Journal of
Linear and Topological Algebra
Vol. 01, No. 01, Summer 2012, 1- 7

n-Jordan homomorphisms on C^*-algebras

Abasalt Bodaghi[a] and Behrouz Shojaee [b]

[a] *Department of Mathematics, Garmsar Branch, Islamic Azad University, Garmsar, Iran*
[b] *Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran*

Abstract. Let $n \in \mathbb{N}$. An additive map $h : \mathcal{A} \longrightarrow \mathcal{B}$ between algebras $\mathcal{A}$ and $\mathcal{B}$ is called n-Jordan homomorphism if $h(a^n) = (h(a))^n$ for all $a \in \mathcal{A}$. We show that every n-Jordan homomorphism between commutative Banach algebras is a n-ring homomorphism when $n < 8$. For these cases, every involutive n-Jordan homomorphism between commutative C^*-algebras is norm continuous.

Keywords: n-homomorphism; n-ring.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras. An n-ring homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a map $h : \mathcal{A} \longrightarrow \mathcal{B}$ that is additive (i.e., $h(a+b) = h(a) + h(b)$ for all $a, b \in \mathcal{A}$) and n-multiplicative (i.e., $h(a_1a_2...a_n) = h(a_1)h(a_1)...h(a_n)$ for all $a_1, a_2, ..., a_n \in \mathcal{A}$). The map $h : \mathcal{A} \longrightarrow \mathcal{B}$ is called n-Jordan homomorphism if it is additive and $h(a^n) = (h(a))^n$ for all $a \in \mathcal{A}$. It is clear that every n-ring homomorphism is n-Jordan homomorphism but the converse is not true. There are some examples of n-Jordan homomorphisms which are not n-ring homomorphisms (for example refer to [2]). It is shown in [2] that every n-Jordan homomorphism between commutative Banach algebras is also n-ring homomorphism when $n \in \{3, 4\}$. For $n = 2$, the proof is simple and routine. For the non-commutative case, Zelazko in [9] showed that if $\mathcal{A}$ is a Banach algebra which need not be commutative, and $\mathcal{B}$ is a semisimple commutative Banach algebra, then each Jordan homomorphism $h : \mathcal{A} \longrightarrow \mathcal{B}$ is a ring homomorphism.

An n-ring homomorphism $h : \mathcal{A} \longrightarrow \mathcal{B}$ between C^*-algebras is said to be $*$-n-ring homomorphism if $h(a^*) = h(a)^*$ for all $a \in \mathcal{A}$. Similarly one can define $*$-n-Jordan homomorphism. If, in addition, h is linear, we say that h is *involutive n-ring (Jordan) homomorphism.*

One of the fundamental results in the study of C^*-algebras is that if $T : \mathcal{A} \longrightarrow \mathcal{B}$ is is a $*$-homomorphism between C^*-algebras, then it is norm contractive [6, theorem 2.1.7]. In [4], authors ask: Is every involutive n-ring homomorphism between C^*-algebras continuous? Park and Trout in [7] answered this question and proved that every involutive n-ring homomorphism between C^*-algebras is in fact norm

*Corresponding author. Email: abasalt.bodaghi@gmail.com

contractive. Some questions of automatic continuity for n-homomorphisms between Banach algebras were also investigated in [1, 5]. After that, Tomforde in [8, theorem 3.6] proved that if $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is a unital $*$-preserving ring homomorphism, then ϕ is contractive. Consequently, ϕ is also continuous.

In this paper, we prove that every n-Jordan homomorphism between commutative Banach algebras is n-ring homomorphism when $n \in \{5, 6, 7\}$(for the case $n = 5$ this had been proved earlier by Eshaghi et al in [3] with a long proof). Finally, using these results, we show that every involutive n-Jordan homomorphism between commutative C^*-algebras is continuous.

2. Main Results

Theorem 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative algebras, and let $h : \mathcal{A} \longrightarrow \mathcal{B}$ be an n-Jordan homomorphism. Then h is an n-ring homomorphism for $n \in \{3, 4, 5, 6, 7\}$.

Proof For the cases $n = 3, 4$, refer to [3]. As for $n = 5$, the map h is additive such that $h(x^5) = (h(x))^5$ for all $x \in \mathcal{A}$. Using this equality, we have

$$h\left(\sum_{k=1}^{4}\binom{5}{k}x^k y^{5-k}\right) = \sum_{k=1}^{4}\binom{5}{k}h(x)^k h(y)^{5-k} \tag{1}$$

for all $x, y \in \mathcal{A}$. Replacing x by $x + z$ in (1), we obtain

$$\begin{aligned} h\Bigg(\left[\sum_{k=0}^{4}\binom{4}{k}x^k z^{4-k}\right]y + 2\left[\sum_{k=0}^{3}\binom{3}{k}x^k z^{3-k}\right]y^2 \\ +2\left[\sum_{k=0}^{2}\binom{2}{k}x^k z^{2-k}\right]y^3 + xy^4 + zy^4\Bigg) \\ = \left[\sum_{k=0}^{4}\binom{4}{k}h(x)^k h(z)^{4-k}\right]h(y) + 2\left[\sum_{k=0}^{3}\binom{3}{k}h(x)^k h(z)^{3-k}\right]h(y)^2 \\ +2\left[\sum_{k=0}^{2}\binom{2}{k}h(x)^k h(z)^{2-k}\right]h(y)^3 + h(x)h(y)^4 + h(z)h(y)^4 \end{aligned} \tag{2}$$

for all $x, y, z \in \mathcal{A}$. Combining (1) and (2) gives

$$\begin{aligned} h(2x^3zy + 3x^2z^2y + 2xz^3y + 3x^2zy^2 + 3xz^2y^2 + 2xzy^3) \\ = 2h(x)^3h(z)h(y) + 3h(x)^2h(z)^2h(y) + 2h(x)h(z)^3h(y) \\ +3h(x)^2h(z)h(y)^2 + 3h(x)h(z)^2h(y)^2 + 2h(x)h(z)h(y)^3 \end{aligned} \tag{3}$$

for all $x, y, z \in \mathcal{A}$. Substituting z by $-x$ in (3), we obtain

$$h(x^4y + 2x^2y^3) = h(x)^4h(y) + 2h(x)^2h(y)^3 \tag{4}$$

for all $x, y \in \mathcal{A}$. Now, if we replace y by $y + w$ in (4) and employ the same equality, we get

$$h(x^2y^2w + x^2yw^2) = h(x)^2h(y)^2h(w) + h(x)^2h(y)h(w)^2 \tag{5}$$

for all $x, y, w \in \mathcal{A}$. Replacing x by $x+u$ in (5), we have

$$h(xuy^2w + xuyw^2) = h(x)h(u)h(y)^2h(w) + h(x)h(u)h(y)h(w)^2 \tag{6}$$

for all $x, y, u, w \in \mathcal{A}$. Now, if we change y to $y+v$ in (6), we conclude

$$h(xuyvw) = h(x)h(u)h(y)h(v)h(w)$$

for all $x, y, u, v, w \in \mathcal{A}$. Therefore h is 5-ring homomorphism.

For the case $n = 6$, we assume that the map h is additive and $h(x^6) = (h(x))^6$ for all $x \in \mathcal{A}$. This fact implies the following equality if we replace x by $x + y$

$$h\left(\sum_{k=1}^{5}\binom{6}{k}x^k y^{6-k}\right) = \sum_{k=1}^{5}\binom{6}{k}h(x)^k h(y)^{6-k} \tag{7}$$

for all $x, y \in \mathcal{A}$. Commuting x by $x+z$ in (7), we obtain

$$\begin{aligned}
&h\left(\left[\sum_{k=0}^{5}\binom{5}{k}x^k z^{5-k}\right]y + 15\left[\sum_{k=0}^{4}\binom{4}{k}x^k z^{4-k}\right]y^2\right.\\
&\quad\left.+20\left[\sum_{k=0}^{3}\binom{3}{k}x^k z^{3-k}\right]y^3 + 15\left[\sum_{k=0}^{2}\binom{2}{k}x^k z^{2-k}\right]y^4 + 6xy^5 + 6zy^5\right)\\
&\quad= 6\left[\sum_{k=0}^{5}\binom{5}{k}h(x)^k h(z)^{5-k}\right]h(y) + 15\left[\sum_{k=0}^{4}\binom{4}{k}h(x)^k h(z)^{4-k}\right]h(y)^2\\
&\quad+20\left[\sum_{k=0}^{3}\binom{3}{k}h(x)^k h(z)^{3-k}\right]h(y)^3 + 15\left[\sum_{k=0}^{2}\binom{2}{k}h(x)^k h(z)^{2-k}\right]h(y)^4\\
&\quad+6h(x)h(y)^5 + 6h(z)h(y)^5
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Combining the above equality and (7), we get

$$\begin{aligned}
&h\left(6\left[\sum_{k=1}^{4}\binom{5}{k}x^k z^{5-k}\right]y + 15\left[\sum_{k=1}^{3}\binom{4}{k}x^k z^{4-k}\right]y^2\right.\\
&\qquad\left.+20\left[\sum_{k=1}^{2}\binom{3}{k}x^k z^{3-k}\right]y^3 + 30zy^4\right)\\
&\qquad= 6\left[\sum_{k=1}^{4}\binom{5}{k}h(x)^k h(z)^{5-k}\right]h(y) + 15\left[\sum_{k=1}^{3}\binom{4}{k}h(x)^k h(z)^{4-k}\right]h(y)^2\\
&\qquad+20\left[\sum_{k=1}^{2}\binom{3}{k}h(x)^k h(z)^{3-k}\right]h(y)^3 + 30h(x)h(z)h(y)^4
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Changing z to $-x$ in the last equality, we obtain

$$h(x^4y^2 + x^2y^4) = h(x)^4h(y)^2 + h(x)^2h(y)^4 \tag{8}$$

for all $x, y \in \mathcal{A}$. Now, if we replace y by $y+t$ in (8), we conclude

$$h(x^4yt + 2x^2yt^3 + 3x^2y^2t^2 + 2x^2y^3t) = h(x)^4h(y)h(t) + 2h(x)^2h(y)h(t)^3$$

$$+3h(x)^2h(y)^2h(t)^2 + 2h(x)^2h(y)^3h(t) \tag{9}$$

for all $x, y, t \in \mathcal{A}$. Substituting t by $t+u$ in (9), we have

$$h(x^2yt^2u+x^2ytu^2+x^2y^2tu) = h(x)^2h(u)h(y)h(t)^2h(u)+h(x)^2h(u)h(y)h(t)h(u)^2 +h(x)^2h(y)^2h(t)h(u) \tag{10}$$

for all $x, y, t, u \in \mathcal{A}$. We replace u by $u + v$ in (10) to obtain

$$h(x^2ytuv) = h(x)^2h(u)h(y)h(t)h(v) \tag{11}$$

for all $x, y, t, u, v \in \mathcal{A}$. Finally if we change x to $x + w$ in (11), we get

$$h(xytuvw) = h(x)h(y)h(t)h(u)h(v)h(w).$$

The above equality shows that the map h is 6-ring homomorphism. Now, for $n = 7$. Replacing x by $x + y$ in equality $h(x^7) = (h(x))^7$, we have

$$h\left(\sum_{k=1}^{6}\binom{7}{k}x^ky^{7-k}\right) = \sum_{k=1}^{6}\binom{7}{k}h(x)^kh(y)^{7-k} \tag{12}$$

for all $x, y \in \mathcal{A}$. Commuting x by $x + z$ in (12), we obtain

$$\begin{aligned}
&h\Bigg(7\left[\sum_{k=0}^{6}\binom{6}{k}x^kz^{6-k}\right]y+21\left[\sum_{k=0}^{5}\binom{5}{k}x^kz^{5-k}\right]y^2\\
&\qquad+35\left[\sum_{k=0}^{4}\binom{4}{k}x^kz^{4-k}\right]y^3+35\left[\sum_{k=0}^{3}\binom{3}{k}x^kz^{3-k}\right]y^4\\
&\qquad+21\left[\sum_{k=0}^{2}\binom{2}{k}x^kz^{2-k}\right]y^5+7xy^6+7zy^6\Bigg)\\
&=7\left[\sum_{k=0}^{6}\binom{6}{k}h(x)^kh(z)^{6-k}\right]h(y)+21\left[\sum_{k=0}^{5}\binom{5}{k}h(x)^kh(z)^{5-k}\right]h(y)^2\\
&+35\left[\sum_{k=0}^{4}\binom{4}{k}h(x)^kh(z)^{4-k}\right]h(y)^3+35\left[\sum_{k=0}^{3}\binom{3}{k}h(x)^kh(z)^{3-k}\right]h(y)^4\\
&+21\left[\sum_{k=0}^{2}\binom{2}{k}h(x)^kh(z)^{2-k}\right]h(y)^5+7h(x)h(y)^6+7h(z)h(y)^6
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Combining (12) and the above equality, we get

$$\begin{aligned}
&h\Bigg(7\left[\sum_{k=1}^{5}\binom{6}{k}x^kz^{6-k}\right]y+21\left[\sum_{k=1}^{4}\binom{5}{k}x^kz^{5-k}\right]y^2\\
&\qquad+35\left[\sum_{k=1}^{3}\binom{4}{k}x^kz^{4-k}\right]y^3+35\left[\sum_{k=1}^{2}\binom{3}{k}x^kz^{3-k}\right]y^4+42xzy^5)\\
&\qquad=7\left[\sum_{k=1}^{5}\binom{6}{k}h(x)^kh(z)^{6-k}\right]h(y)+21\left[\sum_{k=1}^{4}\binom{5}{k}h(x)^kh(z)^{5-k}\right]h(y)^2\\
&\qquad+35\left[\sum_{k=1}^{3}\binom{4}{k}h(x)^kh(z)^{4-k}\right]h(y)^3+35\left[\sum_{k=1}^{2}\binom{3}{k}h(x)^kh(z)^{3-k}\right]h(y)^4\\
&\qquad+42h(x)h(z)h(y)^5
\end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Letting z to be $-x$ in the above, we obtain

$$h(3x^2y^5 + 5x^4y^3 + x^6y) = 3h(x)^2h(y)^5 + 5h(x)^4h(y)^3 + h(x)^6h(y) \tag{13}$$

for all $x, y \in \mathcal{A}$. Now, if we replace y by $y + t$ in (13) and use the same equality, we conclude

$$\begin{aligned} &h(x^2y^4t + 2x^2y^3t^2 + 2x^2y^2t^3 + x^2yt^4 + x^4y^2t + x^4yt^2) \\ &\quad = h(x)^2h(y)^4h(t) + 2h(x)^2h(y)^3h(t)^2 + 2h(x)^2h(y)^2h(t)^3 \\ &\quad +h(x)^2h(y)h(t)^4 + h(x)^4h(y)^2h(t) + h(x)^4h(y)h(t)^2 \end{aligned} \tag{14}$$

for all $x, y, t \in \mathcal{A}$. Substituting t by $t + u$ in (14), we have

$$\begin{aligned} &h(2x^2y^3tu + 3x^2y^2t^2u + 3x^2y^2tu^2 + 2x^2yt^3u + 3x^2yt^2u^2 + 2x^2ytu^3 + x^4ytu) \\ &\quad = 2h(x)^2h(y)^3h(t)h(u) + 3h(x)^2h(y)^2h(t)^2h(u) \\ &\quad +3h(x)^2h(y)^2h(t)h(u)^2 + 2h(x)^2h(y)h(t)^3h(u) \\ &\quad +3h(x)^2h(y)h(t)^2h(u)^2 + 2h(x)^2h(y)h(t)h(u)^3 + h(x)^4h(y)h(t)h(u) \end{aligned}$$

for all $x, y, t, u \in \mathcal{A}$. We replace u by $u + v$ in the last equality to obtain

$$\begin{aligned} &h(x^2y^2tuv + x^2yt^2uv + x^2ytu^2v + x^2ytuv^2) \\ &\quad = h(x)^2h(y)^2h(t)h(u)h(v) + h(x)^2h(u)h(y)h(t)^2h(u)h(v) \\ &\quad +h(x)^2h(y)h(t)h(u)^2h(v) + h(x)^2h(y)h(t)h(u)h(v)^2 \end{aligned} \tag{15}$$

for all $x, y, t, u, v \in \mathcal{A}$. Replacing v by $v + w$ in (15), we deduce

$$h(x^2ytuvw) = h(x)^2h(y)h(t)h(u)h(v)h(w) \tag{16}$$

for all $x, y, t, u, v, w \in \mathcal{A}$. Finally, if we change x to $x + z$ in (16), we get

$$h(xyztuvw) = h(x)h(y)h(z)h(t)h(u)h(v)h(w).$$

for all $x, y, z, t, u, v, w \in \mathcal{A}$. Hence the map h is 7-ring homomorphism. ■

3. Applications

An element a of a C^*-algebra $\mathcal{A}$ is *positive* if a is hermitian, that is $a = a^*$, and $\sigma(a) \subseteq \mathbb{R}^+$, where $\sigma(a)$ is the spectrum of a. We write $a \geqslant 0$ to mean a is positive. Also a linear map $T : \mathcal{A} \longrightarrow \mathcal{B}$ between C^*-algebras is positive if $a \geqslant 0$ implies $T(a) \geqslant 0$ for all $a \in \mathcal{A}$. We say that the map T is *completely positive* if, for any natural number k, the induced map $T_k : M_k(\mathcal{A}) \longrightarrow M_k(\mathcal{B}); T_k((a_{ij})) \mapsto (T(a_{ij}))$, on $k \times k$ matrices is positive.

proposition 3.1. Let $n \in \mathbb{N}$ such that $2 \leqslant n \leqslant 7$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are commutative C^*-algebras. Let α and β be nonnegative real numbers and let r, s be real numbers, f be a map from $\mathcal{A}$ into $\mathcal{B}$, and let r, s be real numbers such that either $(r - 1)(s - 1) > 0$ and $s \geqslant 0$ or $(r - 1)(s - 1) > 0$, $s < 0$, and $f(0) = 0$. Assume that f satisfies the system of functional inequalities

$$\|f(x + y + z^*) - f(x) - f(y) - f(z)^*\| \leqslant \alpha(\|x\|^r + \|y\|^r + \|z\|^r)$$

$$\|f(x^n) - f(x)^n\| \leqslant \beta\|x\|^{ns}$$

for all $x, y \in \mathcal{A}$. Then, there exists a unique $*$-n-ring homomorphism $h : \mathcal{A} \longrightarrow \mathcal{B}$ such that

$$\|f(x) - h(x)\| \leqslant \frac{2\alpha}{|2 - 2^r|}\|x\|^r$$

for all $x \in \mathcal{A}$.

Proof We can deduce the result from [3, theorm 2.1, theorem 2.2] and theorem 2. ■

The following theorem has been proved by Park and Trout in [7, theorm 3.2].

Theorem 3.1. Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be an involutive n-homomorphism between C^*-algebras. If $n \geqslant 3$ is odd, then $\|\phi\| \leqslant 1$, i.e., ϕ is norm-contractive.

corollary 3.1. Let $n \in \{3, 5, 7\}$, and let $\mathcal{A}$ and $\mathcal{B}$ be commutative C^*-algebras. If $h : \mathcal{A} \longrightarrow \mathcal{B}$ is an involutive n-Jordan homomorphism, then $\|h\| \leqslant 1$, i.e., h is norm contractive.

Proof For $n = 3$, The result follows from [2, theorm 2.1] and theorem 2 and for $n = 5, 7$, we can use theorem 2 and theorem 3. ■

For the even case, we need the following theorem which is proved in [7, theorm 2.3].

Theorem 3.2. Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be an involutive n-homomorphism between C^*-algebras. If $n \geqslant 2$ is even, then ϕ is completely positive. Thus, ϕ is bounded.

corollary 3.2. Let $n \in \{4, 6\}$. If $h : \mathcal{A} \longrightarrow \mathcal{B}$ is an involutive n-Jordan homomorphism between commutative C^*-algebras, then h is completely positive. Thus, h is bounded.

Proof By using [2, theorm 2.1] and theorem 2 for $n = 4$ and theorems 2 and 3 for $n = 6$, we obtain the desired result. ■

Question. Let n be an arbitrary and fixed natural number. Is every n-Jordan homomorphism between commutative algebras is also a n-ring homomorphism? If this is true, then every involutive n-Jordan homomorphism between commutative C^*-algebras is norm contractive. Is this true in the non-commutative case?

Acknowledgement. The authors sincerely thank the anonymous reviewer for his careful reading, constructive comments and fruitful suggestions to improve the quality of the manuscript.

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