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A note on power values of generalized derivation in prime ring and noncommutative Banach algebras

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Abstract. Let R be a prime ring with extended centroid C, H a generalized derivation of R and $n \geqslant 1$ a fixed integer. In this paper we study the situations: (1) If $(H(xy))^n = (H(x))^n(H(y))^n$ for all $x, y \in R$; (2) obtain some related result in case R is a noncommutative Banach algebra and H is continuous or spectrally bounded.

Keywords: generalized derivation, prime ring, Banach algebras, Martindale quotient ring.

1. Introduction

Let R be an algebra with center $Z(R)$ and radical Jacobson rad(R). For given $x, y \in R$, the Lie commutator of x, y is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$. A linear mapping $d : R \to R$ is called derivation if it satisfies the Leibniz rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. We recall that an additive map $H : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $H(xy) = H(x)y + xd(y)$ holds for all $x, y \in R$. Many results in literature indicate that global structure of a prime ring R is often lightly connected to the behaviour of additive mappings defined on R. A well-known result of Herstein [10] stated that if R is a prime ring and d is an inner derivation of R such that $d(x)^n = 0$ for all $x \in R$ and n is fixed integer, then $d = 0$. The number of authors extended this theorem in several ways. In [3] Bell and Kappe proved that if d is a derivation of a prime ring R which $d(xy) = d(x)d(y)$ or $d(xy) = d(y)d(x)$ such that for all $x, y \in I$, a non-zero right ideal of R, then $d = 0$ on R. Recently in [19] Rehman studies the case when the derivation d is replaced by generalized derivation H. More precisely, he proves the following: Let R is a 2-torsion free prime ring and $H(xy) = H(x)H(y)$ or $H(xy) = H(y)H(x)$ for all $x, y \in I$, a non-zero ideal of R, then R must be a commutative.

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1.1 *Main result*

In the present paper our motivation is to generalize, all the above results by studying the following theorem:

THEOREM 1.1 *Let R be a prime ring and H a generalized derivation of R. Suppose $(H(xy))^n = (H(x))^n(H(y))^n$ for all $x, y \in R$ and $n \geqslant 1$ is a fixed integer. Then either R is commutative or $d = 0$ and there exists $a \in C$ such that $H(x) = ax$ and $H(y) = ay$ for all $x, y \in R$.*

Finally, in the last section of this paper we apply this result to the study of analogous conditions for continuous generalized derivations on Banach algebras.

2. In case *R* is a prime ring

In this section R denotes a prime ring with extended centroid C, U its two sided Martindale quotient ring. For the definitions and elementary properties of derivation and two sided Martindale quotient ring we refer the reader to [2].

The following results are useful tools needed in the proof of Theorem1.1.

Remark 1 (see [6, Theorem 2]). Let R be a prime ring and I a non-zero ideal of R Then I, R and U satisfy the same generalized polynomial identities with coefficient in U.

Remark 2 (see [16, Theorem 2]). Let R be a prime ring and I a non-zero ideal of R. Then I, R and U satisfy the same differential identities.

Remark 3 Let R be a prime ring and U be the Utumi quotient ring of R and $C = Z(U)$, the center of U. It is well known that any derivation of R can be uniquely extended to a derivation of U, In [16] Lee proved that every generalized derivation H on a dense right ideal of R can be uniquely extended to a generalized derivation of U and assume the form $H(x) = ax + d(x)$ for all $x \in U$, some $a \in U$ and a derivation d of U.

THEOREM 2.1 (Kharchenko [13]). *Let R be a prime ring, d a nonzero derivation of R and I a nonzero ideal of R. If I satisfies the differential identity*

$$f(r_1, r_2, \ldots, r_n, d(r_1), d(r_2), \ldots, d(r_n)) = 0,$$

for any $r_1, r_2, \ldots, r_n \in I$, then one of the following holds:

(i) *first item I satisfies the generalized polynomial identity*

$$f(r_1, r_2, \ldots, r_n, x_1, x_2, \ldots, x_n) = 0.$$

(ii) *d is Q-inner, that is, for some $q \in Q$, $d(x) = [q, x]$ and I satisfies the generalized polynomial identity*

$$f(r_1, r_2, \ldots, r_n, [q, r_1], [q, r_2], \ldots, [q, r_n]) = 0.$$

We establish the following technical result required in the proof of Theorem 1.1.

LEMMA 2.2 *Let R be a prime ring with extended centroid C. Suppose $(axy + [b, x]y + xay + x[b, y])^n - (ax + [b, x])^n(ay + [b, y])^n = 0$, for all $x, y \in R$ and some $a \in R$. Then R is a commutative or $a, b \in C$.*

Proof If R is commutative there is nothing to prove. Suppose R is not commutative. Set

$$f(x,y) = (axy + [b,x]y + xay + x[b,y])^n - (ax + [b,x])^n(ay + [b,y])^n$$

Since R is not commutative, then by Remark *1*, $f(x,y)$ is a nontrivial generalized polynomial identity for R and so for U.
In case C is infinite, we have $f(x,y) = 0$ for all $x,y \in U \bigotimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \bigotimes_C \overline{C}$ are prime and centrally closed [12], we may replace R by U or $U \bigotimes_C \overline{C}$ according to C is finite or infinite. Thus we may assume that R is a centrally closed over C which is either finite or algebraically closed and $f(x,y) = 0$ for all $x,y \in R$. By Martindale's Theorem [17], R is then a primitive ring having nonzero socle H with C as associated division ring. Hence by Jacobson's Theorem [12] R is isomorphic to a dense ring of linear transformations of some vector space V over C, and H consists of the linear transformations in R of finite rank. Let $\dim_C V = k$. Then the density of R on V implies that $R \cong M_k(C)$. If $\dim_C V = 1$, then R is a commutative, which is a contradiction.
Suppose that $\dim_C V \geqslant 2$. We show that for any $v \in V$, v and av are linearly dependent over C. Suppose v and bv are linearly independent for some $v \in V$. By density of R, there exist $x,y \in R$ such that

$$xv = 0,\, xbv = -v,$$

$$yv = 0,\, ybv = -v.$$

Hence we get following contradiction

$$0 = ((axy + [b,x]y + xay + x[b,y])^n - (ax + [b,x])^n(ay + [b,y])^n)v = -v.$$

So we conclude that $\{v, av\}$ are linearly C-dependent. Hence for each $v \in V$, $av = v\alpha_v$ for some $\alpha_v \in C$. Now we prove α_v is not depending on the choice of $v \in V$.
Since $\dim_C V \geqslant 2$ there exists $w \in V$ such that v and w are linearly independent over C. Now there exist $\alpha_v, \alpha_w, \alpha_{v+w} \in C$ such that

$$bv = v\alpha_v, bw = w\alpha_w, b(v+w) = (v+w)\alpha_{(v+w)}.$$

Which implies

$$v(\alpha_v - \alpha_{(v+w)}) + w(\alpha_w - \alpha_{(v+w)}) = 0,$$

and since $\{v, w\}$ are linearly C-independent, it follows $\alpha_v = \alpha_{(v+w)} = \alpha_w$. Therefore there exists $\alpha \in C$ such that $bv = v\alpha$ for all $v \in V$.
Now let $r \in R$, $v \in V$. Since $bv = v\alpha$,

$$[b,r]v = (br)v - (rb)v = b(rv) - r(bv) = (rv)\alpha - r(v\alpha) = 0,$$

that is $[b,r]V = 0$. Hence $[b,r] = 0$ for all $r \in R$, implying $b \in C$. Similarly we get $a \in C$. ■

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Let R be not commutative. By the given hypothesis R satisfies the generalized differential identity

$$(H(x)y + xH(y))^n = (H(x))^n(H(y))^n. \tag{1}$$

By Remark *2*, R and U satisfy the same differential identities, thus U satisfies (1). As we have already remarked in Remark *3*, we may assume that for all $x, y \in U$, $H(x) = ax + d(x)$, $H(y) = ay + d(y)$, for some $a \in U$ and a derivation d of U. Hence U satisfies

$$(axy + d(x)y + xd(y))^n - (ax + d(x))^n(ay + d(y))^n = 0. \tag{2}$$

Assume first that d is inner derivation of U, i.e., there exists $b \in Q$ such that $d(x) = [b, x]$ and $d(y) = [b, y]$ for all $x, y \in U$. Then by (2), we have

$$(axy + [b, x]y + xay + x[b, y])^n - (ax + [b, x])^n(ay + [b, y])^n = 0,$$

for all $x, y \in U$. Now by Lemma 2.2, $a, b \in C$ and so $d = 0$. Hence for some $a \in C$, $H(x) = ax$ and $H(y) = ay$ for all $x, y \in U$ and so for all $x \in R$.
If d is not a U-inner derivation, then by Theorem *2*, (2) becomes

$$(axy + zy + xay + xw)^n - (ax + z)^n(ay + w)^n = 0,$$

for all $x, y, z, w \in U$. In particular U satisfies its blended component $(axy + zy + xay + xw)^n$. This is a polynomial identity and hence there exists a field F such that $U \subseteq M_k(F)$, the ring of $k \times k$ matrices over field F, where $k > 1$. Moreover U and $M_k(F)$ satisfy the same polynomial identity [15, Lemma 1]. But by choosing $x = w = e_{ii}, y = 0$, we get

$$0 = (axy + zy + xay + xw)^n = e_{ii}.$$

which is a contradiction. This complete the proof.

2.1 *Example*

The following example shows the hypothesis of primeness is essential in theorem 1.1.

Example 2.3 Let S be any ring, and $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in S \right\}$. Define $d : R \to R$ as follows:

$$d\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $0 \neq d$ is a derivation of R such that $(d(xy))^n = (d(x))^n(d(y))^n$ for all $x, y \in R$, where $n \geqslant 1$ is a fixed integer, however R is not commutative.

3. In case R is complex Banach algebra

Here R will denote a complex Banach algebra. Let us introduce some well known and elementary definition for a sake of completeness.

By a Banach algebra we shall mean a complex normed algebra R whose underlying vector space is a Banach space. By rad(R) we denote the Jacobson radical of R. Without loss of generality we assume R to be unital. In fact any Banach algebra R without a unity can be embedded into a unital Banach algebra $R_I = R \oplus \mathbb{C}$ as an ideal of codimension one. In particular we may identity R with the ideal $\{(x,0) : x \in R\}$ in R_I via the isometric isomorphism $x \to (x,0)$. We refer the reader for details to [8, 18].

Our first result in this section is about continuous generalized derivations on a Banach algebras:

THEOREM 3.1 *Let R be a non-commutative Banach algebra, $H = L_a + d$ a continuous generalized derivation of R for some $a \in R$ and some derivation d of R. If $(H(xy))^n - (H(x))^n(H(y))^n \in rad(R)$ for all $x \in R$, then $[a,R] \subseteq rad(R)$, for all $x \in R$ and $d(R) \subseteq rad(R)$.*

The following results are useful tools needed in the proof of Theorem 3.1.

Remark 1 (see [20]). Any continuous derivation of Banach algebra leaves the primitive ideals invariant.

Remark 2 (see [21]). Any continuous linear derivation on a commutative Banach algebra maps the algebra into its radical.

Remark 3 (see [11]). Any linear derivation on semisimple Banach algebra is continuous.

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. Under the assumption that H is continuous, and since it is well known that the left multiplication map L_a is also continuous, we have the derivation d is continuous. As we have already remarked in Remark *1*, we may assume that for any primitive ideal P of R, $H(P) \subseteq aP + d(P) \subseteq P$, that is, also the continuous generalized derivation H leaves the primitive ideals invariant. Denote $\frac{R}{P} = \overline{R}$ for any primitive ideals P. Hence we may introduce the generalized derivation $H_P : \overline{R} \to \overline{R}$ by $H_P(\overline{x}) = H_p(x+P) = H(x)+P = ax+d(x)+P$ for all $x \in R$ and $\overline{x} = x+P$. Moreover by $H_P(\overline{y}) = H_p(y+P) = H(y)+P = ay+d(y)+P$ for all $y \in R$ and $\overline{y} = y+P$. Now by our assumption we have

$$(H(\overline{xy}))^n - (H(\overline{x}))^n(H(\overline{y}))^n = \overline{0},$$

for all $\overline{x}, \overline{y} \in \overline{R}$. Since $\overline{R}$ is primitive, a fortiori it is prime. Thus by Theorem 1.1, we get that either $\overline{R}$ is commutative, i.e., $[R,R] \subseteq P$ or $d = \overline{0}$ and $\overline{a} \in Z(\overline{R})$, i.e., $d(R) \subseteq P$ and $[a,R] \subseteq P$. Now let P be a primitive ideal such that $\overline{R}$ is commutative, By Remarks *2* and *3*, there are no non-zero linear continuous derivations on commutative semisimple Banach algebras. Therefore $d = \overline{0}$ in $\overline{R}$, and since $[R,R] \subseteq P$ follows by the commutativity of $\overline{R}$, we also have $[a,R] \subseteq P$. Hence in any case $d(R) \subseteq P$ and $[a,R] \subseteq P$ for all primitive ideal P of R. Since $rad(R)$ is the intersection of all primitive ideals, we get the required conclusion.
In the special case when R is a semisimple Banach algebra we have:

COROLLARY 3.2 *Let R be a non-commutative semisimple Banach algebra, $H = L_a + d$ a continuous generalized derivation of R for some $a \in R$ and some derivation d of R. If $(H(xy))^n - (H(x))^n(H(y))^n = 0$ for all $x, y \in R$, then $H(x) = ax$ and $H(y) = ay$ for some $a \in Z(R)$.*

Proof For proof we use the fact that $rad(R) = 0$, since R is a semisimple. ■

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