^{k-1}\psi(P_n) - x.$ ---- (12)

The inequality in (11) contradicts the inequality in (12). So, we can always send $\psi(P_n) - x$ pebbles to v_0 at a cost of $2^{k-1}[\psi(P_n) - x]$ pebbles (at most). So, we cover dominate the path P_n. Now, we have to cover dominate C_m. In C_m, we have at least $2^{n-1}\psi(C_m) + \psi(P_{n-2}) - x - 2^{k-1}[\psi(P_n) - x]$ pebbles. We need at most $\psi(C_m)$ pebbles to cover dominate the vertices of C_m. But,

$$2^{n-1}\psi(C_m) + \psi(P_{n-2}) - x - 2^{k-1}[\psi(P_n) - x] - \psi(C_m)$$

$$= (2^{n-1} - 1)\psi(C_m) + \psi(P_{n-2}) - x - 2^{k-1}[\psi(P_n) - x]$$

$$= (2^{n-1} - 1)[\psi(P_k) + \psi(P_{k-1}) - 1] + \psi(P_{n-2}) - x - 2^{k-1}[\psi(P_n) - x]$$

$$\geq (2^{n-1} - 1)\left[\frac{2^{k+1}-1}{7} + \frac{2^k+3}{7} - 1\right] + \psi(P_{n-2}) - x - 2^{k-1}\left[\frac{2^{n+1}+3}{7} - x\right]$$

$$= (2^{n-1})\left(\frac{2^k-5}{7}\right) - \left(\frac{9(2^{k-1})}{7}\right) + \psi(P_{n-2}) + (2^{k-1}-1)x + \frac{5}{7}$$

$$\geq (2^{n-1})\left(\frac{2^k-5}{7}\right) - \left(\frac{9(2^{k-1})}{7}\right) + \left(\frac{2^{n-1}-2}{7}\right)$$

$$\geq (2^{n-1})\left(\frac{2^k-5}{7} - \frac{9(2^{k-1})}{7(8)} + \frac{1}{7} - \frac{2}{7(8)}\right)$$

$$= (2^{n-1})\left(\frac{7(2^{k-1}) - 40 + 6}{56}\right) > 0,$$

$$= (2^{n-1})\left(\frac{7(2^{k-1}) - 40 + 6}{56}\right) > 0,$$

where the second equality follows since m=2k-2, third inequality follows $since\, \alpha_k = 0, \alpha_{k-`1} = 2\; and\, \psi(P_n) \le \frac{2^{n+1}+3}{7}$,the fifth inequality follows since k$\geq$5,the seventh inequality follows since n$\geq$4 and for n=3 it is obvious, and the eighth equality follows since k $\geq$5 and n$\geq$3.

Thus, we have enough pebbles to cover dominate C_m and hence we are done.

Case1.2: C_m contains y<ψ(C_m) pebbles.

This implies that, P_C contains $2^{n-1}\psi(C_m) + \psi(P_{n-2}) - y$ pebbles. We use at most ψ(P_{n-1}) pebbles to cover dominate the vertices of P_C. Thus, we have at least $2^{n-1}\psi(C_m) + \psi(P_{n-2}) - y - \psi(P_{n-1})$ pebbles in P_C.

We need at most 2^{n-1} [ψ(C_m)-y] pebbles from P_C to cover dominate the vertices of C_m. But,

$$2^{n-1}\psi(C_m) + \psi(P_{n-2}) - y - \psi(P_{n-1}) - 2^{n-1}\left[\psi(C_m) - y\right]$$

$$\ge 2^{n-1}y + \frac{2^{n-1}-2}{7} - y - \frac{2^n+3}{7}$$

$$\ge 2^{n-1}\left[\frac{21y-9}{28}\right] > 0,$$

where the second inequality follows since n$\geq$3. Thus, we can send ψ(C_m)-y pebbles to v_0 and already C_m contains y pebbles and so C_m contains ψ(C_m) pebbles and we are done.

So, $\psi(L(m,n)) \le 2^{n-1}\psi(C_m) + \psi(P_{n-2})$.

Therefore, $\psi(L(m,n)) = 2^{n-1}\psi(C_m) + \psi(P_{n-2})$, if α_k=0.

Case2: Let α_k=1. Then k≥3.

Consider the distribution of ψ(L(m,n))-1pebbles at $v_{p_{n-1}}$. Clearly, we cannot cover dominate at least one of the vertices of L(m,n). Thus, $\psi(L(m,n)) \geq 2^{n-1}\psi(C_m) + \psi(P_{n-2})$.

Now, consider the distribution of ψ(L(m,n)) pebbles on the vertices of L(m,n).

Case2.1: C_m contains at least $\psi(C_m)$ pebbles.

If P_C contains at least $\psi(P_{n-1})$ pebbles, then clearly we are done(by our assumption). So assume that P_C contains $x< \psi(P_{n-1})$ pebbles. This implies that, C_m contains $2^{n-1}\psi(C_m) + \psi(P_{n-2}) - x$ pebbles. Suppose, we cannot move $\psi(P_n)$-x pebbles to v_0, then we must have,

$$\hat{f}(P_A) + \hat{f}(P_B) + 2\left\lfloor \frac{\hat{f}(v_{k-1})}{2} \right\rfloor \leq 2^{k-1}\left[\psi(P_n) - x\right] - 2. \qquad \text{---- (13)}$$

To minimize the L.H.S of (13), it is sufficient to assume that $\hat{f}(P_A)$=0= $\hat{f}(P_B)$. That is, we may assume that all the pebbles are at v_{k-1}.

From (13), we get

$$\hat{f}(v_{k-1}) \leq 2^{k-1}\left[\psi(P_n) - x\right] - 1. \qquad \text{---- (14)}$$

But, we have $\hat{f}(v_{k-1}) \geq 2^{n-1}\psi(C_m) + \psi(P_{n-2}) - x$

$$\geq 2^{n-1}\left(\frac{2^{k+1} - 2 + 2^k - 1}{7}\right) + \frac{2^{n-1} - 2}{7} - x$$

$$= 2^{n-1}\left(\frac{3(2^k) - 3}{7}\right) + \frac{2^{n-1} - 2}{7} - x$$

$$= \frac{2^{n+1}(2^k) - 3(2^{n-1}) - 2^k(2^{n-1})}{7} + \frac{2^{n-1} - 2}{7} - x$$

$$\geq 2^k\psi(P_n) - \left(\frac{2(2^{n-1}) + 2^k(2^{n-1} + 3)}{7}\right) - \frac{2}{7} - x$$

$$= 2^{k-1}\psi(P_n) + \left[2^{k-1}\psi(P_n) - \left(\frac{2(2^{n-1}) + 2^k(2^{n-1}+3)}{7}\right)\right] - \frac{2}{7} - x$$

$$\geq 2^{k-1}\psi(P_n) + 2^{k-1}\left[\left(\frac{2^{n+1}-2}{7}\right) - \left(\frac{2(2^{n-1})}{7(4)}\right) - \left(\frac{2^n+6}{7}\right)\right] - \frac{2}{7} - x$$

$$= 2^{k-1}\psi(P_n) + 2^{k-1}\left[\frac{3(2^n)-32}{28}\right] - \frac{2}{7} - x$$

$$\geq 2^{k-1}\psi(P_n) - 1 - x,$$

where the second inequality follows since m = 2k-2, $\alpha_k = 1$, $and\, \psi(P_n) \geq \frac{2^{n+1}-2}{7}$, seventh inequality follows $since\; k \geq 3\; and\; \psi(P_n) \geq \frac{2^{n+1}-2}{7}$ and the ninth inequality follows $since\; n \geq 4, k \geq 3\; and\; for\; n = 3, it\; is\; obvious..$

That is, $\hat{f}(v_{k-1}) \geq 2^{k-1}\psi(P_n) - 1 - x.$ ---- (15)

The inequality in (14) contradicts the inequality in (15). So, we can always send $\psi(P_n) - x$ pebbles to v_0 at a cost of $2^{k-1}[\psi(P_n) - x]$ pebbles(at most). So, we cover dominate the path P_n. Now, we have to cover dominate C_m. In C_m, we have at least $2^{n-1}\psi(C_m) + \psi(P_{n-2}) - x - 2^{k-1}[\psi(P_n) - x]$ pebbles. We need at most $\psi(C_m)$ pebbles to cover dominate C_m. But,

$$2^{n-1}\psi(C_m) + \psi(P_{n-2}) - x - 2^{k-1}[\psi(P_n) - x] - \psi(C_m)$$

$$= (2^{n-1}-1)\psi(C_m) + \psi(P_{n-2}) - x - 2^{k-1}[\psi(P_n) - x]$$

$$= (2^{n-1}-1)[\psi(P_k) + \psi(P_{k-1}) - 1] + \psi(P_{n-2}) - x - 2^{k-1}[\psi(P_n) - x]$$

$$\geq (2^{n-1}-1)\left(\frac{3(2^k)-3}{7}\right)+\frac{2^{n-1}-2}{7}+(2^{k-1}-1)x-2^{k-1}\left(\frac{2^{n+1}+3}{7}\right)$$

$$\geq (2^{n-1})\left(\frac{3(2^k)-3}{7}-\frac{3(2^k)}{7(2^{n-1})}-\frac{2}{7(2^{n-1})}-\frac{2(2^k)}{7}-3\left(\frac{2^{k-1}}{7(2^{n-1})}\right)+\frac{1}{7}\right)$$

$$\geq (2^{n-1})\left(\frac{2^k-3}{7}-\frac{9(2^{k-1})}{7(8)}-\frac{2}{7(8)}+\frac{1}{7}\right)$$

$$=(2^{n-1})\left(\frac{16(2^{k-1})-9(2^{k-1})-24-2+8}{56}\right)$$

$$=(2^{n-1})\left(\frac{7(2^{k-1})-18}{56}\right)>0,$$

where the second equality follows since m-2k-2, the third inequality follows $since\,\alpha_k=1,\alpha_{k-`1}=0\ and\ \frac{2^{n+1}-2}{7}\leq\psi(P_n)\leq\frac{2^{n+1}+3}{7}$,the fifth inequality follows since n $\geq$ 4 and for n=3 it is obvious, and the seventh inequality follows since k$\geq$3 and n$\geq$ 3.

Thus, we have enough pebbles to cover dominate C_m and hence we are done.

Case2.2: C_m contains $y<\psi(C_m)$ pebbles.

This implies that, P_C contains $2^{n-1}\psi(C_m)+\psi(P_{n-2})-y$ pebbles. We use at most $\psi(P_{n-1})$ pebbles to cover dominate the vertices of P_C. Thus, we have at least $2^{n-1}\psi(C_m)+\psi(P_{n-2})-y-\psi(P_{n-1})$ pebbles in P_C.

We need at most 2^{n-1} [$\psi(C_m)$-y] pebbles from P_C to cover dominate the vertices of C_m. But,

$$2^{n-1}\psi(C_m)+\psi(P_{n-2})-y-\psi(P_{n-1})-2^{n-1}\left[\psi(C_m)-y\right]$$

$$=2^{n-1}y+\psi(P_{n-2})-y-\psi(P_{n-1})$$

$$\geq 2^{n-1}y+\frac{2^{n-1}-2}{7}-y-\frac{2^{n}+3}{7}$$

$$\geq 2^{n-1}\left[y-\frac{5+7y}{7(4)}-\frac{1}{7}\right]$$

$$=2^{n-1}\left[\frac{21y-9}{28}\right]>0,$$

where the third inequality follows since n$\geq$3. Thus, we can send ψ(C_m)-y pebbles to v_0 and already C_m contains y pebbles and so C_m contains ψ(C_m) pebbles and we are done.

So, $\psi(L(m,n)) \leq 2^{n-1}\psi(C_m)+\psi(P_{n-2})$.

Therefore, $\psi(L(m,n)) = 2^{n-1}\psi(C_m)+\psi(P_{n-2})$, if α_k=1.

Case3: Let α_k=2. Then k≥4.

Consider the distribution of ψ(L(m,n))-1pebbles at $v_{p_{n-1}}$. Clearly, we cannot cover dominate at least one of the vertices of L(m,n). Thus, $\psi(L(m,n)) \geq 2^{n-1}\psi(C_m)+\psi(P_{n-1})$.

Now, consider the distribution of ψ(L(m,n)) pebbles on the vertices of L(m,n).

Case3.1: C_m contains at least ψ(C_m) pebbles.

If P_C contains at least ψ(P_{n-1}) pebbles, then clearly we are done(by our assumption). So assume that P_C contains x< ψ(P_{n-1}) pebbles. This implies that, C_m contains $2^{n-1}\psi(C_m)+\psi(P_{n-1})-x$ pebbles. Suppose, we cannot move ψ(P_n)-x pebbles to v_0, then we must have,

$$\hat{f}(P_A)+\hat{f}(P_B)+2\left\lfloor\frac{\hat{f}(v_{k-1})}{2}\right\rfloor \leq 2^{k-1}\left[\psi(P_n)-x\right]-2. \qquad \text{---- (16)}$$

To minimize the L.H.S of (16), it is sufficient to assume that $\hat{f}(P_A)$=0= $\hat{f}(P_B)$.

That is, we may assume that all the pebbles are at v_{k-1}.

From (16), we get

$$\hat{f}(v_{k-1}) \leq 2^{k-1}[\psi(P_n) - x] - 1. \qquad \text{---- (17)}$$

But, we have $\hat{f}(v_{k-1}) \geq 2^{n-1}\psi(C_m) + \psi(P_{n-1}) - x$

$= 2^{n-1}[\psi(P_k) + \psi(P_{k-1}) - |\alpha_k - 1||\alpha_{k-1} - 1|] + \psi(P_{n-1}) - x$

$\geq 2^{n-1}\left(\frac{3(2^k)+1}{7}\right) + \frac{2^n - 2}{7} - x$

$\geq 2^{n+1}\left(\frac{2^k}{7}\right) - 2^{n-1}\left(\frac{2^k}{7}\right) + \frac{3(2^{n-1})}{7} - 1 - x$

$\geq 2^{k-1}\psi(P_n) + 2^{k-1}\left[\psi(P_n) - \left(\frac{3(2^{n-1})}{7(2^{k-1})}\right) - \left(\frac{2^k(2^{n-1}+3)}{7(2^{k-1})}\right)\right] - 1 - x$

$\geq 2^{k-1}\psi(P_n) + 2^{k-1}\left[\frac{13(2^{n-1}) - 64}{28}\right] - 1 - x$

$\geq 2^{k-1}\psi(P_n) - 1 - x,$

where the second inequality follows since m=2k-2, the third inequality follows since $\alpha_k = 2, \alpha_{k-1} = 1 \; and \psi \; (P_n) \geq \frac{2^{n+1}-2}{7}$, the fourth inequality follows $since \; k \geq 4 \; and \; \psi(P_n) \geq \frac{2^{n+1}-2}{7}$, and the seventh inequality follows $since \; n \geq 4, k \geq 4 \; and \; for \; n = 3, it \; is \; obvious.$

That is, $\hat{f}(v_{k-1}) \geq 2^{k-1}\psi(P_n) - 1 - x.$ ---- (18)

The inequality in (17) contradicts the inequality in (18). So, we can always send $\psi(P_n) - x$ pebbles to v_0 at a cost of at most $2^{k-1}[\psi(P_n) - x]$ pebbles. So, we cover dominate the path P_n. Now, we have to cover dominate C_m. In C_m, we have at least $2^{n-1}\psi(C_m) + \psi(P_{n-1}) - x - 2^{k-1}[\psi(P_n) - x]$ pebbles. We need at most $\psi(C_m)$ pebbles to cover dominate the vertices of C_m. But,

$since\ n \geq 4, k \geq 4\ and\ for\ n = 3, it\ is\ obvious.$

That is, $\hat{f}(v_{k-1}) \geq 2^{k-1}\psi(P_n) - 1 - x.$ ---- (18)

The inequality in (17) contradicts the inequality in (18). So, we can always send $\psi(P_n) - x$ pebbles to v_0 at a cost of at most $2^{k-1}[\psi(P_n) - x]$ pebbles. So, we cover

dominate the path P_n. Now, we have to cover dominate C_m. In C_m, we have at least $2^{n-1}\psi(C_m) + \psi(P_{n-1}) - x - 2^{k-1}[\psi(P_n) - x]$ pebbles. We need at most $\psi(C_m)$ pebbles to cover dominate the vertices of C_m. But,

$$2^{n-1}\psi(C_m) + \psi(P_{n-1}) - x - 2^{k-1}[\psi(P_n) - x] - \psi(C_m)$$

$$= (2^{n-1} - 1)\psi(C_m) + \psi(P_{n-1}) - x - 2^{k-1}[\psi(P_n) - x]$$

$$= (2^{n-1} - 1)[\psi(P_k) + \psi(P_{k-1}) - 1] + \psi(P_{n-2}) - x - 2^{k-1}[\psi(P_n) - x]$$

$$\geq (2^{n-1} - 1)\left(\frac{3(2^k) + 1}{7}\right) + \frac{2^n - 2}{7} + (2^{k-1} - 1)x - 2^{k-1}\left(\frac{2^{n+1} + 3}{7}\right)$$

$$= (2^{n-1})\left(\frac{2^k + 3}{7} - \frac{9(2^{k-1})}{7(2^{n-1})} - \frac{3}{7(2^{n-1})}\right)$$

$$\geq (2^{n-1})\left(\frac{16(2^{k-1}) - 9(2^{k-1}) + 24 - 3}{56}\right)$$

$$= (2^{n-1})\left(\frac{7(2^{k-1}) + 21}{56}\right) > 0,$$

where the second equality follows since m=2k-2, the third inequality follows $since\ \alpha_k = 2, \alpha_{k-1} = 1\ and\ \frac{2^{n+1} - 2}{7} \leq \psi(P_n) \leq \frac{2^{n+1} + 3}{7}$,the fifth inequality follows $since\ n \geq 4\ and\ for\ n = 3\ it\ is\ obvious$, and the sixth inequality follows $since\ k \geq 4\ \ and\ \ n \geq 3.$

Thus, we have enough pebbles to cover dominate C_m and hence we are done.

Case3.2: C_m contains $y<\psi(C_m)$ pebbles.

This implies that, P_C contains at least $2^{n-1}\psi(C_m) + \psi(P_{n-1}) - y$ pebbles. We use at most $\psi(P_{n-1})$ pebbles to cover dominate the vertices of P_C. Thus, we have at least $2^{n-1}\psi(C_m) + \psi(P_{n-1}) - y - \psi(P_{n-1})$ pebbles in P_C.

We need at most 2^{n-1} [$\psi(C_m)$-y] pebbles from P_C to cover dominate the vertices of C_m. But,

$$2^{n-1}\psi(C_m)+\psi(P_{n-1})-y-\psi(P_{n-1})-2^{n-1}\left[\psi(C_m)-y\right]$$

$= 2^{n-1}y-y = (2^{n-1}-1)y > 0$, since $n \geq 3$.

Thus, we can send $\psi(C_m)$-y pebbles to v_0 and already C_m contains y pebbles and so C_m contains $\psi(C_m)$ pebbles and we are done.

So, $\psi(L(m,n)) \leq 2^{n-1}\psi(C_m)+\psi(P_{n-1})$.

Therefore, $\psi(L(m,n)) = 2^{n-1}\psi(C_m)+\psi(P_{n-1})$, if $\alpha_k=2$.

Hence, $\psi(L(m,n)) = \begin{cases} 2^{n-1}\psi(C_m)+\psi(P_{n-2}), if\ \alpha_k = 0\ or\ 1 \\ 2^{n-1}\psi(C_m)+\psi(P_{n-1}), if\ \alpha_k = 2 \end{cases}$,

where m=2k-2 and k-2=α_k(mod 3). ■

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