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The odd 2t-pebbling property of graphs

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Abstract : The t-pebbling number, $f_t(G)$, of a connected graph G, is the smallest positive integer such that from every placement of $f_t(G)$ pebbles, t pebbles can be moved to a specified target vertex by a sequence of pebbling moves, each move taking two pebbles off a vertex and placing one on an adjacent vertex. We say a graph G satisfies the odd 2t–pebbling property if, for any arrangement of pebbles with at least $2f_t(G) - r + 1$ pebbles, where r is the number of vertices with an odd number of pebbles in the arrangement, it is possible to put 2t pebbles on any target vertex using pebbling moves. We study the odd 2t-pebbling property of graphs.

Key words : Pebbling, Graham's Conjecture, Direct products, Graph parameters.

1. Introduction :

Let G be a simple connected graph. The pebbling number of G is the smallest number f(G) such that however these f(G) pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex [Chu 89].

The t-pebbling number of a vertex v in a graph G is the smallest number $f_t(v,G)$ with the property that from every placement of $f_t(v,G)$ pebbles on G, it is possible to move t pebbles to v by a sequence of pebbling moves where a pebbling move consists of the removal of two pebbles from a vertex, and the placement of one of those pebbles on an adjacent vertex [HH 98]. The t-pebbling number of the graph G, denoted by $f_t(G)$ is the maximum of $f_t(v,G)$ over all vertices v in G [HH 98]

Note that t is a positive integer here and $f_1(G) = f(G)$.

$\lfloor x \rfloor$ stands for the largest integer $\leq x$ and

$\lceil x \rceil$ stands for the smallest integer $\geq x$

2. Known Results

We find the following results with regard to the t-pebbling number of a graph in [LS 06].

Theorem 2.1 Let G be a connected graph on n vertices where $n \geq 2$. Let there be a vertex v such that $d(v) = n-1$. Then $f_t(v, G) = 2t+n-2$. ■

Theorem 2.2 Let K_n be the complete graph on n vertices where $n \geq 2$.

Then $f_t(K_n) = 2t + n-2$. ■

Theorem 2.3. Let $K_{1,n}$ be an n- star where $n > 1$. Then $f_t(K_{1,n}) = 4t + n-2$. ■

Theorem 2.4 Let C_n denote a simple cycle with n vertices where $n \geq 3$. If n is even, then

$f_t(C_n) = t(2^{n/2})$. If n is odd, then $f_t(C_n) = 1 + (t-1)2^{\lfloor n/2 \rfloor} + 2\lceil \frac{2}{3}(2^{\lfloor n/2 \rfloor} - 1) \rceil$. ■

Herscovici [Her 03] gives the t-pebbling number of all cycles as given in Theorem 2.5.

Theorem 2.5. The t-pebbling number of the cycles C_{2k} and C_{2k+1} satisfy $f_t(C_{2k}) = 2^k.t$,

$$f_t(C_{2k+1}) = \frac{2^{k+2} - (-1)^{k+2}}{3} + 2^k(t-1).$$ ■

Snevily and Foster [SF 00] give the t- pebbling number of odd cycles as given in Theorem 2.6.

Theorem 2.6 The t-pebbling number of C_{2k+1} satisfies $f_t(C_{2k+1}) \le 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1 + (t-1)2^k$. ■

Theorem 2.7 Let P_n be a path on n vertices. Then $f_t(P_n) = t(2^{n-1})$. ■

Theorem 2.8. Let Q_n be the n- cube. Then $f_t(Q_n) = t(2^n)$. ■

3. Generalization of two – pebbling property.

Fan R.K.Chung [Chu 89] defined the 2–pebbling property as follows:

Definition 3.1. [Chu 89]. Suppose p pebbles are placed on a graph G in such a way that q vertices of G are occupied, i.e., there are exactly q vertices which have one pebble or more. We say the graph G satisfies the 2–pebbling property if we can put two pebbles on any specified vertex of G starting from every configuration in which $p \geq 2f(G) - q + 1$ or equivalently $(p+q) > 2f(G)$.

S.S. Wang [Wan 01] referred to this as 2-pebbling graph and he defined odd 2–pebbling graph in the same way, except that q is the number of vertices with an odd number of pebbles.

The following theorems of [Chu 89] are used here.

Theorem 3.2. [Chu 89]. All paths satisfy the 2-pebbling property. ■

Theorem 3.3. [Chu 89]. The n-cube Q_n satisfies the 2-pebbling property. ■

We also find the following theorem in [HH 98].

Theorem 3.4 [HH 98]. The 5 cycle C_5 satisfies the 2-pebbling property. ■

We now state the following results from [Her 03].

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Theorem 3.2. [Chu 89]. All paths satisfy the 2-pebbling property. ■
Theorem 3.3. [Chu 89]. The n-cube Qn satisfies the 2-pebbling property. ■
We also find the following theorem in [HH 98].
Theorem 3.4 [HH 98]. The 5 cycle C5 satisfies the 2-pebbling property. ■
We now state the following results from [Her 03].

Natation 3.5 [Her 03]. Let the vertices of C_n be $\{x_0, x_1, ..., x_{n-1}\}$ in order. Without loss of generality, assume x_0 is the target vertex in C_n. Given a configuration of pebbles on C_n, let p_i represent the number of pebbles on x_i. If n is even, we suppose n = 2k, and if n is odd, we let n = 2k+1. In either case, we define the vertex sets A and B by $A = \{x_1, x_2, ..., x_{k-1}\}$, $B = \{x_{n-1}, x_{n-2}, ..., x_{n-k+1}\}$.

Theorem 3.6. [Her 03]. C_{2k} satisfies the 2-pebbling property for all $k \geq 2$. ■

Theorem 3.7. [Her 03]. C_{2k+1} satisfies the 2-pebbling property for all $k \geq 2$. ■

In [LS 006], we see the generalization of the concept, the 2-pebbling property as the 2t-pebbling property.

Definition 3.8 [LS 006]. Given a t-pebbling of G, let p be the number of pebbles on G, let q be the number of vertices with at least one pebble. We say that G satisfies *the 2t-pebbling property* if it is possible to move 2t pebbles to any specified target vertex of G starting from every configuration in which $p \geq 2\ f_t(G) - q+1$ or equivalently $(p+q)>2f_t(G)$. In this case we also say G is a *2t–pebbling graph*.

If q stands for the number of vertices with an odd number of pebbles, we call the property the odd 2t-pebbling property.

Definition 3.9[LS 006]. We say a graph G satisfies *the odd 2t–pebbling property* if, for any arrangement of pebbles with at least $2f_t\ (G) - r + 1$ pebbles, where r is the number of vertices with an odd number of pebbles in the arrangement, it is possible to put 2t pebbles on any target vertex using pebbling moves. In this case we also say that G is *an odd 2t–pebbling graph.*

It is easy to see that a graph which satisfies the 2t-pebbling property also satisfies the odd 2t-pebbling property.

We find **Lemma 3.10, Corollary 3.11** and **Corollary 3.12** in [LS 006].

Lemma 3.10. Let G satisfy the 2-pebbling property. If $f_t\ (G) = t\ f\ (G)$ then G satisfies the 2t-pebbling property.

Proof : Since G satisfies the 2-pebbling property, if $(p+q)>2t\ f(G)$ then we can put 2t pebbles on any target vertex. We are given that $f_t\ (G) = t\ f\ (G)$. We now consider a configuration of pebbles on G in which p pebbles occupy q vertices where $(p+q)>2f_t\ (G)$. Since G satisfies the 2-pebbling property, we can move 2t pebbles to any target vertex. Hence G satisfies the 2t–pebbling property. ■

Corollary 3.11. All paths satisfy the 2t-pebbling property. ■

Corollary 3.12. All even cycles satisfy the 2t–pebbling property. ■

Corollary 3.13. The n-cube Q_n satisfies the 2t-pebbling property. ■

Since a graph which satisfies the 2t-pebbling property also satisfies the odd 2t-pebbling property, all paths, all even cycles and the n-cube satisfy the odd 2t–pebbling property.

Let us now look at the odd 2t–pebbling property of odd cycles.

Theorem 3.14. C_3 satisfies the odd 2t-pebbling property. ■

Theorem 3.15. C_5 satisfies the odd 2t-pebbling property.

We prove Theorem 3.15. Proving Theorem 3.14 is straight forward and hence it is left to the reader.

Proof of Theorem 3.15: Let $C_5=(x_1,x_2,x_3,x_4,x_5)$. Assume the target vertex to be x_3. Consider the paths $P_1=\{x_1,x_2,x_3\}$ and $P_2 = \{x_3,x_4,x_5\}$.

Consider a configuration of pebbles on C_5 in which p pebbles occupy r vertices with an odd number of pebbles where $p\geq 2f_t(C_5)-r+1=8t+3-r$. Clearly $r\leq 5$. So we get the following cases:

Case 1: $r \leq 2$

Now $p \geq 4(2t) + 1$. So 2t pebbles can be moved to x_3.

Case 2: r=3

No $p\geq 2f_t(C_5) - 2 = 8t$. Let us now use induction on t to prove that 2t pebbles can be moved to x_3.

For t = 1, the result is true by Theorem 3.4.

For $t > 1$, $p \geq 2f_{t-1}(C_5) + 6$. We claim that there are at least eight pebbles on one of the paths P_1 and P_2. If not, each path contains at most seven pebbles and so there are at most fourteen pebbles on C_5 which is a contradiction. So using eight pebbles lying on one of the paths we can put two pebbles on x_3. After using these eight pebbles the remaining number of pebbles lying on the graph are at least $2f_{t-1}(C_5) - 2$. When we start with an odd number of pebbles on a particular vertex, after some pebbling moves, the vertex should have at least one pebble as every pebbling move uses two pebbles. So there are still three vertices with an odd number of pebbles. Now by induction the remaining pebbles are sufficient to put 2(t-1) additional pebbles on x_3.

Case 3: r = 4

Now $p \geq 2f_t(C_5) - 3$. Let us use induction on t to prove that we can put 2t pebbles on x_3.

For t = 1, the result is true by Theorem 3.4.

For t > 1, $p \geq 2f_{t-1}(C_5) + 5$. We claim that there are at least eight pebbles on one of the paths P_1 and P_2. If not, each path contains at most seven pebbles and so there are at most fourteen pebbles on C_5 which is a contradiction. So using eight pebbles lying on one of the paths we can put two pebbles on x_3. After using these eight pebbles the remaining number of pebbles lying on the graph are at least $2f_{t-1}(C_5) - 3$. We note that there are still four vertices with an odd number of pebbles as each move uses two pebbles. Now by induction the remaining pebbles are sufficient to put 2(t-1) additional pebbles on x_3.

Case 4: r=5. Now $p \geq 2f_t(C_5)-4$. Let us use induction on t to prove that 2t pebbles can be put on x_3. For t = 1, the result is true by Theorem 3.4.

For t > 1, $p \geq 2f_{t-1}(C_5) + 4$. That is, $p \geq 14$. We claim that there are at least eight pebbles on one of the paths P_1 and P_2. Suppose not, then both P_1 and P_2 have at most seven pebbles. Now we note that x_3 lies on both paths and x_3 has at least one pebble since r = 5. So the total number of pebbles on C_5 are at most thirteen which is a contradiction. So there are at least eight pebbles on one of the paths. Using these eight pebbles lying on one of the paths we can put two pebbles on x_3. After using these eight pebbles, the remaining number of pebbles on C_5 are at least $2f_{t-1}(C_5) - 4$. We note that there are still five vertices with an odd number of pebbles as each move uses two pebbles. By induction, the remaining pebbles are sufficient to put 2(t-1) additional pebbles on x_3. ■

Theorem 3.16. C_{2k+1} satisfies the odd 2t–pebbling property for all $k \geq 3$.

Proof : Consider a configuration of pebbles on C_{2k+1} in which p pebbles occupy r vertices with an odd number of pebbles where $(p+r) \geq 2f_t(C_{2k+1})+1$. Without loss of generality we may assume that x_0 has zero pebbles.

The proof is by induction on t. For t = 1 we get the result by Theorem 3.7.

For t > 1, $(p+r) \geq 2f_{t-1}(C_{2k+1})+2^{k+1}+1$. We claim that either $A \cup \{x_k\}$ or $B \cup \{x_{k+1}\}$ has at least 2^{k+1} pebbles. If not, then the total number of pebbles placed is less than 2^{k+2}. Then $2^{k+2} + r > p+r \geq 2f_t(C_{2k+1}) + 1$. That is, $r > 3 + (t-3)2^{k+1} + 4\left\lceil \frac{2}{3}(2^k - 1) \right\rceil$ for some $t \geq 2$ and for some $k \geq 3$. This implies $r > 2k+1$ for some $k \geq 3$. This is a contradiction since $r \leq 2k+1$.

So either $A \cup \{x_k\}$ or $B \cup \{x_{k+1}\}$ has at least 2^{k+1} pebbles. Using only 2^{k+1} pebbles of either $A \cup \{x_k\}$ or $B \cup \{x_{k+1}\}$ we can put two pebbles on x_0. Then there remain at least $2f_{t-1}(C_{2k+1}) - r+1$ pebbles. When we start with an odd number of pebbles on a particular vertex, after some pebbling moves, the vertex should have at least one pebble as every pebbling move uses two pebbles. So there are still r vertices with an odd number of pebbles. By induction the remaining pebbles are sufficient to put 2(t-1) additional pebbles on x_0. ■

Theorem 3.17. All cycles satisfy the odd 2t-pebbling property.

Proof : Follows from Corollary 3.12, Theorem 3.14, Theorem 3.15, and Theorem 3.16. ■

We find the following definitions, Example 3.20, and Theorem 3.21 in [Moe 92].

3.17 Path - partition of a rooted tree. Let T be a tree and v be a vertex of T. Let T_v be the rooted tree obtained from T by directing all edges towards v, which becomes the root. For a rooted tree U, we shall call a vertex v of U a leaf it is of indegree 0. We shall call v, a parent of w if there is a directed edge from w to v, and an ancestor of w if there is a directed path from w to v. We call v, a vertex of level n if the directed path from v to the root has n edges; the height of a tree is the maximum level of its vertices. A path-partition of a rooted tree U is a partition of the edges of U such that each set of edges in the partition forms a directed path.

3.18. Maximum Path-partition of a rooted tree. Path - partitions of a rooted tree U with height h can be formed in the following way. First we consider the subtree U^1 of U induced by all leaves of level h and their ancestors and construct a path-partition P^1 of U^1 such that every path in P^1 touches a leaf. Then we let U^{11} be the subtree of U induced by all leaves of level h or h-1 and their ancestors and extend P^1 to a path-partition P^{11} of U^{11} by adding paths, which touch the level h-1 leaves of U. We continue in this manner until we have a path-partition P of all of U. A path-partition constructed in this way is called maximum.

3.19. Path - size sequence. The path-size sequence of a path-partition $\{P_1,P_2,....,P_n\}$ is an n-tuple $(a_1, a_2,...,a_n)$, where a_j is the length of P_j (i.e., the number of edges in it).

Example 3.20. Let us construct a maximum path-partition of the tree U in figure 1. We start with the subtree U^1 of U induced by the vertex i, the unique vertex of U of level 4, and its ancestors b,e, f and h. There is a unique path-partition of U^1 such that every path touches a leaf, namely the path-partition with just one path, {{ib, be, ef, fh}}. Now we extend this path-partition to a path-partition of the subtree of U induced by the set {a,e,i,b,f,h} of all vertices of level 3 or 4 and their ancestors. This produces the path-partition {{ae}, {ib,be,ef,fh}}. Another extension gives us {{cg,gh},{ae},{ib,be,ef,fh}}, and another extension gives us the maximum path-partition of U, namely {{cg, gh},{ae},{ib,be,ef,fh},{dh}}. In this case, the maximum path-partition is unique, but this is not always the case. For example, if the vertex i and the edge ib were removed from U,U would have two maximum path-partitions {{ae,ef,fh},{be},{cg, gh},{dh}} and {{be,ef,fh}, {ae},{cg,gh},{dh}}.

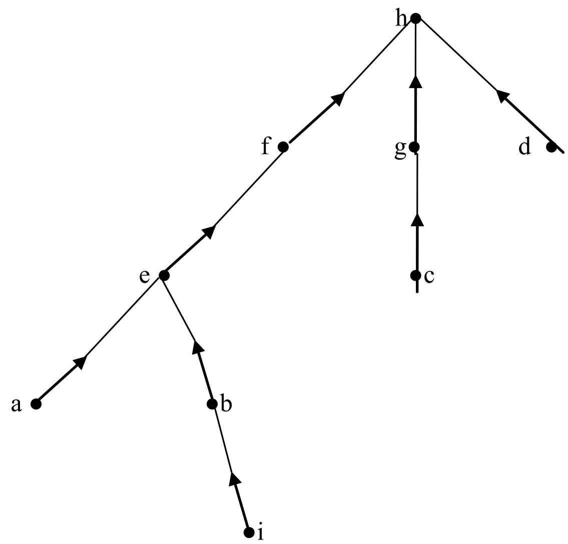

Figure 1. A rooted Tree

Theorem 3.21. [Moe 92] Let U be a rooted tree and v be the root of U. If the path-size sequence of some maximum path-partition for U is $(a_1,a_2,...,a_n)$, then

$$f(v,U)=\sum_{i=1}^{n}2^{a_i}-n+1. \blacksquare$$

We find Theorem3.22 in [LS 06].

Theorem 3.22. Let U be a rooted tree and v be the root of U. Let $(a_1, a_2,...,a_n)$, be the path-size sequence for some maximum path-partition for U. Without loss of generality a_1 can be taken to be h where h is the height of the tree. Then

$$f_t(v,U)=t2^h+\sum_{i=2}^{n}2^{a_i}-n+1. \blacksquare$$

We find Definition 2.7 in [HH 98].

Definition 3.23. Given a pebbling of G, a transmitting subgraph of G is a path $x_0,x_1,...,x_k$ such that there are at least two pebbles on x_0 and at least one pebble on each of the other vertices in the path, except possibly x_k. In this case, we can transmit a pebble from x_0 to x_k.

The following theorem of [Chu 89] is also used here.

Theorem 3.24[Chu 89]. A Tree satisfies the 2-pebbling property. ■

We will now prove that a tree satisfies the odd 2t-pebbling property.

Theorem 3.25. A tree satisfies the odd 2t-pebbling property.

Proof : Let T be a tree and v be a vertex of T. Let U be the rooted tree obtained from T by directing all edges towards v, which becomes the root.

Let $(a_1,a_2,...,a_n)$ be the path-size sequence for some maximum path-partition for U. Without loss of generality a_1 can be taken to be h where h is the height of the tree.

Then by Theorem 3.22,

$$f_t(v,U) = t2^h + \sum_{i=2}^{n} 2^{a_i} - n + 1$$

Consider a configuration of $2f_t(v,U)$-q+1 pebbles where q is the number of vertices with an odd number of pebbles. We use induction on t to prove that v satisfies the odd 2t-pebbling property. For t=1, the result is true by Theorem 3.24.

For t>1, the number of pebbles on the tree will be at least

$$2^{h+2} + \sum_{i=2}^{n} 2^{a_i+1} - 2n + 3 - q = 2f_{t-1}(v,U) - q + 1 + 2^{h+1}$$

where q is the number of vertices with an odd number of pebbles. Let p be the number of pebbles on U. We claim that there will be at least one P_i with at least 2^{a_i} pebbles. Otherwise, the total number of pebbles placed on T will be at most $2^{h+1} + \sum_{i=2}^{n} 2^{a_i+1} - n$.

Then $2f_t(v,U) + 1 \le p + q \le 2^{h+1} + \sum_{i=2}^{n} 2^{a_i+1} + q - n$

That is, $t2^{h+1} + \sum_{i=2}^{n} 2^{a_i+1} - 2n + 2 \le 2^{h+1} + \sum_{i=2}^{n} 2^{a_i+1} + q - n$

That is, $(t-1)\, 2^{h+1} - 2n+3 \leq q-n$

That is, $(t-1)\, 2^{h+1} + 3 \leq q+n$

That is, $(t-1)\, 2^{h+1} + 3 \leq 2\, |V(U)|$ since $n \leq |V(U)|$ and $q \leq |V(U)|$

That is, $(t-1)\, 2^{h} + (3/2) \leq |V(U)|$ for all $t > 1$.

This is a contradiction.

So we can put two pebbles on v using 2^{a_i} pebbles lying on P_i. So at most 2^{h+1} pebbles will be used to put two pebbles on v. Then the remaining number of pebbles on U will be at least $2 f_{t-1}(v,U)-q+1$ where q is the number of vertices with an odd number of pebbles. By induction, these pebbles would suffice to put 2(t-1) additional pebbles on v.

As v is arbitrary, every vertex in T satisfies the odd 2t-pebbling property. Hence T satisfies the odd 2t-pebbling property. ■

4. t-pebbling the product of graphs.

We now define the direct product of two graphs, and discuss some results on the t-pebbling number of direct product of two graphs.

Definition 4.1 [HH98]. If $G=(V_G,E_G)$ and $H=(V_H,E_H)$are two graphs, the direct product of G and H is the graph, $G\times H$, whose vertex set is the cartesian product $V_{G\times H} = V_G \times V_H = \{(x,y): x \in V_G, y \in V_H\}$ and whose edges are given by $E_{G\times H} = \{((x,y),(x^1,y^1)) : x = x^1 \text{ and } (y,y^1) \in E_H \text{ or } (x,x^1) \in E_G \text{ and } y = y^1\}$.

We write $\{x\}\times H$ (respectively $G\times\{y\}$ for the subgraph of vertices whose projection onto V_G is the vertex x (respectively whose projection onto V_H is y). If the vertices of G are labeled x_i then for any distribution of pebbles on $G\times H$, we write p_i for the number of pebbles on $\{x_i\}\times H$ and q_i for the number of occupied vertices of $\{x_i\}\times H$.

Fan R.K. Chung [Chu 89] credited Conjecture 4.2 to Graham.

Conjecture 4.2. For any connected graphs G and H, we have $f(G\times H) \leq f(G)\, f(H)$ where $G\times H$ represents the direct product of graphs. ■

We find from [LS 0006], the generalization of Graham's conjecture as follows:

Conjecture 4.3[LS 0006]**. [Generalization of Graham's Conjecture]**. For any connected graphs G and H, We have $f_t(G\times H \leq f(G)\ f_t(H)$ where $G\times H$ represents the direct product of graphs G and H. ■

We take Lemma 4.4 from [HH 98]. It describes how many pebbles we can transfer from one copy of H to an adjacent copy of H in $G\times H$. It is also called transfer Lemma.

Lemma 4.4 [Transfer Lemma]. Let (x_i, x_j) be an edge in G. Suppose that in $G\times H$, we have p_i pebbles occupying q_i vertices of $\{x_i\} \times H$. If $(q_i - 1) \leq k \leq p_i$ and if k and p_i have the same parity then k pebbles can be retained on $\{x_i\}\times H$ while moving $(p_i\text{-}k)/2$ pebbles onto $\{x_j\}\times H$. If k and p_i have opposite parity we must leave k+1 pebbles on $\{x_i\}\times H$, so we can only move $(p_i - (k + 1))/2$ pebbles onto $\{x_j\}\times H$. In particular we can always move at least $(p_i\text{-}q_i)/2$ pebbles onto $\{x_j\}\times H$. ■

We find Theorem 4.5 in [LS 006] which will prove Conjecture 4.3 when G is a path and H satisfies the 2t–pebbling property.

Theorem 4.5. Let P_m be a path on m vertices. When G satisfies the 2t-pebbling property, $f_t(P_m \times G) \leq 2^{m-1}\ f_t(G)$. ■

We find Theorem 4.6 and Theorem4.7 in [LS 006]

Theorem 4.6. Let P_m be a path on m vertices. The $f_t(P_m \times P_n) \leq t\ 2^{m+n-1}$. ■

Theorem 4.7. Suppose G satisfies the 2t–pebbling property. Let $P_m = \{x_1, x_2, ..., x_m\}$ be a path on m vertices where m is odd. Consider the graph $P_m\times G$. Let $k = (m+1)/2$. Then $f_t(\{x_k\} \times G \leq f(x_k, P_m) f_t(G) = (2^k\text{-}1)\ f_t\ (G)$. ■

5. Open problems

The Generalization of Graham's Conjecture can be seen in the following forms also. Now we state the following conjectures for all connected graphs G and H.

Conjecture 4.4. $f_t(G\times H) \le f_t(G)\, f(H)$. ■

Conjecture 4.5. $f(G\times H) \le \min\{f(G)\, f_t(H), f_t(G)\, f(H)\}$. ■

Conjecture 4.3 discusses the t-pebbling number of the graph as a whole. To discuss the t-pebbling number of a specific vertex, we state Conjecture 4.6 which is a stronger form of Conjecture 4.3.

Conjecture 4.6. The t-pebbling number of every vertex (v,w) in G×H satisfies

$f_t((v,w),G\times H) \le f(v,G)\, f_t(w,H)$. ■

Conjecture 4.7. Conjecture 4.3 is true for a graph which is the direct product of a tree with a tree. ■

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