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OD-characterization of almost simple groups related to $U_3(11)$

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Abstract. Let $L := U_3(11)$. In this article, we classify groups with the same order and degree pattern as an almost simple group related to L. In fact, we prove that L, $L.2$ and $L.3$ are OD-characterizable, and $L.S_3$ is 5-fold OD-characterizable.

Keywords: prime graph, recognition, linear group, finite simple group, degree pattern

1. Introduction

Let G be a finite group. Denote by $\pi(G)$ the set of all prime divisors of the order of G. The prime graph $\Gamma(G)$ of a finite group G is a simple graph with vertex set $\pi(G)$ in which two distinct vertices p and q are joined by an edge if and only if G has an element of order pq.

Definition 1.1 *Let G be a finite group and $|G| = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1 < p_2 < \ldots < p_k$. For $p \in \pi(G)$, let $deg(p) = |\{q \in \pi(G)|p \sim q\}|$ be the degree of p in the graph $\Gamma(G)$, we define $D(G) = (deg(p_1), deg(p_2), \ldots, deg(p_k))$, which is called the degree pattern of G.*

Given a finite group G, denote by $h_{OD}(G)$ the number of isomorphism classes of finite groups S such that $|G| = |S|$ and $D(G) = D(S)$. In terms of the function h_{OD}, groups G are classified as follows:

Definition 1.2 *A group G is called k-fold OD-characterizable if there exist exactly k non-isomorphic group S such that $|G| = |S|$ and $D(G) = D(S)$. Moreover, a 1-fold OD-characterizable group is simply called an OD-characterizable.*

Definition 1.3 *A group G is said to be an almost simple related to S if and only if $S \trianglelefteq G \trianglelefteq Aut(S)$ for some non-abelian simple group S.*

Definition 1.4 *Let p be a prime number. The set of all non-abelian finite simple groups G such that $p \in \Pi(G) \subseteq \{2, 3, 5, \ldots, p\}$ is denoted by $\mathfrak{S}_p$. It is clear that the set of all non-abelian finite simple groups is the disjoint union of the finite sets $\mathfrak{S}_p$ for all primes p.*

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2. Preliminaries

For any group G, let $w(G)$ be the set of orders of elements in G, where each possible order element occurs once in $w(G)$ regardless of how many elements of that order G has. This set is closed and partially ordered by divisibility, hence it is uniquely determined by its maximal elements. The set of maximal elements of $w(G)$ is denoted by $\mu(G)$. The number of connected components of $\Gamma(G)$ is denoted by $t(G)$. Let $\pi_i = \pi_i(G)$, $1 \leqslant i \leqslant t(G)$, be the ith connected components of $\Gamma(G)$. For a group of even order we let $2 \in \pi_1(G)$. We denote by $\pi(n)$ the set of all primes divisors of n, where n is a natural number. Then $|G|$ can be expressed as a product of $m_1, m_2, \ldots, m_{t(G)}$, where m_i's are positive integers with $\pi(m_i) = \pi_i$. These m_i's are called the order components of G. We write $OC(G) = \{m_1, m_2, \ldots, m_{t(G)}\}$ and call it the set of order components of G. The set of prime graph components of G is denoted by $T(G) = \{\pi_i(G) | i = 1, 2, \ldots, t(G)\}$.

Definition 2.1 *Let n be a natural number. We say that a finite simple group G is a simple K_n-group if $|\pi(G)| = n$.*

Definition 2.2 *Suppose that $K \trianglelefteq G$ and $G/K \cong H$. Then we shall call G an extension of K by H.*

3. Elementary Results

Lemma 3.1 [5]*Let G be a finite group and $|\pi(G)| \geqslant 3$. If there exist prime numbers $r, s, t \in \pi(G)$ such that $\{tr, ts, rs\} \cap \omega(G) = \varnothing$, then G is non-solvable.*

Definition 3.2 *A group G is called a 2-Frobenius group, if there exists a normal series $1 \lhd H \lhd K \lhd G$, such that K and $\frac{G}{H}$ are Frobenius groups with kernels H and $\frac{K}{H}$, respectively.*

Lemma 3.3 [1]*Let G be a 2-Frobenius group of even order which has a normal series $1 \lhd H \lhd K \lhd G$, such that K and $\frac{G}{H}$ are Frobenius groups with kernels H and $\frac{K}{H}$, respectively. Then*

(1) $t(G) = 2$ and $T(G) = \{\pi_1(G) = \pi(H) \cup \pi(\frac{G}{K}), \pi_2(G) = \pi(\frac{K}{H})\}$.
(2) $\frac{G}{K}$ and $\frac{K}{H}$ are cyclic groups, $|\frac{G}{K}| \mid |Aut(\frac{K}{H})|$, and $(|\frac{G}{K}|, |\frac{K}{H}|) = 1$.
(3) H is a nilpotent group and G is a solvable group.

The following lemmas are useful when dealing with a Frobenius group.

Lemma 3.4 [3], [8]*Let G be a Frobenius group with complement H and kernel K. Then the following assertions hold:*

(1) K is a nilpotent group;
(2) $|K| \equiv 1 (mod |H|)$;
(3) Every subgroup of H of order pq, with p, q (not necessarily distinct)primes, is cyclic. In particular, every Sylow Subgroup of H of odd order is cyclic and a 2-Sylow subgroup of H is either cyclic or a generalized quaternion group. If H is a non-solvable group, then H has a subgroup of index at most 2 isomorphic to $Z \times SL(2,5)$, where Z has cyclic Sylow p-subgroups and $\pi(Z) \cap \{2,3,5\} = \varnothing$. In particular, $15, 20 \notin \omega(H)$.

Lemma 3.5 [1]*Let G be a Frobenius group of even order where H and K are Frobenius complement and Frobenius kernel of G, respectively. Then $t(G) = 2$ and $T(G) = \{\pi(H), \pi(K)\}$.*

The structure of a finite group with non-connected prime graph is described in the following lemma.

LEMMA 3.6 [4], [9]*Let G be a finite group with $t(G) \geqslant 2$. Then G is one of the following groups:*

(1) G is a Frobenius or a 2-Frobenius group;
(2) G has a normal series $1 \trianglelefteq H \lhd K \trianglelefteq G$, such that H and $\frac{G}{K}$ are π_1-groups and $\frac{K}{H}$ is a non-abelian simple group, where π_1 is the prime graph component containing 2, H is a nilpotent group, and $|\frac{G}{H}| \mid |Aut(\frac{K}{H})|$. Moreover, any odd order component of G is also an odd order component of $\frac{K}{H}$.

The following lemma is taken from [10].

LEMMA 3.7 *Let $S = P_1 \times P_2 \times \ldots \times P_r$, where P_i's are isomorphic non-abelian simple groups. Then $Aut(S) \cong (Aut(P_1) \times Aut(P_2) \times \ldots \times Aut(P_r)) \cdot S_r$.*

4. Main Results

THEOREM 4.1 *If G is a finite group such that $D(G) = D(M)$ and $|G| = |M|$, where M is an almost simple group related to $L := U_3(11)$, then the following assertions holds:*

(1) If $M = L$, then, $G \cong L$,
(2) If $M = L.2$, then, $G \cong L.2$,
(3) If $M = L.3$, then, $G \cong L.3$,
(4) If $M = L.S_3$, then, $G \cong L.S_3$, $Z_3 \times (L.2)$ or $Z_3.(L.2)$, $(Z_3 \times L).Z_2$, $(Z_3.L).Z_2$.
In particular, L, $L.2$ and $L.3$ are OD-characterizable; and $L.S_3$ is 5-fold OD-characterizable.

Proof We break the proof into a number of separate cases:
Case 1: If $M = L$, then, $G \cong L$ by [7].
Case 2: If $M = L.2$, then, $G \cong L.2$.
If $M = L.2$, by [2], we have $\mu(L.2) = \{24, 37, 40, 44\}$ from which we deduce that $D(L.2) = (3, 1, 1, 1, 0)$. The prime graph of $L.2$ has the following form:

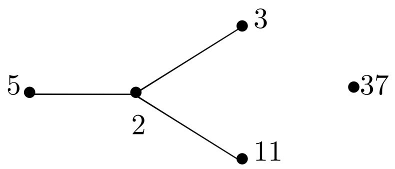

Figure 1: The prime graph of $L.2$

As $|G| = |L.2| = 2^6 \cdot 3^2 \cdot 5 \cdot 11^3 \cdot 37$ and $D(G) = D(L.2) = (3, 1, 1, 1, 0)$, then, $\Gamma(G) = \Gamma(M) = \{2 \sim 3, 2 \sim 5, 2 \sim 11; 37\}$.

G is non-solvable. Since $\{3 \cdot 37, 5 \cdot 37, 3 \cdot 5\} \cap \omega(G) = \varnothing$, therefore by lemma 3.1, G is not solvable. Therefore, by lemma 3.2(iii), G is not a 2-Frobenius group.

Suppose that G is a non-solvable Frobenius group with H and K as its Frobenius complement and Frobenius kernel, respectively. Using the same notations as in lemma 3.3(iii),we obtain $11 \in \pi(Z)$, it follows that H_0 has an element of order $11 \cdot 5$, a contradiction.

By lemma 3.5(ii), G has a normal series $1 \trianglelefteq H \lhd K \trianglelefteq G$, such that H is a nilpotent π_1-group, K/H is a non-abelian simple group and G/K is a solvable π_1-group. Therefore, $K/H \leqslant G/H \leqslant Aut(K/H)$. Since $37 \nmid |H|$, we have $37 \in \pi(K/H)$.

Therefore, $K/H \in \mathfrak{S}_{37}$ and $\{7, 13, 17, 19, 23, 29, 31\} \nsubseteq \pi(K/H)$. Using [11] we listed the possibilities for K/H in Table 1.

By Table 1, we obtain that K/H isomorphic to A_5, A_6, $L_2(11)$, M_{11} or L.

If $K/H \cong A_5$ we get $A_5 \leqslant G/H \leqslant Aut(A_5)$, because $G/H \leqslant Aut(K/H)$. It follows that $|H| = 2^4 \cdot 3 \cdot 11^3 \cdot 37$ or $|H| = 2^3 \cdot 3 \cdot 11^3 \cdot 37$. By nilpotency of H, $11 \sim 37$ in $\Gamma(G)$, a contradiction. Similarly, we can prove that $K/H \ncong A_6$, $L_2(11)$ or M_{11}.

Therefore, $K/H \cong L$. As $|G| = 2|L|$, we deduce $|H| = 1$ or 2.

If $|H| = 1$, then, $G \cong L.2$.

If $|H| = 2$, then, $G/C_G(H) \leqslant Aut(H) \cong Z_2^{\times} = 1$, so $G = C_G(H)$. Therefore, $H \leqslant Z(G)$. It follows that $2 \sim 37$ in $\Gamma(G)$, a contradiction.

Table 1: Non-abelian simple group $S \in \mathfrak{S}_{37}$ with $\pi(S) \subseteq \{2, 3, 5, 11, 37\}$

S	$\|S\|$	$\|out(S)\|$	S	$\|S\|$	$\|out(S)\|$
A_5	$2^2 \cdot 3 \cdot 5$	2	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1
A_6	$2^3 \cdot 3^2 \cdot 5$	4	M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2
$U_4(2) \cong S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	2	$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	$U_3(11)$	$2^5 \cdot 3^2 \cdot 5 \cdot 11^3 \cdot 37$	6

Case 3: If $M = L.3$, then $G \cong L.3$.

If $M = L.3$, by [2], we have $\mu(L.3) = \{111, 120, 132\}$ from which we deduce that $D(L.3) = (3, 3, 2, 2, 1)$. The prime graph of $L.3$ has the following form:

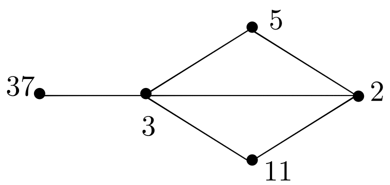

Figure 2: The prime graph of $L.3$

As $|G| = |L.3| = 2^5 \cdot 3^3 \cdot 5 \cdot 11^3 \cdot 37$ and $D(G) = D(L.3) = (3, 4, 2, 2, 1)$, then, $\Gamma(G) = \Gamma(M) = \{2 \sim 3, 2 \sim 5, 2 \sim 11, 3 \sim 5, 3 \sim 11, 3 \sim 37\}$.

LEMMA 4.2 *Let K be the maximal normal solvable subgroup of G. Then K is a $\{2, 3\}$-group. In particular, G is non-solvable.*

Proof First assume that $\{5, 11\} \subseteq \pi(K)$. Let H be a Hall $\{5, 11\}$-subgroup of K. It is easy to see that H is a subgroup of order $5 \cdot 11^3$. H is nilpotent, since $H = H_5.H_{11}$, $5 \nsim 11$, therefore $H_5 \cap H_{11} = \{1\}$. We have $H_5 \trianglelefteq H$ and $N_{11} = 11k+1 \mid |H| = 5.11^3$, where N_{11} is the number of 11- Sylow subgroups from H, and $(N_{11}, 11) = 1$ then $11k+1 \mid 5$, hence $k = 0$ and, by Sylow's Lemma, $H_{11} \trianglelefteq H$. Therefore $H \cong H_5 \times H_{11}$ and by Tampson's Lemma, we have H_{11} is nilpotent, hence H is nilpotent.
Since H is nilpotent, which implies that $5 \cdot 11 \in \omega(K) \subseteq \omega(G)$, a contradiction. Thus $\{5\} \subseteq \pi(K) \subseteq \{2, 3, 5, 37\}$. Let $K_5 \in Syl_5(K)$, by Frattini argument $G = KN_G(K_5)$. Therefore, the normalizer $N_G(K_5)$ contains an element of order 11, say x. Similar to H we can prove that $< x > K_5$ is a nilpotent subgroup of G of order $5 \cdot 11$. Hence $5 \cdot 11 \in \omega(G)$, a contradiction. Similarly, we can prove that $\{11, 37\} \cap \pi(K) = \varnothing$. Therefore, K is a $\{2, 3\}$-group. In addition, since $K \neq G$, it follows that G is non-solvable. This completes the proof. ■

LEMMA 4.3 *The quotient G/K is an almost simple group. In fact, $S \leqslant G/K \leqslant Aut(S)$, where $S \cong L$.*

Proof Let $\overline{G} := G/K$, $S := Soc(\overline{G})$, where $Soc(\overline{G})$ denotes the socle of the group $\overline{G}$, i.e., the subgroup of $\overline{G}$ generated by the set of all the minimal normal subgroups of $\overline{G}$. Then, $S \cong P_1 \times P_2 \times \ldots \times P_r$, where P_i's are non-abelian simple groups and $S \leqslant \overline{G} \leqslant Aut(S)$. In what follows, we will show that $r = 1$ and $P_1 \cong L$.

Suppose that $r \geqslant 3$, then, there exists distinct P_i and P_j such that $\pi(P_i) \neq \pi(P_j)$, because $|G|_5 = 5$, $|G|_{11} = 11^3$ and $|G|_{37} = 37$, where n_p denotes the p-part of the integer $n \in N$. If $|\pi(P_i)| = 5$ or $|\pi(P_j)| = 5$, then, $37 \in \pi(P_i)$ or $37 \in \pi(P_j)$. It follows that $2.37 \in \omega(G)$, a contradiction. Hence, without loss of generality, by Table 1, we can suppose that $\{2, 3\} \subseteq \pi(P_i) \subseteq \{2, 3, p, q\}$ and $\{2, 3\} \subseteq \pi(P_j) \subseteq \{2, 3, r, s\}$, where $\{r, s\}$, $\{p, q\} \subseteq \{\{5, 11\}, \{5, 37\}, \{11, 37\}$ and $\{r, s\} \neq \{p, q\}$. As $S \cong P_1 \times \ldots \times P_i \times \ldots \times P_j \times \ldots \times P_r$, we have $\{pr, ps, qr, qs\} \subseteq \omega(S)$. Thus, $\{pr, ps, qr, qs\} \subseteq \omega(G)$, which is a contradiction because there exists no edge between 5, 11 and 37 in $\Gamma(G)$.

Hence, $r = 2$ if $r > 1$. Recall that $|G| = 2^5 \cdot 3^3 \cdot 5 \cdot 11^3 \cdot 37$ and $S \cong P_1 \times P_2 \times \ldots \times P_r$, where $P_i's$ are finite non-abelian simple groups. By Table 1, we have $5 \in \pi(P_i)$, therefore, if $S \cong P_i \times P_j$, then, $5^2 \mid |S|$, a contradiction. Thus, $r = 1$ and $S = P_1$.

By Table 1, $\{2, 3\} \subseteq \pi(S)$ and $\pi(Out(S)) \subseteq \{2, 3\}$. Therefore, by Lemma 4.7, it is evident that $|S| = 2^a \cdot 3^b \cdot 5 \cdot 11^3 \cdot 37$, where $2 \leqslant a \leqslant 5$ and $1 \leqslant b \leqslant 3$. Now, using collected results contained in Table 1, we deduce that $S \cong U_3(11)$ and the proof is completed. ∎

Lemma 4.4 $G \cong L.3$.

Proof By Lemma 4.8, $L \leqslant G/K \leqslant Aut(L)$. Hence, $|K| = 1$ or 3.

If $|K| = 1$, then, $G \cong L.3$.

If $|K| = 3$, then, $G/K \cong L$. In this case we have $G/C_G(K) \leqslant Aut(K) \cong Z_2$. Thus $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then $K \leqslant Z(G)$. It follows that $3 \sim 37$ in $\Gamma(G)$, a contradiction. If $|G/C_G(K)| = 2$, then $K \subset C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$. Thus, we obtain $G = C_G(K)$ because L is simple, which is a contradiction. ∎

Case 4: If $M = L.S_3$, then, $G \cong L.S_3$, $Z_3 \times (L.2)$, $Z_3 \cdot (L.2)$, $(Z_3 \times L).Z_2$, $(Z_3 \cdot L).Z_2$.

If $M = L.S_3$, by [2], we have $\mu(L.S_3) = \{111, 120, 132\}$ from which we deduce that $D(L.S_3) = (3, 3, 2, 2, 1)$. The prime graph of $L.S_3$ has the following form:

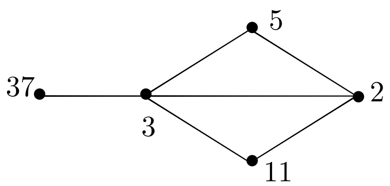

Figure 3: The prime graph of $L.S_3$

As $|G| = |L.S_3| = 2^6 \cdot 3^3 \cdot 5 \cdot 11^3 \cdot 37$ and $D(G) = D(L.S_3) = (3, 4, 2, 2, 1)$, then, $\Gamma(G) = \Gamma(M) = \{2 \sim 3, 2 \sim 5, 2 \sim 11, 3 \sim 5, 3 \sim 11, 3 \sim 37\}$.

Similarly to Lemma 4.7 in Case 3, we can prove that, if K be the maximal normal solvable subgroup of G, then K is a $\{2, 3\}$-group and G is non-solvable. Also, similarly to Lemma 4.8 in case 3, we can prove that, the quotient G/K is an almost simple group. In fact, $S \leqslant G/K \leqslant Aut(S)$, where $S \cong L$.

Now, we proof that $G \cong L.S_3$, $Z_3 \times (L.2)$, $Z_3 \cdot (L.2)$, $(Z_3 \times L).Z_2$, $(Z_3 \cdot L).Z_2$.

Since $L \leqslant G/K \leqslant Aut(L)$, then, $|K| = 1$, 2, 3 or 6.

If $|K| = 1$, then, $G \cong L.S_3$.

If $|K| = 2$, then, $K \leqslant Z(G)$. It follows that $2 \sim 37$ in $\Gamma(G)$, a contradiction.

If $|K| = 3$, then, $G/K \cong L.2$. In this case we have $G/C_G(K) \leqslant Aut(K) \cong Z_2$. Thus, $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then, $K \leqslant Z(G)$, i.e., G is a central extension of Z_3 by $L.2$. If G splits over K we obtain $G \cong Z_3 \times (L.2)$, otherwise, we have $G \cong Z_3 \cdot (L.2)$. If $|G/C_G(K)| = 2$, then, $K \subset C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L.2$, and we obtain that $C_G(K)/K \cong L$. Because $K \leqslant Z(C_G(K))$, $C_G(K)$ is a central extension of K by L. If G splits over K, we obtain that $C_G(K) \cong Z_3 \times L$. Otherwise, we have $C_G(K) = Z_3 \cdot L$. Thus, $G \cong (Z_3 \times L).Z_2$ or $G \cong (Z_3 \cdot L).Z_2$.

If $|K| = 6$, then, $G/K \cong L$ and $K \cong Z_6$ or S_3.

Subcase 1: If $K \cong Z_6$, then, $G/C_G(K) \leqslant Aut(Z_6) = Z_6^{\times} \cong Z_2$ and so $|G/C_G(K)| = 1$ or 2. If $|G/C_G(K)| = 1$, then, $Z_6 \cong K \leqslant Z(G)$. It follows that $2 \sim 37$ in $\Gamma(G)$, a contradiction. If $|G/C_G(K)| = 2$, then, $K \subset C_G(K)$ and $1 \neq C_G(K)/K \trianglelefteq G/K \cong L$, which is a contradiction since L is simple.

Subcase 2: If $K \cong S_3$, then, $K \cap C_G(K) = 1$ and $G/C_G(K) \leqslant S_3$. Thus, $C_G(K) \neq 1$. Hence, $1 \neq C_G(K) \cong C_G(K)K/K \trianglelefteq G/K \cong L$. It follows that $L \cong G/K \cong C_G(K)$ because L is simple. Therefore, $G \cong S_3 \times L$, Which implies that $2 \sim 37$ in $\Gamma(G)$, a contradiction. ■

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