HB135-147
UDK: 330.42

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Mathematical Models of Dynamics of Indicators of Budgetary and Taxation System

ABSTRACT
The complex of mathematical models of investigated indicators has been developed to study the dynamics of budget indicators, forecast their change and plan revenues and expenditures of state and local budgets appropriately. This is a linear dynamic model with constant and variable coefficients, nonlinear nonstationary and stationary models of all studied indicators and also models of the dynamics of the total amount of state and local budgets. Methods of establishing parameters of these models have been described.
Key words: *model, budget, dynamics, approximation, differential equation, forecast, spline.*

The empirical analysis of reported data is carried out to establish patterns of dynamics of revenues to the budget and planning of budget expenditures. However, on the basis of such analysis of reported data it is difficult to establish projected values of budget indicators, to reveal the dependence of dynamics of budget indicators on certain selected values. Therefore, the objective is to develop mathematical models of dynamics of investigated budget indicators.

55 indicators that describe revenues and expenditures of local and state budgets and their transfers, have been selected to study the dynamics and structure of revenues and expenditures.

Revenues and expenditures of state and local budgets and also their transfers are interrelated. Such dependence between them is caused by planning of budget expenditures and their incidence on commercial activities which in turns impact on budget revenues. It gives reasons to believe that indicators $z_i(t)$ that describe revenues and expenditures of local budgets ($i=\overline{1,28}$), revenues and expenditures of state budgets ($i=\overline{34,53}$) and their transfers ($i=\overline{54,55}$) and revenues and expenditures of consolidated budget ($i=\overline{29,33}$) are interdependent. Therefore, it is appropriate for indicators $z_i(t)$, ($i=\overline{1,55}$) to be simulated with the system of differential equations in which each of indicators depends on the rest of them.

In the simplest approximation the system of linear differential equations with constant coefficients describes the dynamic link between indicators $z_i(t)$, ($i=\overline{1,55}$):

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_{55} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1\,55} \\ a_{21} & a_{22} & \dots & a_{2\,55} \\ \dots & \dots & \dots & \dots \\ a_{55\,1} & a_{55\,2} & \dots & a_{55\,55} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_{55} \end{bmatrix} \tag{1}$$

where: a_{ij} ($ij=\overline{1,n}$) — constant parameters, found by reported data, n = 55 — number of variables of the model (1).

We consider the identification problem of determining the parameters a_{ij} ($ij=\overline{1,n}$) of model (1). Values of indicators $z_i(t)$, ($i=\overline{1,55}$) are known from reported data at time points t_k ($k=\overline{1,m}$), where m — number values of consolidated data of different calendar reporting periods. We denote such well-known discrete functional dependencies with symbol

$$\bar{z}_i(t_k),\ (i=\overline{1,n},\ k=\overline{1,m}). \tag{2}$$

We approximate the dependencies (2) using the exponential splines

$$\tilde{z}_i(t_k) = \bar{z}_i(t_k),\ (i=\overline{1,n},\ k=\overline{1,m}). \tag{3}$$

where $\tilde{z}_i(t_k)$ — spline approximation of dependency i indicator $\bar{z}_i(t_k)$ on time on the domain $t \in [t_1, t_m]$.

We take derivatives of splines (3) using analytical differentiation

$$\tilde{z}_i'(t) = \frac{d}{dt}\tilde{z}_i(t)\ \ (i=\overline{1,n}\ ;\ t \in [t_1, t_m])$$

and calculate the value $\tilde{z}_i'(t_k)$ of these derivatives at the points t_k ($k=\overline{1,m}$)

$$\tilde{z}_i'(t_k) = \frac{d}{dt}\tilde{z}_i(t)\Big|_{t=t_k}\ \ (i=\overline{1,n}\ ;\ k=\overline{1,m}\ ) \tag{4}$$

We substitute the value i indicator $\bar{z}_i(t_k)$, $(i=\overline{1,n};\ k=\overline{1,m})$, that is known from reported data (2), and the value of its derivative $\tilde{z}_i(t_k)(i=\overline{1,n};\ k=\overline{1,m})$, that is defined by (4), in i line of equation (1). We describe n systems of linear algebraic equations, each of which has m equations in n unknown a_{ij} $(ij=\overline{1,n};\ n<m)$

$$\begin{cases} \tilde{z}_i'(t_k) = \sum_{j=1}^{n} a_{ij}\bar{z}_j(t_k) \\ k=\overline{1,m} \end{cases} \quad (i=\overline{1,n}) \tag{5}$$

The least squares method that is given in the form of minimization of Tikhonov's regularizing functional is applied to solve the equations (5).

$$\min_a \left[\sum_{k=1}^{m} \left(\tilde{z}_i'(t_k) - \sum_{j=1}^{n} a_{ij}\bar{z}_j(t_k) \right)^2 + \alpha \sum_{j=1}^{n} a_{ij}^2 \right] \tag{6}$$

where: $i=\overline{1,n}$; a — a conventional sign of all parameters a_{ij} $(i,j=\overline{1,n})$, α — a parameter of regularization. Having solved problems (6), we find the parameters a_{ij} $(i,j=\overline{1,n})$ of the system of differential equations (1).

The system of differential equations (1) with initial conditions

$$z_i^0 = z_i(t_0),\ (i=\overline{1,n}) \tag{7}$$

describes the linear dynamic model of budget indicators. This model is suitable for the calculation of short-term forecasts of values of all budget indicators. The duration of such forecast is about 1-3 reporting periods.

It is necessary to choose initial conditions (7) at the last reported data point $t_0 = t_m$ and solve the equation (1) on the domain $t \in [t_m, t_M]$, where t_M — the final time point of determination of projected values of the studied indicators, in order to calculate projected values of budget indicators.

It is also easy to determine the reaction of studied dynamical system to possible changes of initial conditions with the aid of the model (1), (7). If in this model planned, projected or expected value of the initial conditions (7) are set, then its solution $z_i(t)$, $(i=\overline{1,n})$ will simulate a change of budget indicators that corresponds to the chosen values of the initial conditions. In this way, the model (1), (7) can be conveniently used in decision support system during planning the structure of revenues and expenditures of budget.

It should be noted that the parameters a_{ij} $(i,j=\overline{1,n})$ of the model (1), determined from problem (6) reflect the structure of impact of the budget indicators on their dynamics. Therefore, qualitative analysis of these parameters complements the results of research of budget indicators, which are made through the analysis of their dynamics, structure and functional dependencies between them.

The model (1) — (7), described above, represents only the simplest linear dependencies between the studied indicators. However, in reality some of the indicators vary rapidly over time, nonlinearly dependent on others. In this regard, the challenge is to investigate the nonlinear dependence of the budget indicators on the time and other budget indicators.

We first consider the model with variable parameters that are functions of time. The value of budget indicators $z_i(t)$ are defined on a discrete set t_k, $(k=\overline{1,m})$, which belongs to the continuous area $t \in [t_1, t_m]$. This area covers all reporting periods, taken for the study. Analysis of budget indicators shows that the parameters a_{ij} $(i,j=\overline{1,n})$ should not be set according to all data values $z_i(t)$ on the domain $t \in [t_1, t_m]$, but on parts of this domain, which correspond to the same type of budget planning expenditures.

Therefore, in order to build a model of the form (1) with parameters that are functions of time $a_{ij} = a_{ij}(t)$, $(i,j=\overline{1,n})$, we divide the domain $t \in [t_1, t_m]$ into L equal parts, such that the intervals t_l, t_{l+1} $(l=\overline{1,L})$ corresponded to the duration of the reporting period $(t_{L1}+t_m)$. We will establish $\hat{m}$ values of indicators and their derivatives with the aid of previously established spline approximation (3)

$$\tilde{z}_i^l(t_k),\ \tilde{z}_i'^l(t_k),\ (i=\overline{1,n};\ k=\overline{1,\hat{m}};\ l=\overline{1,L}) \tag{8}$$

where: $\tilde{z}_i^l(t_k)$ — values of i indicator in k node of l segment of general domain of the studied indicators; $\tilde{z}_i^l(t_k)$ — value of derivative of the indicator at the same point.

Having taken $m=\hat{m}$, we will solve problems of identification of parameters of the model (1) separately for each of the time intervals $t \in [t_l, t_{l+1}]$, $(l=\overline{1,L-1})$. Having solved these problems we find coefficients

$$a_{ij}^l,\ (i,j=\overline{1,n};\ l=\overline{1,L}) \tag{9}$$

Here the values of parameters , ( ) of the model (1) are identified on the l domain $t \in [t_l, t_{l+1}]$, $(l=\overline{1,L-1})$. They reflect the typical link between dynamic variables of the model (1) during this time.

We define the middle point of a segment $t \in [t_l, t_{l+1}]$ $\theta_l = (t_l + t_{l+1})/2$, $l=\overline{1,L}$. Than the discrete functional dependence

$$\theta_l, a_{ij}^l \quad i,j=\overline{1,n} \quad l=\overline{1,L} \tag{10}$$

describes dependence of the model parameters (1) on the time. We approximate the dependence (10) with the aid of nonlinear functions on the domain $\theta \in [\theta_1, \theta_L]$

$$\tilde{a}_{ij}(\theta_l) = a_{ij}^l,\ (i,j=\overline{1,n};\ l=\overline{1,L}), \tag{11}$$

where $\tilde{a}_{ij}(\theta_l)$ — the value of ij model parameter (1) at the time point θ_l $(l=\overline{1,L})$. Without significant limitation on statements of problem of approximation of data (3) caused by functions (11), we assume that the functions $\tilde{a}_{ij}(\theta_l)$ reflect the dependence of the parameters of the model (1) in the time period $t \in [t_1, t_m]$, and also — in a given neighborhood outside of it. This assumption follows from the properties of nonlinear approximation. That is, functions $\tilde{a}_{ij}(t)$ $(i,j=\overline{1,n})$ are defined on the domain $t \in [t_1, t_m]$.

Having substituted (11) into the equation (1), we obtain a linear dynamic model parameters of the budget with variable coefficients.

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_n \end{bmatrix} = \begin{bmatrix} \tilde{a}_{11}(t) & \tilde{a}_{12}(t) & \dots & \tilde{a}_{1n}(t) \\ \tilde{a}_{21}(t) & \tilde{a}_{22}(t) & \dots & \tilde{a}_{2n}(t) \\ \dots & \dots & \dots & \dots \\ \tilde{a}_{n1}(t) & \tilde{a}_{n2}(t) & \dots & \tilde{a}_{nn}(t) \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_n \end{bmatrix} \tag{12}$$

The system of equations (12) with initial conditions (7) models the dynamics of budget indicators with consideration of structural changes that occur in the correlations between these indicators. Model (12) in comparison with the system (1) has better forecasting qualities and is appropriate for forecasting dynamics of the budget indicators, and also — for calculating the reaction of the complex of these indicators on planned change of their values, simulated with the choice of initial conditions (7).

It should be noted that a nonlinear dependence between various parameters of the budget is not taken into account in such a model.

We analyze the nonlinear model of the budget indicators. Reported data $\bar{z}_i(t_k)$, $(i=\overline{1,n};\ k=\overline{1,m})$ reflect the change of corresponding indicators over time. The same dependence is described with continuous spline approximation $\tilde{z}_i(t)$, $(i=\overline{1,n})$ of these indicators (3), that is defined on the domain $t \in [t_1,t_m]$. Derivatives (4) of these indicators $\tilde{z}'_i(t)$, $(i=\overline{1,n})$ are nonlinear functions of time, defined on the same domain $t \in [t_1,t_m]$.

We approximate known values of derivatives of the budget indicators $\tilde{z}'_i(t)$, $(i=\overline{1,n})$ with the aid of nonlinear functions $f_i(t)$ on the domain $t \in [t_1,t_m]$. It is necessary to use a non-linear least-square method for such an approximation. By definition of the derivative we get the correlation.

$$\begin{cases} \dfrac{d}{dt}\tilde{z}_i(t) = f_i(t) \\ i = \overline{1,n} \end{cases} \tag{13}$$

The system of equations (13) with initial conditions (7) is Cauchy problem, that simulates the dynamics of budget indicators

Because of the use of nonlinear approximating functions $f_i(t)$, $(i=\overline{1,n})$ model (13) more exactly reproduces the dynamics of budget indicators. This gives promises that the extrapolation model solutions, found by the initial conditions (7), close to the right domain boundary $[t_1,t_m]$, will give more accurate forecasts than the solutions of linear models described above. In addition, comparison of three forecasting solutions, found with the aid of models (1), (12), (13) provides additional tools for analysis and planning of current values of the budget.

Indicators of the state budget, which are selected for the study, are dependent on other indicators of the budget. Also — all selected indicators of the local budget for the study dynamically depend on other indicators of this budget. Thus, it is appropriate to develop nonlinear models that reflect the nonlinear dynamic dependency of some indicators on the others in order to analyze and forecast planning indicators of state and local budgets.

We denote the set of indices of the state budget by symbol $I_a = \{1,\dots,28,54\}$. We denote the set of indices of local budget indicators by $I_b = \{34,\dots,53,55\}$. It was found out above that each of the indicators $\tilde{z}_i(t)$ $(i \in I_a)$ of state budget dynamically depends on all the indicators $\tilde{z}_i(t)$ $(i \in I_b)$ of this budget. It means that the rate of change of each indicator nonlinearly depends on the rest of the budget indicators. This dependence is described with certain nonlinear functions

$$\frac{d}{dt}z_i(t) = F_i\left(z_1(t),\dots,z_{28}(t),z_{54}(t)\right)\ (i \in I_a).$$

We approximate these functions using power polynomials

$$F_i(z_1,\dots,z_{28},z_{54}) = P_i(z_1,\dots,z_{28},z_{54}),$$

where: $P_i(x_1,\dots,x_\mu) = \sum\limits_{i_1+\dots+i_\mu \le r} c_{i_1\dots i_\mu} x_1^{i_1} \cdot \dots \cdot x_\mu^{i_\mu}$; $c_{i_1\dots i_\mu}$ — coefficients of this polynomial; r — a degree of polynomial $(r > 0)$; μ — the number of its arguments.

Thus, the dynamic of indicators of the state budget is described with the aid of the differential equation system:

$$\frac{d}{dt}z_i(t) = P_i\left(z_1(t),\dots,z_{28}(t),z_{54}(t)\right),\ (i \in I_a). \tag{14}$$

Analogously, the dynamics of the local budget indicators is described with the aid of the differential equation system:

$$\frac{d}{dt}z_i(t) = P_i\left(z_{34}(t),\dots,z_{53}(t),z_{55}(t)\right),\ (i \in I_b). \tag{15}$$

We consider the identification problem for determining the parameters of polynomials in equations (14) and (15). The value of derivatives $\tilde{z}_i(t_k)$ $(i \in I_a, I_b;\ k=\overline{1,m})$ are known. They are found with the aid of analytical differentiation. Substituting the values of indicators $\tilde{z}_i(t_k)$ and derivatives $\tilde{z}'_i(t_k)$, $(i \in I_a;\ k=\overline{1,m})$ in equation (14), we obtain a system of linear algebraic equations with regard to the coefficients of approximation:

$$\begin{cases} \tilde{z}'_i(t_k) = P_i\left(z_1(t_k),\dots,z_{28}(t_k),z_{54}(t_k)\right); \\ k = \overline{1,m}, \end{cases}$$

where $i \in I_a$. We apply the least squares method in order to solve this system of equations:

$$\min_{c^i_{J_a}} \left\{ \sum_{k=1}^{m} \left[\tilde{z}'_i(t_k) - P_i\left(z_1(t_k),\ldots,z_{28}(t_k),z_{54}(t_k)\right)\right]^2 + \alpha \sum_{k=1}^{m} \left[c^i_{J_a} \right]^2 \right\}, \tag{16}$$

where: $c^i_{J_a}$ — coefficients of the polynomial P_i $(i \in I_a)$; $J_a = i_1 \ldots i_{28} i_{54}$ — indices near the polynomial coefficients in equations (14); α — a regularization parameter.

Analogously we set identification problem to determine the parameters of the model (15) of local budget indicators:

$$\min_{c^i_{J_b}} \left\{ \sum_{k=1}^{m} \left[\tilde{z}'_i(t_k) - P_i\left(z_{34}(t_k),\ldots,z_{53}(t_k),z_{55}(t_k)\right)\right]^2 + \alpha \sum_{k=1}^{m} \left[c^i_{J_b} \right]^2 \right\}, \tag{17}$$

where: $c^i_{J_a}$ — coefficients of the polynomial P_i $(i \in I_b)$; $J_b = i_{34} \ldots i_{53} i_{55}$ — indices near the polynomial coefficients in equations (14); α — a regularization parameter. Having solved identification problems (16), (17), we find coefficients of polynomials ($i \in I_a$, I_b).

Equations (14), (15) with initial conditions

$$z^0_i(t) = z_i(t_0) \quad (i \in I_a, I_b) \tag{18}$$

reflect the dynamics of indicators of revenues and expenditure in accordance with state and local budgets taking into account the interdependence between these indicators. Models (14), (15) with initial conditions (18) are suitable for the analysis of short-term forecasting trends of change of budget indicators, taking fully into account their structure. Also, these models are easily applied to study the influence of the structure of budget revenues and expenditures on the dynamics of these indicators. It's enough to solve the equations (14), (15) with forecasted or planned values of the initial conditions (18) and perform a qualitative analysis of found solutions $z_i(t_k)$, $i \in I_a$, I_b.

Although the models (14), (15) reflect the dynamics of budget indicators $z_i(t_k)$, taking into account structural dependencies between them, they have restricted domain of applicability. Because they provide acceptable solutions only on that domain of values $z_i(t_k)$, ($i \in I_a$, I_b), on which identification problems (16), (17) were resolved. Therefore, it is appropriate for these models to be used for research and planning structures of budget indicators under the condition of small deviations of these indicators from their previous values.

A dynamic linear model with invariables (1), a linear dynamic model with variable parameters (12), a nonlinear nonstationary model (13), nonlinear stationary models (14), (15) of studied indicators of state and local budgets with different accuracy, described above, reflect projected values of budget indicators and the influence of initial conditions on the change of the budget indicators. Taken as a whole, these models serve as a basis for decision support system during the study of budget indicators and planning their values.

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