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ECONOMIC INVESTIGATION OF LAGRANGE MULTIPLIER IF COST OF INPUTS AND BUDGET SIZE OF A FIRM INCREASE: A PROFIT MAXIMIZATION ENDEAVOR

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Abstract: *In this study method of Lagrange multiplier is considered to investigate profit maximization policy. In the twenty first century global economy faces serious complexities. Sensitivity analysis of economic firms becomes essential part of sustainable economic environment. The method of Lagrange multiplier is a very useful and powerful technique in multivariable calculus that is applied in economic models to obtain higher dimensional unconstrained problem from the lower dimensional constrained problems. This paper proceeds with Cobb-Douglas production function, where 6×6 bordered Hessian and 6×6 Jacobian are used to examine sensitivity analysis efficiently and elaborately.*

Keywords: *budget, inputs, Lagrange multiplier, profit maximization*

JEL Classification: C51, C52, , C53, C61, C67, D21, D24, H32, I31

1. Introduction

Mathematical modeling is a highly abstract discipline that covers many fields of social sciences, such as economics, sociology, psychology, political science, etc. [Carter, 2001]. It is widely used to solve optimization problems and many other problems of welfare economics [Samuelson, 1947]. In economics it makes relationships among prices, production, employment, saving, investment, etc. [Zheng & Liu, 2022].

Every firm wants to earn the maximum profit with minimum cost [Eaton & Lipsey, 1975; Dixit, 1990]. To obtain maximum profit, an organization must be very sincere in every step of its operation, such as in production, financial balance, transportation, total management system, etc. [Mohajan & Mohajan, 2022a]. In this study we have used Cobb-Douglas production function as our profit function [Cobb & Douglas, 1928]. Lagrange multipliers method is a very useful and powerful practice in multivariable calculus. We have used this method as a device for transforming a constrained problem to a higher dimensional unconstrained problem [Baxley & Moorhouse, 1984]. In this study, we have worked with the 6×6 bordered Hessian matrix and 6×6 Jacobian matrix [Mohajan & Mohajan, 2022e,f].

2. Literature Review

The literature review is an introductory section of a scholarly research that tries to designate the contributions of other scholars in the same research field [Polit & Hungler, 2013]. Two US distinguish professors; mathematician John V. Baxley and economist John C. Moorhouse have discussed the profit maximization method subject to a budget constraint [Baxley & Moorhouse, 1984]. Mathematician Jamal Nazrul Islam and his coworkers have discussed profit maximization problem by providing reasonable interpretatio n of the Lagrange multiplier [Islam et al., 2010a, 2011]. Lia Roy and her coauthors have studied with cost minimization problem of an industry [Roy et al., 2021].

Jannatul Ferdous and Haradhan Kumar Mohajan have worked on a profit maximization problem [Ferdous & Mohajan, 2022]. Pahlaj Moolio and his coworkers have used Cobb-Douglas production function to analyze output maximization [Moolio et al., 2009]. Devajit Mohajan and Haradhan Kumar Mohajan have run the profit maximization problem with the sensitivity analysis among commodities, coupons, and prices [Mohajan & Mohajan, 2022a,c,f].

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3. Research Methodology of the Study

Research is an essential and influential tool to the professors to lead the academic atmosphere [Pandey & Pandey, 2015]. Methodology is an organized and meaningful procedural works that tries to describe the types of research and the types of data [Ojo, 2003; Somekh & Lewin, 2005]. Therefore, research methodology is the science and philosophy behind all researches for organizing, planning, designing and conducting a good research [Remenyi et al., 1998; Legesse, 2014]. To prepare this paper we have depended on the mathematical secondary data sources of profit maximization that are collected from published research papers, and books of famous authors; and also essential materials are managed using internet, websites, etc. [Islam et al., 2009a,b; Mohajan, 2017a, 2018b, 2020].

4. Objective of the Study

The leading objective of this paper is to discuss the economic procedures of Lagrange multiplier when cost of various inputs and budget size of a firm increase. Some other related trivial objectives are as follows:

- to offer the economic forecasts precisely,
- to show the mathematical calculations elaborately, and
- to enhance the global economic research area.

5. Economic Model of the Firm

Let us consider that an emerging economic firm uses χ_1 quantity of capital, χ_2 quantity of labor, χ_3 quantity of principal raw materials, and χ_4 quantity of irregular inputs to run its industrial sector successfully. Now we can introduce Cobb-Douglas production function as a four-dimensional profit function for this study in the form [Islam et al., 2011; Mohajan, 2018a],

$$P = f\left(\chi_1, \chi_2, \chi_3, \chi_4\right) = A\chi_1^i \chi_2^j \chi_3^k \chi_4^l \tag{1}$$

where A is the technical process of economic system that indicates total factor productivity. Here i, j, k, and l are parameters; i indicates the output of elasticity of capital measures the percentage change in P for 1% change in χ_1, while χ_2, χ_3, and χ_4 are held constants. Similar properties bear parameters j, k, and l. The firm wants to maximize the profit function (1) for the sustainability in the global economic environment. The values of *i*, *j*, *k*, and *l* are determined by the available technologies, and must satisfy the following four inequalities [Roy et al., 2021; Mohajan, 2021a]:

$$0 < i < 1\,,\; 0 < j < 1\,,\; 0 < k < 1\,, \text{and } 0 < l < 1\,. \tag{2}$$

A strict Cobb-Douglas production function, in which $\Phi = i + j + k + l = 1$ indicates constant returns to scale, $\Phi < 1$ indicates decreasing returns to scale, and $\Phi > 1$ indicates increasing returns to scale [Moolio et al., 2009; Mohajan, 2022].

Now we consider that the four-dimensional budget constraint,

$$B(\chi_1, \chi_2, \chi_3, \chi_4) = b_1\chi_1 + b_2\chi_2 + b_3\chi_3 + b_4\chi_4\,, \tag{3}$$

where b_1 is rate of interest or services of capital per unit of capital χ_1; b_2 is the wage rate per unit of labor χ_2; b_3 is the cost per unit of principal raw material χ_3; and b_4 is the cost per unit of other inputs χ_4.

Now we introduce a single Lagrange multiplier η, as a device; and by using equations (1) and (3) we can represent the Lagrangian function $Z(\chi_1, \chi_2, \chi_3, \chi_4, \eta)$, in a five-dimensional unconstrained problem as follows [Mohajan & Mohajan, 2022d; Mohajan, 2021c]:

$$Z(\chi_1, \chi_2, \chi_3, \chi_4, \eta) = A\chi_1^i \chi_2^j \chi_3^k \chi_4^l + \eta\,(B - b_1\chi_1 - b_2\chi_2 - b_3\chi_3 - b_4\chi_4), \tag{4}$$

where $\frac{\partial B}{\partial \chi_1} = B_1$, $\frac{\partial B}{\partial \chi_2} = B_2$, $\frac{\partial Z}{\partial \chi_1} = Z_1$, $\frac{\partial^2 Z}{\partial \chi_1 \partial \chi_3} = Z_{31}$, $\frac{\partial^2 Z}{\partial \chi_2^2} = Z_{22}$, etc. are partial derivatives. Let us consider the determinant of the 5×5 bordered Hessian matrix as [Ferdous & Mohajan, 2022];

$$H = \begin{vmatrix} 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ -B_1 & Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ -B_2 & Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ -B_3 & Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ -B_4 & Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{vmatrix}. \tag{5}$$

Taking first-order partial differentiations of (3) we get,

$$B_1 = b_1\,,\; B_2 = b_2\,,\; B_3 = b_3\,, \text{and } B_4 = b_4\,. \tag{6}$$

Taking second-order and cross partial derivatives of (4) we get,

$$
\begin{aligned}
Z_{11} &= i(i-1)A\chi_1^{i-2}\chi_2^{j}\chi_3^{k}\chi_4^{l},\\
Z_{22} &= j(j-1)A\chi_1^{i}\chi_2^{j-2}\chi_3^{k}\chi_4^{l},\\
Z_{33} &= k(k-1)A\chi_1^{i}\chi_2^{j}\chi_3^{k-2}\chi_4^{l},\\
Z_{44} &= l(l-1)A\chi_1^{i}\chi_2^{j}\chi_3^{k}\chi_4^{l-2},\\
Z_{12} &= Z_{21} = ijA\chi_1^{i-1}\chi_2^{j-1}\chi_3^{k}\chi_4^{l},\\
Z_{13} &= Z_{31} = ikA\chi_1^{i-1}\chi_2^{j}\chi_3^{k-1}\chi_4^{l},\\
Z_{14} &= Z_{41} = ilA\chi_1^{i-1}\chi_2^{j}\chi_3^{k}\chi_4^{l-1},\\
Z_{23} &= Z_{32} = jkA\chi_1^{i}\chi_2^{j-1}\chi_3^{k-1}\chi_4^{l},\\
Z_{24} &= Z_{42} = jlA\chi_1^{i}\chi_2^{j-1}\chi_3^{k}\chi_4^{l-1},\\
Z_{34} &= Z_{43} = klA\chi_1^{i}\chi_2^{j}\chi_3^{k-1}\chi_4^{l-1}.
\end{aligned}
\tag{7}
$$

Now we expand the bordered Hessian (5) as,

$$
|H| = \frac{A^3B^2ijkl\chi_1^{3i}\chi_2^{3j}\chi_3^{3k}\chi_4^{3l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4^2\Phi} > 0 \tag{8}
$$

where $A>0$, $i,j,k,l>0$, and budget, $B>0$, therefore, $|H|>0$. Hence, the profit is maximized [Mohajan et al., 2013; Mohajan & Mohajan, 2022b].

6. Briefs on Matrix Operations

We have observed that the second order condition is satisfied, so that the determinant of (5) survives at the optimum, i.e., $|J|=|H|$; hence, we can apply the implicit function theorem. Let $\mathbf{G}$ be the vector-valued function of ten variables $\eta^*, \chi_1^*, \chi_2^*, \chi_3^*, \chi_4^*, b_1, b_2, b_3, b_4$, and *B*, and we define the function $\mathbf{G}$ for the point $(\eta^*, \chi_1^*, \chi_2^*, \chi_3^*, \chi_4^*, b_1, b_2, b_3, b_4, B) \in R^{10}$, and take the values in R^5. By the Implicit Function Theorem of multivariable calculus the equation,

$$
F(\eta^*, \chi_1^*, \chi_2^*, \chi_3^*, \chi_4^*, b_1, b_2, b_3, b_4, B) = 0, \tag{9}
$$

may be solved in the form of

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$$\begin{bmatrix} \eta \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{bmatrix} = \mathbf{G}(b_1, b_2, b_3, b_4, B). \tag{10}$$

Now the 5×5 Jacobian matrix for $\mathbf{G}$; regarded as $J_G = \frac{\partial(\eta, \chi_1, \chi_2, \chi_3, \chi_4)}{\partial(b_1, b_2, b_3, b_4, B)}$, and is presented by;

$$J_G = \begin{bmatrix} \frac{\partial \eta}{\partial b_1} & \frac{\partial \eta}{\partial b_2} & \frac{\partial \eta}{\partial b_3} & \frac{\partial \eta}{\partial b_4} & \frac{\partial \eta}{\partial B} \\ \frac{\partial \chi_1}{\partial b_1} & \frac{\partial \chi_1}{\partial b_2} & \frac{\partial \chi_1}{\partial b_3} & \frac{\partial \chi_1}{\partial b_4} & \frac{\partial \chi_1}{\partial B} \\ \frac{\partial \chi_2}{\partial b_1} & \frac{\partial \chi_2}{\partial b_2} & \frac{\partial \chi_2}{\partial b_3} & \frac{\partial \chi_2}{\partial b_4} & \frac{\partial \chi_2}{\partial B} \\ \frac{\partial \chi_3}{\partial b_1} & \frac{\partial \chi_3}{\partial b_2} & \frac{\partial \chi_3}{\partial b_3} & \frac{\partial \chi_3}{\partial b_4} & \frac{\partial \chi_3}{\partial B} \\ \frac{\partial \chi_4}{\partial b_1} & \frac{\partial \chi_4}{\partial b_2} & \frac{\partial \chi_4}{\partial b_3} & \frac{\partial \chi_4}{\partial b_4} & \frac{\partial \chi_4}{\partial B} \end{bmatrix}. \tag{11}$$

$$= -J^{-1} \begin{bmatrix} -\chi_1 & -\chi_2 & -\chi_3 & -\chi_4 & 1 \\ -\eta & 0 & 0 & 0 & 0 \\ 0 & -\eta & 0 & 0 & 0 \\ 0 & 0 & -\eta & 0 & 0 \\ 0 & 0 & 0 & -\eta & 0 \end{bmatrix}.$$

The inverse of Jacobian matrix is, $J^{-1}=\frac{1}{|J|}C^T$, where $C=\left(C_{ij}\right)$, the matrix of cofactors of J, where T indicates transpose, then (11) becomes [Mohajan, 2017a; Mohajan & Mohajan, 2022f],

$$=-\frac{1}{|J|}\begin{bmatrix} C_{11} & C_{21} & C_{31} & C_{41} & C_{51} \\ C_{12} & C_{22} & C_{32} & C_{42} & C_{52} \\ C_{13} & C_{23} & C_{33} & C_{43} & C_{53} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{54} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} \end{bmatrix}\begin{bmatrix} -x_1 & -x_2 & -x_3 & -x_4 & 1 \\ -\eta & 0 & 0 & 0 & 0 \\ 0 & -\eta & 0 & 0 & 0 \\ 0 & 0 & -\eta & 0 & 0 \\ 0 & 0 & 0 & -\eta & 0 \end{bmatrix}$$

$$J_G=-\frac{1}{|J|}\begin{bmatrix} -\chi_1C_{11}-\eta C_{21} & -\chi_2C_{11}-\eta C_{31} & -\chi_3C_{11}-\eta C_{41} & -\chi_4C_{11}-\eta C_{51} & C_{11} \\ -\chi_1C_{12}-\eta C_{22} & -\chi_2C_{12}-\eta C_{32} & -\chi_3C_{12}-\eta C_{42} & -\chi_4C_{12}-\eta C_{52} & C_{12} \\ -\chi_1C_{13}-\eta C_{23} & -\chi_2C_{13}-\eta C_{33} & -\chi_3C_{13}-\eta C_{43} & -\chi_4C_{13}-\eta C_{53} & C_{13} \\ -\chi_1C_{14}-\eta C_{24} & -\chi_2C_{14}-\eta C_{34} & -\chi_3C_{14}-\eta C_{44} & -\chi_4C_{14}-\eta C_{54} & C_{14} \\ -\chi_1C_{15}-\eta C_{25} & -\chi_2C_{15}-\eta C_{35} & -\chi_3C_{15}-\eta C_{45} & -\chi_4C_{15}-\eta C_{55} & C_{15} \end{bmatrix}. \tag{12}$$

In (11) there are total 25 comparative statics, and for sensitivity analysis we will try only four of these to predict the economic analysis for the profit maximization [Baxley & Moorhouse, 1984; Mohajan & Mohajan, 2022a].

7. Sensitivity Analysis of the Model

Now we analyze the economic effects of the firm on Lagrange multiplier η when per unit cost, interest or services of capital increases. Taking T_{11} (i.e., term of 1st row and 1st column) from both sides of (12) we get [Islam et al., 2011; Roy et al., 2021],

$$\frac{\partial\eta}{\partial b_1}=\frac{\chi_1}{|J|}\left[C_{11}\right]+\frac{\eta}{|J|}\left[C_{21}\right]$$

$$=\frac{\chi_1}{|J|}\text{Cofactor of } C_{11}+\frac{\eta}{|J|}\text{Cofactor of } \mathrm{C}_{21}$$

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$$=\frac{\chi_1}{|J|}\begin{vmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{vmatrix} - \frac{\eta}{|J|}\begin{vmatrix} -B_1 & -B_2 & -B_3 & -B_4 \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{vmatrix}$$

$$=\frac{\chi_1}{|J|}\left\{ Z_{11}\begin{vmatrix} Z_{22} & Z_{23} & Z_{24} \\ Z_{32} & Z_{33} & Z_{34} \\ Z_{42} & Z_{43} & Z_{44} \end{vmatrix} - Z_{12}\begin{vmatrix} Z_{21} & Z_{23} & Z_{24} \\ Z_{31} & Z_{33} & Z_{34} \\ Z_{41} & Z_{43} & Z_{44} \end{vmatrix} + Z_{13}\begin{vmatrix} Z_{21} & Z_{22} & Z_{24} \\ Z_{31} & Z_{32} & Z_{34} \\ Z_{41} & Z_{42} & Z_{44} \end{vmatrix} \right.$$

$$\left. - Z_{14}\begin{vmatrix} Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \\ Z_{41} & Z_{42} & Z_{43} \end{vmatrix} \right\}$$

$$-\frac{\eta}{|J|}\left\{ -B_1\begin{vmatrix} Z_{22} & Z_{23} & Z_{24} \\ Z_{32} & Z_{33} & Z_{34} \\ Z_{42} & Z_{43} & Z_{44} \end{vmatrix} + B_2\begin{vmatrix} Z_{21} & Z_{23} & Z_{24} \\ Z_{31} & Z_{33} & Z_{34} \\ Z_{41} & Z_{43} & Z_{44} \end{vmatrix} - B_3\begin{vmatrix} Z_{21} & Z_{22} & Z_{24} \\ Z_{31} & Z_{32} & Z_{34} \\ Z_{41} & Z_{42} & Z_{44} \end{vmatrix} \right.$$

$$\left. + B_4\begin{vmatrix} Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \\ Z_{41} & Z_{42} & Z_{43} \end{vmatrix} \right\}$$

$$=\frac{\chi_1}{|J|}\left[Z_{11}\left\{Z_{22}\left(Z_{33}Z_{44}-Z_{43}Z_{34}\right)+Z_{23}\left(Z_{42}Z_{34}-Z_{32}Z_{44}\right)+Z_{24}\left(Z_{32}Z_{43}-Z_{42}Z_{33}\right)\right\}\right.$$

$$-Z_{12}\left\{Z_{21}\left(Z_{33}Z_{44}-Z_{43}Z_{34}\right)+Z_{23}\left(Z_{41}Z_{34}-Z_{31}Z_{44}\right)+Z_{24}\left(Z_{31}Z_{43}-Z_{41}Z_{33}\right)\right\}$$

$$+Z_{13}\left\{Z_{21}\left(Z_{32}Z_{44}-Z_{42}Z_{34}\right)+Z_{22}\left(Z_{41}Z_{34}-Z_{31}Z_{44}\right)+Z_{24}\left(Z_{31}Z_{42}-Z_{41}Z_{32}\right)\right\}$$

$$\left.-Z_{14}\left\{Z_{21}\left(Z_{32}Z_{43}-Z_{42}Z_{33}\right)+Z_{22}\left(Z_{41}Z_{33}-Z_{31}Z_{43}\right)+Z_{23}\left(Z_{31}Z_{42}-Z_{41}Z_{32}\right)\right\}\right]$$

$$-\frac{\eta}{|J|}\left[-B_1\left\{Z_{22}\left(Z_{33}Z_{44}-Z_{43}Z_{34}\right)+Z_{23}\left(Z_{42}Z_{34}-Z_{32}Z_{44}\right)+Z_{24}\left(Z_{32}Z_{43}-Z_{42}Z_{33}\right)\right\}\right.$$

$$+B_2\left\{Z_{21}\left(Z_{33}Z_{44}-Z_{43}Z_{34}\right)+Z_{23}\left(Z_{41}Z_{34}-Z_{31}Z_{44}\right)+Z_{24}\left(Z_{31}Z_{43}-Z_{41}Z_{33}\right)\right\}$$

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$$-B_3\{Z_{21}(Z_{32}Z_{44}-Z_{42}Z_{34})+Z_{22}(Z_{41}Z_{34}-Z_{31}Z_{44})+Z_{24}(Z_{31}Z_{42}-Z_{41}Z_{32})\}$$
$$+B_4\{Z_{21}(Z_{32}Z_{43}-Z_{42}Z_{33})+Z_{22}(Z_{41}Z_{33}-Z_{31}Z_{43})+Z_{23}(Z_{31}Z_{42}-Z_{41}Z_{32})\}]$$

$$=\frac{\chi_1}{|J|}\{Z_{11}Z_{22}Z_{33}Z_{44} \quad -Z_{11}Z_{22}Z_{43}Z_{34} \quad +Z_{11}Z_{23}Z_{42}Z_{34} \quad -Z_{11}Z_{23}Z_{32}Z_{44}$$
$$+Z_{11}Z_{24}Z_{32}Z_{43} \; -Z_{11}Z_{24}Z_{42}Z_{33} \; -Z_{12}Z_{21}Z_{33}Z_{44} \; +Z_{12}Z_{21}Z_{43}Z_{34} \; -Z_{12}Z_{23}Z_{41}Z_{34}$$
$$+Z_{12}Z_{23}Z_{31}Z_{44} \; -Z_{12}Z_{24}Z_{31}Z_{43} \; +Z_{12}Z_{24}Z_{41}Z_{33} \; +Z_{13}Z_{21}Z_{32}Z_{44} \; -Z_{13}Z_{21}Z_{42}Z_{34}$$
$$+Z_{13}Z_{22}Z_{41}Z_{34} \; -Z_{13}Z_{22}Z_{31}Z_{44} \; +Z_{13}Z_{24}Z_{31}Z_{42} \; -Z_{13}Z_{24}Z_{41}Z_{32} \; -Z_{14}Z_{21}Z_{32}Z_{43}$$
$$+Z_{14}Z_{21}Z_{42}Z_{33} -Z_{14}Z_{22}Z_{41}Z_{33} +Z_{14}Z_{22}Z_{31}Z_{43} -Z_{14}Z_{23}Z_{31}Z_{42} +Z_{14}Z_{23}Z_{41}Z_{32}\}$$

$$-\frac{\eta}{|J|}\{-B_1Z_{22}Z_{33}Z_{44} \; +B_1Z_{22}Z_{43}Z_{34} -B_1Z_{23}Z_{42}Z_{34} +B_1Z_{23}Z_{32}Z_{44} -B_1Z_{24}Z_{32}Z_{43}$$
$$+B_1Z_{24}Z_{42}Z_{33} \; +B_2Z_{21}Z_{33}Z_{44} \; -B_2Z_{21}Z_{43}Z_{34} \; +B_2Z_{23}Z_{41}Z_{34} \; -B_2Z_{23}Z_{31}Z_{44}$$
$$+B_2Z_{24}Z_{31}Z_{43} \; -B_2Z_{24}Z_{41}Z_{33} \; -B_3Z_{21}Z_{32}Z_{44} \; +B_3Z_{21}Z_{42}Z_{34} \; -B_3Z_{22}Z_{41}Z_{34}$$
$$+B_3Z_{22}Z_{31}Z_{44} \; -B_3Z_{24}Z_{31}Z_{42} \; +B_3Z_{24}Z_{41}Z_{32} \; +B_4Z_{21}Z_{32}Z_{43} \; -B_4Z_{21}Z_{42}Z_{33}$$
$$+B_4Z_{22}Z_{41}Z_{33} -B_4Z_{22}Z_{31}Z_{43} +B_4Z_{23}Z_{31}Z_{42} -B_4Z_{23}Z_{41}Z_{32}\}$$

$$=\frac{\chi_1}{|J|}\frac{A^4\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4^2}\{i(i-1)j(j-1)k(k-1)l(l-1) \quad -i(i-1)j(j-1)k^2l^2$$
$$+i(i-1)j^2k^2l^2 \quad -i(i-1)j^2k^2l(l-1) \quad +i(i-1)j^2k^2l^2 \quad -i(i-1)j^2k(k-1)l^2$$
$$-i^2j^2k(k-1)l(l-1) \quad +i^2j^2k^2l^2 \quad -i^2j^2k^2l^2 \quad +i^2j^2k^2l(l-1) \quad -i^2j^2k^2l^2$$
$$+i^2j^2k(k-1)l^2 \; +i^2j^2k^2l(l-1) \; -i^2j^2k^2l^2 \; +i^2j(j-1)k^2l^2 \; -i^2j(j-1)k^2l(l-1)$$
$$+i^2j^2k^2l^2 \quad -i^2j^2k^2l^2 \quad -i^2j^2k^2l^2 \quad +i^2j^2k(k-1)l^2 \quad -i^2j(j-1)k(k-1)l^2$$
$$+i^2j(j-1)k^2l^2 \quad -i^2j^2k^2l^2 \quad +i^2j^2k^2l^2\} \; -\frac{1}{|J|}\frac{A^3\chi_1^{3i}\chi_2^{3j}\chi_3^{3k}\chi_4^{3l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4^2}\frac{A\chi_1^i\chi_2^j\chi_3^k\chi_4^l\Phi}{B}$$
$$\{-b_1\chi_1^2j^2k^2l^2 \quad -b_1\chi_1^2j(j-1)k(k-1)l(l-1) \quad +b_1\chi_1^2j(j-1)k^2l^2 \quad -b_1\chi_1^2j^2k^2l^2$$
$$+b_1\chi_1^2j^2k^2l(l-1) \qquad +b_1\chi_1^2j^2k^2(k-1)l^2 \qquad +b_2\chi_1\chi_2ijk(k-1)l(l-1)$$
$$-b_2\chi_1\chi_2ijk^2l(l-1) \qquad -b_2\chi_1\chi_2ijk^2l^2 \qquad +b_2\chi_1\chi_2ijk^2l^2 \qquad +b_2\chi_1\chi_2ijk^2l^2$$

$$-b_2\chi_1\chi_2 ijk(k-1)l^2 \quad -b_3\chi_1\chi_3 ij^2kl(l-1) \quad +b_3\chi_1\chi_3 ij^2kl^2 \quad -b_3\chi_1\chi_3 ij(j-1)kl^2$$
$$+b_3\chi_1\chi_3 ij(j-1)kl(l-1) \quad -b_3\chi_1\chi_3 ij^2kl^2 \quad +b_3\chi_1\chi_3 ij^2kl^2 \quad +b_4\chi_1\chi_4 ij^2k^2l$$
$$-b_4\chi_1\chi_4 ij^2k(k-1)l \quad +b_4\chi_1\chi_4 ij(j-1)k(k-1)l \quad -b_4\chi_1\chi_4 ij(j-1)k^2l$$
$$+b_4\chi_1\chi_4 ij^2k^2l - b_4\chi_1\chi_4 ij^2k^2l\}$$
$$=\frac{\chi_1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4^2}\{(i-1)(j-1)(k-1)(l-1) -(i-1)(j-1)kl +2(i-1)jkl$$
$$-(i-1)jk(l-1) \quad -(i-1)j(k-1)l \quad -ij(k-1)(l-1) \quad -3ijkl \quad +2ij(k-1)l$$
$$+2ijk(l-1) \quad -i(j-1)k(l-1) \quad -i(j-1)(k-1)l \quad +2i(j-1)kl\}$$
$$-\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1\chi_2^2\chi_3^2\chi_4^2} \quad \{-(j-1)(k-1)(l-1) \quad +(j-1)(k-1)l \quad -(j-1)kl$$
$$+(j-1)k(l-1) + j(k-1)(l-1) - j(k-1)l + jkl - jk(l-1)\}$$
$$=\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1\chi_2^2\chi_3^2\chi_4^2}(1-\Phi)-\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1\chi_2^2\chi_3^2\chi_4^2}$$
$$\frac{\partial\eta}{\partial b_1}=-\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}\Phi}{\chi_1\chi_2^2\chi_3^2\chi_4^2}<0\,. \tag{13}$$

From the relation (13) we have realized that when rate of interest or services of capital χ_1 increases, the value of the Lagrange multiplier is decreased. In this situation the firm should take attempts to decrease the production level if it has no sufficient demand of its products.

Now we study the economic effects on Lagrange multiplier η when wage rate of the workers of the firm increases. Taking T_{12} (i.e., term of 1st row and 2nd column) from both sides of (12) we get [Islam et al., 2010a,b; Moolio et al., 2009],

$$\frac{\partial\eta}{\partial b_2}=\frac{\chi_2}{|J|}[C_{11}]+\frac{\eta}{|J|}[C_{31}]$$

$$=\frac{\chi_2}{|J|}\text{Cofactor of } C_{11}+\frac{\eta}{|J|}\text{Cofactor of } \text{C}_{31}$$

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$$
=\frac{\chi_2}{|J|}\begin{vmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{vmatrix}+\frac{\eta}{|J|}\begin{vmatrix} -B_1 & -B_2 & -B_3 & -B_4 \\ Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{vmatrix}
$$

$$
=\frac{\chi_2}{|J|}\left\{Z_{11}\begin{vmatrix} Z_{22} & Z_{23} & Z_{24} \\ Z_{32} & Z_{33} & Z_{34} \\ Z_{42} & Z_{43} & Z_{44} \end{vmatrix} - Z_{12}\begin{vmatrix} Z_{21} & Z_{23} & Z_{24} \\ Z_{31} & Z_{33} & Z_{34} \\ Z_{41} & Z_{43} & Z_{44} \end{vmatrix} + Z_{13}\begin{vmatrix} Z_{21} & Z_{22} & Z_{24} \\ Z_{31} & Z_{32} & Z_{34} \\ Z_{41} & Z_{42} & Z_{44} \end{vmatrix}\right.
$$

$$
\left.- Z_{14}\begin{vmatrix} Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \\ Z_{41} & Z_{42} & Z_{43} \end{vmatrix}\right\}
$$

$$
+\frac{\eta}{|J|}\left\{-B_1\begin{vmatrix} Z_{12} & Z_{13} & Z_{14} \\ Z_{32} & Z_{33} & Z_{34} \\ Z_{42} & Z_{43} & Z_{44} \end{vmatrix} + B_2\begin{vmatrix} Z_{11} & Z_{13} & Z_{14} \\ Z_{31} & Z_{33} & Z_{34} \\ Z_{41} & Z_{43} & Z_{44} \end{vmatrix} - B_3\begin{vmatrix} Z_{11} & Z_{12} & Z_{14} \\ Z_{31} & Z_{32} & Z_{34} \\ Z_{41} & Z_{42} & Z_{44} \end{vmatrix}\right.
$$

$$
\left.+ B_4\begin{vmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{31} & Z_{32} & Z_{33} \\ Z_{41} & Z_{42} & Z_{43} \end{vmatrix}\right\}
$$

$$
=\frac{\chi_2}{|J|}\left[Z_{11}\left\{Z_{22}\left(Z_{33}Z_{44}-Z_{43}Z_{34}\right)+Z_{23}\left(Z_{42}Z_{34}-Z_{32}Z_{44}\right)+Z_{24}\left(Z_{32}Z_{43}-Z_{42}Z_{33}\right)\right\}\right.
$$

$$
-Z_{12}\left\{Z_{21}\left(Z_{33}Z_{44}-Z_{43}Z_{34}\right)+Z_{23}\left(Z_{41}Z_{34}-Z_{31}Z_{44}\right)+Z_{24}\left(Z_{31}Z_{43}-Z_{41}Z_{33}\right)\right\}
$$

$$
+Z_{13}\left\{Z_{21}\left(Z_{32}Z_{44}-Z_{42}Z_{34}\right)+Z_{22}\left(Z_{41}Z_{34}-Z_{31}Z_{44}\right)+Z_{24}\left(Z_{31}Z_{42}-Z_{41}Z_{32}\right)\right\}
$$

$$
\left.-Z_{14}\left\{Z_{21}\left(Z_{32}Z_{43}-Z_{42}Z_{33}\right)+Z_{22}\left(Z_{41}Z_{33}-Z_{31}Z_{43}\right)+Z_{23}\left(Z_{31}Z_{42}-Z_{41}Z_{32}\right)\right\}\right]
$$

$$
+\frac{\eta}{|J|}\left[-B_1\left\{Z_{12}\left(Z_{33}Z_{44}-Z_{43}Z_{34}\right)+Z_{13}\left(Z_{42}Z_{34}-Z_{32}Z_{44}\right)+Z_{14}\left(Z_{32}Z_{43}-Z_{42}Z_{33}\right)\right\}\right.
$$

$$
+B_2\left\{Z_{11}\left(Z_{33}Z_{44}-Z_{43}Z_{34}\right)+Z_{13}\left(Z_{41}Z_{34}-Z_{31}Z_{44}\right)+Z_{14}\left(Z_{31}Z_{43}-Z_{41}Z_{33}\right)\right\}
$$

$$
\begin{aligned}
&- B_3\{Z_{11}(Z_{32}Z_{44}-Z_{42}Z_{34})+Z_{12}(Z_{41}Z_{34}-Z_{31}Z_{44})+Z_{14}(Z_{31}Z_{42}-Z_{41}Z_{32})\}\\
&+ B_4\{Z_{11}(Z_{32}Z_{43}-Z_{42}Z_{33})+Z_{12}(Z_{41}Z_{33}-Z_{31}Z_{43})+Z_{13}(Z_{31}Z_{42}-Z_{41}Z_{32})\}]\\
&=\frac{\chi_2}{|J|}\{Z_{11}Z_{22}Z_{33}Z_{44} - Z_{11}Z_{22}Z_{43}Z_{34} + Z_{11}Z_{23}Z_{42}Z_{34} - Z_{11}Z_{23}Z_{32}Z_{44}\\
&+Z_{11}Z_{24}Z_{32}Z_{43} - Z_{11}Z_{24}Z_{42}Z_{33} - Z_{12}Z_{21}Z_{33}Z_{44} + Z_{12}Z_{21}Z_{43}Z_{34} - Z_{12}Z_{23}Z_{41}Z_{34}\\
&+Z_{12}Z_{23}Z_{31}Z_{44} - Z_{12}Z_{24}Z_{31}Z_{43} + Z_{12}Z_{24}Z_{41}Z_{33} + Z_{13}Z_{21}Z_{32}Z_{44} - Z_{13}Z_{21}Z_{42}Z_{34}\\
&+Z_{13}Z_{22}Z_{41}Z_{34} - Z_{13}Z_{22}Z_{31}Z_{44} + Z_{13}Z_{24}Z_{31}Z_{42} - Z_{13}Z_{24}Z_{41}Z_{32} - Z_{14}Z_{21}Z_{32}Z_{43}\\
&+Z_{14}Z_{21}Z_{42}Z_{33} - Z_{14}Z_{22}Z_{41}Z_{33} + Z_{14}Z_{22}Z_{31}Z_{43} - Z_{14}Z_{23}Z_{31}Z_{42} + Z_{14}Z_{23}Z_{41}Z_{32}\}\\
&+\frac{\eta}{|J|}\{-B_1Z_{12}Z_{33}Z_{44} + B_1Z_{12}Z_{43}Z_{34} - B_1Z_{13}Z_{42}Z_{34} + B_1Z_{13}Z_{32}Z_{44} - B_1Z_{14}Z_{32}Z_{43}\\
&+B_1Z_{14}Z_{42}Z_{33} + B_2Z_{11}Z_{33}Z_{44} - B_2Z_{11}Z_{43}Z_{34} + B_2Z_{13}Z_{41}Z_{34} - B_2Z_{13}Z_{31}Z_{44}\\
&+B_2Z_{14}Z_{31}Z_{43} - B_2Z_{14}Z_{41}Z_{33} - B_3Z_{11}Z_{32}Z_{44} + B_3Z_{11}Z_{42}Z_{34} - B_3Z_{12}Z_{41}Z_{34}\\
&+B_3Z_{12}Z_{31}Z_{44} - B_3Z_{14}Z_{31}Z_{42} + B_3Z_{14}Z_{41}Z_{32} + B_4Z_{11}Z_{32}Z_{43} - B_4Z_{11}Z_{42}Z_{33}\\
&+B_4Z_{12}Z_{41}Z_{33} - B_4Z_{12}Z_{31}Z_{43} + B_4Z_{13}Z_{31}Z_{42} - B_4Z_{13}Z_{41}Z_{32}\}\\
&=\frac{\chi_2}{|J|}\frac{A^4\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4^2}\{i(i-1)j(j-1)k(k-1)l(l-1) - i(i-1)j(j-1)k^2l^2\\
&+i(i-1)j^2k^2l^2 - i(i-1)j^2k^2l(l-1) + i(i-1)j^2k^2l^2 - i(i-1)j^2k(k-1)l^2\\
&-i^2j^2k(k-1)l(l-1) + i^2j^2k^2l^2 - i^2j^2k^2l^2 + i^2j^2k^2l(l-1) - i^2j^2k^2l^2\\
&+i^2j^2k(k-1)l^2 + i^2j^2k^2l(l-1) - i^2j^2k^2l^2 + i^2j(j-1)k^2l^2 - i^2j(j-1)k^2l(l-1)\\
&+i^2j^2k^2l^2 - i^2j^2k^2l^2 - i^2j^2k^2l^2 + i^2j^2k(k-1)l^2 - i^2j(j-1)k(k-1)l^2\\
&+i^2j(j-1)k^2l^2 - i^2j^2k^2l^2 + i^2j^2k^2l^2\}\\
&+\frac{1}{|J|}\frac{A^3\chi_1^{3i}\chi_2^{3j}\chi_3^{3k}\chi_4^{3l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4^2}\frac{A\chi_1^i\chi_2^j\chi_3^k\chi_4^l\Phi}{B}\{-b_1\chi_1\chi_2ijk(k-1)l(l-1) + b_1\chi_1\chi_2ijk^2l^2\\
&-b_1\chi_1\chi_2ijk^2l^2 + b_1\chi_1\chi_2ijk^2l(l-1) - b_1\chi_1\chi_2ijk^2l^2 + b_1\chi_1\chi_2ijk(k-1)l^2\\
&+b_2\chi_2^2i(i-1)k(k-1)l(l-1) - b_2\chi_2^2i(i-1)k^2l^2 + b_2\chi_2^2i^2k^2l^2 - b_2\chi_2^2i^2k^2l(l-1)
\end{aligned}
$$

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$$+b_2\chi_2^2 i^2k^2l^2 \; -b_2\chi_2^2 i^2k(k-1)l^2 \; -b_3\chi_2\chi_3 i(i-1)jkl(l-1) \; +b_3\chi_2\chi_3 i(i-1)jkl^2$$
$$-b_3\chi_2\chi_3 i^2 jkl^2 \quad +b_3\chi_2\chi_3 i^2 jkl(l-1) \quad -b_3\chi_2\chi_3 i^2 jkl^2 \quad +b_3\chi_2\chi_3 i^2 jkl^2$$
$$+b_4\chi_2\chi_4 i(i-1)jk^2l \quad -b_4\chi_2\chi_4 i(i-1)jk(k-1)l \quad +b_4\chi_2\chi_4 i^2 jk(k-1)l$$
$$-b_4\chi_2\chi_4 i^2 jk^2l \; +b_4\chi_2\chi_4 i^2 jk^2l \; -b_4\chi_2\chi_4 i^2 jk^2l\}$$
$$=\frac{1}{|J|}\frac{A^4ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2\chi_3^2\chi_4^2}\{(i-1)(j-1)(k-1)(l-1) \; -(i-1)(j-1)kl \; +2(i-1)jkl$$
$$-(i-1)jk(l-1) \quad -(i-1)j(k-1)l \quad -ij(k-1)(l-1) \quad -3ijkl \quad +2ij(k-1)l$$
$$+2ijk(l-1) \quad -i(j-1)k(l-1) \quad -i(j-1)(k-1)l \quad +2i(j-1)kl\}$$
$$+\frac{1}{|J|}\frac{A^4ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2\chi_3^2\chi_4^2} \quad \{-i(k-1)(l-1) \quad +ik(l-1) \quad +i(k-1)l-ikl$$
$$+(i-1)(k-1)(l-1)-(i-1)k(l-1)+(i-1)kl \; -(i-1)(k-1)l\}$$
$$=\frac{1}{|J|}\frac{A^4ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2\chi_3^2\chi_4^2}(1-\Phi)-\frac{1}{|J|}\frac{A^4ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2\chi_3^2\chi_4^2}$$
$$\frac{\partial\eta}{\partial b_2}=-\frac{1}{|J|}\frac{A^4ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}\Phi}{\chi_1^2\chi_2\chi_3^2\chi_4^2}<0. \qquad (14)$$

From (14) we have realized that when the wage rate of the workers increases, the level of marginal product decreases. This seems that due to income effect workers use fewer working hours; consequently, production rate of the firm decreases.

Now we analyze the economic effect on Lagrange multiplier η when per unit cost of principal raw material increases. Taking T_{13} (i.e., term of 1st row and 3rd column) from both sides of (12) we get [Islam et al., 2011; Mohajan, 2017a],

$$\frac{\partial\eta}{\partial b_3}=\frac{\chi_3}{|J|}[C_{11}]+\frac{\eta}{|J|}[C_{41}]$$
$$=\frac{\chi_3}{|J|}\text{Cofactor of } C_{11}+\frac{\eta}{|J|}\text{Cofactor of C}_{41}$$

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$$=\frac{\chi_3}{|J|}\begin{vmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{vmatrix} - \frac{\eta}{|J|}\begin{vmatrix} -B_1 & -B_2 & -B_3 & -B_4 \\ Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{vmatrix}$$

$$=\frac{\chi_3}{|J|}\left\{ Z_{11}\begin{vmatrix} Z_{22} & Z_{23} & Z_{24} \\ Z_{32} & Z_{33} & Z_{34} \\ Z_{42} & Z_{43} & Z_{44} \end{vmatrix} - Z_{12}\begin{vmatrix} Z_{21} & Z_{23} & Z_{24} \\ Z_{31} & Z_{33} & Z_{34} \\ Z_{41} & Z_{43} & Z_{44} \end{vmatrix} + Z_{13}\begin{vmatrix} Z_{21} & Z_{22} & Z_{24} \\ Z_{31} & Z_{32} & Z_{34} \\ Z_{41} & Z_{42} & Z_{44} \end{vmatrix} \right.$$

$$\left. - Z_{14}\begin{vmatrix} Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \\ Z_{41} & Z_{42} & Z_{43} \end{vmatrix} \right\}$$

$$-\frac{\eta}{|J|}\left\{ -B_1\begin{vmatrix} Z_{12} & Z_{13} & Z_{14} \\ Z_{22} & Z_{23} & Z_{24} \\ Z_{42} & Z_{43} & Z_{44} \end{vmatrix} + B_2\begin{vmatrix} Z_{11} & Z_{13} & Z_{14} \\ Z_{21} & Z_{23} & Z_{24} \\ Z_{41} & Z_{43} & Z_{44} \end{vmatrix} - B_3\begin{vmatrix} Z_{11} & Z_{12} & Z_{14} \\ Z_{21} & Z_{22} & Z_{24} \\ Z_{41} & Z_{42} & Z_{44} \end{vmatrix} \right.$$

$$\left. + B_4\begin{vmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{41} & Z_{42} & Z_{43} \end{vmatrix} \right\}$$

$$=\frac{\chi_3}{|J|}\left[Z_{11}\left\{Z_{22}\left(Z_{33}Z_{44}-Z_{43}Z_{34}\right)+Z_{23}\left(Z_{42}Z_{34}-Z_{32}Z_{44}\right)+Z_{24}\left(Z_{32}Z_{43}-Z_{42}Z_{33}\right)\right\}\right.$$
$$-Z_{12}\left\{Z_{21}\left(Z_{33}Z_{44}-Z_{43}Z_{34}\right)+Z_{23}\left(Z_{41}Z_{34}-Z_{31}Z_{44}\right)+Z_{24}\left(Z_{31}Z_{43}-Z_{41}Z_{33}\right)\right\}$$
$$+Z_{13}\left\{Z_{21}\left(Z_{32}Z_{44}-Z_{42}Z_{34}\right)+Z_{22}\left(Z_{41}Z_{34}-Z_{31}Z_{44}\right)+Z_{24}\left(Z_{31}Z_{42}-Z_{41}Z_{32}\right)\right\}$$
$$\left.-Z_{14}\left\{Z_{21}\left(Z_{32}Z_{43}-Z_{42}Z_{33}\right)+Z_{22}\left(Z_{41}Z_{33}-Z_{31}Z_{43}\right)+Z_{23}\left(Z_{31}Z_{42}-Z_{41}Z_{32}\right)\right\}\right]$$

$$-\frac{\eta}{|J|}\left[-B_1\left\{Z_{12}\left(Z_{23}Z_{44}-Z_{43}Z_{24}\right)+Z_{13}\left(Z_{42}Z_{24}-Z_{22}Z_{44}\right)+Z_{14}\left(Z_{22}Z_{43}-Z_{42}Z_{23}\right)\right\}\right.$$
$$+B_2\left\{Z_{11}\left(Z_{23}Z_{44}-Z_{43}Z_{24}\right)+Z_{13}\left(Z_{41}Z_{24}-Z_{21}Z_{44}\right)+Z_{14}\left(Z_{21}Z_{43}-Z_{41}Z_{23}\right)\right\}$$

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$$- B_3\{Z_{11}(Z_{22}Z_{44}-Z_{42}Z_{24})+Z_{12}(Z_{41}Z_{24}-Z_{21}Z_{44})+Z_{14}(Z_{21}Z_{42}-Z_{41}Z_{22})\}$$
$$+ B_4\{Z_{11}(Z_{22}Z_{43}-Z_{42}Z_{23})+Z_{12}(Z_{41}Z_{23}-Z_{21}Z_{43})+Z_{13}(Z_{21}Z_{42}-Z_{41}Z_{22})\}]$$

$$= \frac{\chi_3}{|J|}\{Z_{11}Z_{22}Z_{33}Z_{44} - Z_{11}Z_{22}Z_{43}Z_{34} + Z_{11}Z_{23}Z_{42}Z_{34} - Z_{11}Z_{23}Z_{32}Z_{44}$$
$$+ Z_{11}Z_{24}Z_{32}Z_{43} - Z_{11}Z_{24}Z_{42}Z_{33} - Z_{12}Z_{21}Z_{33}Z_{44} + Z_{12}Z_{21}Z_{43}Z_{34} - Z_{12}Z_{23}Z_{41}Z_{34}$$
$$+ Z_{12}Z_{23}Z_{31}Z_{44} - Z_{12}Z_{24}Z_{31}Z_{43} + Z_{12}Z_{24}Z_{41}Z_{33} + Z_{13}Z_{21}Z_{32}Z_{44} - Z_{13}Z_{21}Z_{42}Z_{34}$$
$$+ Z_{13}Z_{22}Z_{41}Z_{34} - Z_{13}Z_{22}Z_{31}Z_{44} + Z_{13}Z_{24}Z_{31}Z_{42} - Z_{13}Z_{24}Z_{41}Z_{32} - Z_{14}Z_{21}Z_{32}Z_{43}$$
$$+ Z_{14}Z_{21}Z_{42}Z_{33} - Z_{14}Z_{22}Z_{41}Z_{33} + Z_{14}Z_{22}Z_{31}Z_{43} - Z_{14}Z_{23}Z_{31}Z_{42} + Z_{14}Z_{23}Z_{41}Z_{32}\}$$

$$- \frac{\eta}{|J|}\{- B_1Z_{12}Z_{23}Z_{44} + B_1Z_{12}Z_{43}Z_{24} - B_1Z_{13}Z_{42}Z_{24} + B_1Z_{13}Z_{22}Z_{44} - B_1Z_{14}Z_{22}Z_{43}$$
$$+ B_1Z_{14}Z_{42}Z_{23} + B_2Z_{11}Z_{23}Z_{44} - B_2Z_{11}Z_{43}Z_{24} + B_2Z_{13}Z_{41}Z_{24} - B_2Z_{13}Z_{21}Z_{44}$$
$$+ B_2Z_{14}Z_{21}Z_{43} - B_2Z_{14}Z_{41}Z_{23} - B_3Z_{11}Z_{22}Z_{44} + B_3Z_{11}Z_{42}Z_{24} - B_3Z_{12}Z_{41}Z_{24}$$
$$+ B_3Z_{12}Z_{21}Z_{44} - B_3Z_{14}Z_{21}Z_{42} + B_3Z_{14}Z_{41}Z_{22} + B_4Z_{11}Z_{22}Z_{43} - B_4Z_{11}Z_{42}Z_{23}$$
$$+ B_4Z_{12}Z_{41}Z_{23} - B_4Z_{12}Z_{21}Z_{43} + B_4Z_{13}Z_{21}Z_{42} - B_4Z_{13}Z_{41}Z_{22}\}$$

$$= \frac{\chi_3}{|J|}\frac{A^4\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4^2}\{i(i-1)j(j-1)k(k-1)l(l-1) - i(i-1)j(j-1)k^2l^2$$
$$+ i(i-1)j^2k^2l^2 - i(i-1)j^2k^2l(l-1) + i(i-1)j^2k^2l^2 - i(i-1)j^2k(k-1)l^2$$
$$- i^2j^2k(k-1)l(l-1) + i^2j^2k^2l^2 - i^2j^2k^2l^2 + i^2j^2k^2l(l-1) - i^2j^2k^2l^2$$
$$+ i^2j^2k(k-1)l^2 + i^2j^2k^2l(l-1) - i^2j^2k^2l^2 + i^2j(j-1)k^2l^2 - i^2j(j-1)k^2l(l-1)$$
$$+ i^2j^2k^2l^2 - i^2j^2k^2l^2 - i^2j^2k^2l^2 + i^2j^2k(k-1)l^2 - i^2j(j-1)k(k-1)l^2$$
$$+ i^2j(j-1)k^2l^2 - i^2j^2k^2l^2 + i^2j^2k^2l^2\} - \frac{1}{|J|}\frac{A^3\chi_1^{3i}\chi_2^{3j}\chi_3^{3k}\chi_4^{3l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4^2}\frac{A\chi_1^i\chi_2^j\chi_3^k\chi_4^l\Phi}{B}$$
$$\{- b_1\chi_1\chi_3ij^2kl(l-1) + b_1\chi_1\chi_3ij^2kl^2 - b_1\chi_1\chi_3ij^2kl^2 + b_1\chi_1\chi_3ij(j-1)kl(l-1)$$
$$- b_1\chi_1\chi_3ij(j-1)kl^2 + b_1\chi_1\chi_3ij^2kl^2 + b_2\chi_2\chi_3i(i-1)jkl(l-1) - b_2\chi_2\chi_3i(i-1)jkl^2$$
$$+ b_2\chi_2\chi_3i^2jkl^2 - b_2\chi_2\chi_3i^2jkl(l-1) + b_2\chi_2\chi_3i^2jkl^2 - b_2\chi_2\chi_3i^2jkl^2$$

$$-b_3\chi_3^2 i(i-1)j(j-1)l(l-1) \quad +b_3\chi_3^2 i(i-1)j^2l^2 \quad -b_3\chi_3^2 i^2 j^2 l^2 \quad +b_3\chi_3^2 i^2 j^2 l(l-1)$$
$$-b_3\chi_3^2 i^2 j^2 l^2 \quad +b_3\chi_3^2 i^2 j(j-1)l^2 \quad +b_4\chi_3\chi_4 i(i-1)j(j-1)kl \quad -b_4\chi_3\chi_4 i(i-1)j^2kl$$
$$+b_4\chi_3\chi_4 i^2 j^2 kl - b_4\chi_3\chi_4 i^2 j^2 kl + b_4\chi_3\chi_4 i^2 j^2 kl - b_4\chi_3\chi_4 i^2 j(j-1)kl\}$$
$$=\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3\chi_4^2}\{(i-1)(j-1)(k-1)(l-1)-(i-1)(j-1)kl+2(i-1)jkl$$
$$-(i-1)jk(l-1) \quad -(i-1)j(k-1)l \quad -ij(k-1)(l-1) \quad -3ijkl \quad +2ij(k-1)l$$
$$+2ijk(l-1) \quad -i(j-1)k(l-1) \quad -i(j-1)(k-1)l \quad +2i(j-1)kl\}$$
$$-\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3\chi_4^2}\{i(j-1)(l-1) \quad -i(j-1)l \quad +(i-1)j(l-1) \quad -(i-1)jl$$
$$+ijl \ -ij(l-1)-(i-1)(j-1)(l-1)+(i-1)(j-1)l\}$$
$$=\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1\chi_2^2\chi_3^2\chi_4^2}(1-\Phi)-\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3\chi_4^2}$$
$$\frac{\partial\eta}{\partial b_3}=-\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}\Phi}{\chi_1\chi_2^2\chi_3^2\chi_4^2}<0\,. \quad (15)$$

From the relation (15) we have seen that when per unit cost of principal raw material increases, the level of marginal product decreases, which is reasonable. This seems that due to increase of the cost of principal raw material, the firm will purchase less of it; consequently, production of the firm will decrease. Moreover, it seems that the principal raw material may have no substitutes.

Now we analyze the economic effect on Lagrange multiplier η when per unit cost of irregular inputs increases. Taking T_{14} (i.e., term of 1st row and 4th column) from both sides of (12) we get [Mohajan et al., 2013; Mohajan, 2021a, 2022],

$$\frac{\partial\eta}{\partial b_4}=\frac{\chi_4}{|J|}[C_{11}]+\frac{\eta}{|J|}[C_{51}]$$
$$=\frac{\chi_4}{|J|}\text{Cofactor of } C_{11}+\frac{\eta}{|J|}\text{Cofactor of } C_{51}$$

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$$=\frac{\chi_4}{|J|}\begin{vmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{vmatrix} + \frac{\eta}{|J|}\begin{vmatrix} -B_1 & -B_2 & -B_3 & -B_4 \\ Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \end{vmatrix}$$

$$=\frac{\chi_4}{|J|}\left\{ Z_{11}\begin{vmatrix} Z_{22} & Z_{23} & Z_{24} \\ Z_{32} & Z_{33} & Z_{34} \\ Z_{42} & Z_{43} & Z_{44} \end{vmatrix} - Z_{12}\begin{vmatrix} Z_{21} & Z_{23} & Z_{24} \\ Z_{31} & Z_{33} & Z_{34} \\ Z_{41} & Z_{43} & Z_{44} \end{vmatrix} + Z_{13}\begin{vmatrix} Z_{21} & Z_{22} & Z_{24} \\ Z_{31} & Z_{32} & Z_{34} \\ Z_{41} & Z_{42} & Z_{44} \end{vmatrix} - Z_{14}\begin{vmatrix} Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \\ Z_{41} & Z_{42} & Z_{43} \end{vmatrix} \right\}$$

$$+\frac{\eta}{|J|}\left\{ -B_1\begin{vmatrix} Z_{12} & Z_{13} & Z_{14} \\ Z_{22} & Z_{23} & Z_{24} \\ Z_{32} & Z_{33} & Z_{34} \end{vmatrix} + B_2\begin{vmatrix} Z_{11} & Z_{13} & Z_{14} \\ Z_{21} & Z_{23} & Z_{24} \\ Z_{31} & Z_{33} & Z_{34} \end{vmatrix} - B_3\begin{vmatrix} Z_{11} & Z_{12} & Z_{14} \\ Z_{21} & Z_{22} & Z_{24} \\ Z_{31} & Z_{32} & Z_{34} \end{vmatrix} + B_4\begin{vmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{vmatrix} \right\}$$

$$=\frac{\chi_4}{|J|}\left[Z_{11}\{Z_{22}(Z_{33}Z_{44}-Z_{43}Z_{34})+Z_{23}(Z_{42}Z_{34}-Z_{32}Z_{44})+Z_{24}(Z_{32}Z_{43}-Z_{42}Z_{33})\}\right.$$
$$-Z_{12}\{Z_{21}(Z_{33}Z_{44}-Z_{43}Z_{34})+Z_{23}(Z_{41}Z_{34}-Z_{31}Z_{44})+Z_{24}(Z_{31}Z_{43}-Z_{41}Z_{33})\}$$
$$+Z_{13}\{Z_{21}(Z_{32}Z_{44}-Z_{42}Z_{34})+Z_{22}(Z_{41}Z_{34}-Z_{31}Z_{44})+Z_{24}(Z_{31}Z_{42}-Z_{41}Z_{32})\}$$
$$\left.-Z_{14}\{Z_{21}(Z_{32}Z_{43}-Z_{42}Z_{33})+Z_{22}(Z_{41}Z_{33}-Z_{31}Z_{43})+Z_{23}(Z_{31}Z_{42}-Z_{41}Z_{32})\}\right]$$
$$+\frac{\eta}{|J|}\left[-B_1\{Z_{12}(Z_{23}Z_{34}-Z_{33}Z_{24})+Z_{13}(Z_{32}Z_{24}-Z_{22}Z_{34})+Z_{14}(Z_{22}Z_{33}-Z_{32}Z_{23})\}\right.$$
$$+B_2\{Z_{11}(Z_{23}Z_{34}-Z_{33}Z_{24})+Z_{13}(Z_{31}Z_{24}-Z_{21}Z_{34})+Z_{14}(Z_{21}Z_{33}-Z_{31}Z_{23})\}$$

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$$
\begin{aligned}
& -B_3\{Z_{11}(Z_{22}Z_{34}-Z_{32}Z_{24})+Z_{12}(Z_{31}Z_{24}-Z_{21}Z_{34})+Z_{14}(Z_{21}Z_{32}-Z_{31}Z_{22})\} \\
& +B_4\{Z_{11}(Z_{22}Z_{33}-Z_{32}Z_{23})+Z_{12}(Z_{31}Z_{23}-Z_{21}Z_{33})+Z_{13}(Z_{21}Z_{32}-Z_{31}Z_{22})\}] \\
& =\frac{\chi_4}{|J|}\{Z_{11}Z_{22}Z_{33}Z_{44} \quad -Z_{11}Z_{22}Z_{43}Z_{34} \quad +Z_{11}Z_{23}Z_{42}Z_{34} \quad -Z_{11}Z_{23}Z_{32}Z_{44} \\
& +Z_{11}Z_{24}Z_{32}Z_{43} \ -Z_{11}Z_{24}Z_{42}Z_{33} \ -Z_{12}Z_{21}Z_{33}Z_{44} \ +Z_{12}Z_{21}Z_{43}Z_{34} \ -Z_{12}Z_{23}Z_{41}Z_{34} \\
& +Z_{12}Z_{23}Z_{31}Z_{44} \ -Z_{12}Z_{24}Z_{31}Z_{43} \ +Z_{12}Z_{24}Z_{41}Z_{33} \ +Z_{13}Z_{21}Z_{32}Z_{44} \ -Z_{13}Z_{21}Z_{42}Z_{34} \\
& +Z_{13}Z_{22}Z_{41}Z_{34} \ -Z_{13}Z_{22}Z_{31}Z_{44} \ +Z_{13}Z_{24}Z_{31}Z_{42} \ -Z_{13}Z_{24}Z_{41}Z_{32} \ -Z_{14}Z_{21}Z_{32}Z_{43} \\
& +Z_{14}Z_{21}Z_{42}Z_{33} -Z_{14}Z_{22}Z_{41}Z_{33} +Z_{14}Z_{22}Z_{31}Z_{43} -Z_{14}Z_{23}Z_{31}Z_{42} +Z_{14}Z_{23}Z_{41}Z_{32}\} \\
& +\frac{\eta}{|J|}\{-B_1Z_{12}Z_{23}Z_{34} \ +B_1Z_{12}Z_{33}Z_{24} \ -B_1Z_{13}Z_{32}Z_{24} \ +B_1Z_{13}Z_{22}Z_{34} \ -B_1Z_{14}Z_{22}Z_{33} \\
& +B_1Z_{14}Z_{32}Z_{23} \ +B_2Z_{11}Z_{23}Z_{34} \ -B_2Z_{11}Z_{33}Z_{24} \ +B_2Z_{13}Z_{31}Z_{24} \ -B_2Z_{13}Z_{21}Z_{34} \\
& +B_2Z_{14}Z_{21}Z_{33} \ -B_2Z_{14}Z_{31}Z_{23} \ -B_3Z_{11}Z_{22}Z_{34} \ +B_3Z_{11}Z_{32}Z_{24} \ -B_3Z_{12}Z_{31}Z_{24} \\
& +B_3Z_{12}Z_{21}Z_{34} \ -B_3Z_{14}Z_{21}Z_{32} \ +B_3Z_{14}Z_{31}Z_{22} \ +B_4Z_{11}Z_{22}Z_{33} \ -B_4Z_{11}Z_{32}Z_{23} \\
& +B_4Z_{12}Z_{31}Z_{23} -B_4Z_{12}Z_{21}Z_{33} +B_4Z_{13}Z_{21}Z_{32} -B_4Z_{13}Z_{31}Z_{22}\} \\
& =\frac{\chi_4}{|J|}\frac{A^4\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4^2}\{i(i-1)j(j-1)k(k-1)l(l-1) \quad -i(i-1)j(j-1)k^2l^2 \\
& +i(i-1)j^2k^2l^2 \quad -i(i-1)j^2k^2l(l-1) \quad +i(i-1)j^2k^2l^2 \quad -i(i-1)j^2k(k-1)l^2 \\
& -i^2j^2k(k-1)l(l-1) \quad +i^2j^2k^2l^2 \quad -i^2j^2k^2l^2 \quad +i^2j^2k^2l(l-1) \quad -i^2j^2k^2l^2 \\
& +i^2j^2k(k-1)l^2 \ +i^2j^2k^2l(l-1) \ -i^2j^2k^2l^2 \ +i^2j(j-1)k^2l^2 \ -i^2j(j-1)k^2l(l-1) \\
& +i^2j^2k^2l^2 \quad -i^2j^2k^2l^2 \quad -i^2j^2k^2l^2 \quad +i^2j^2k(k-1)l^2 \quad -i^2j(j-1)k(k-1)l^2 \\
& +i^2j(j-1)k^2l^2 \qquad -i^2j^2k^2l^2 \qquad +i^2j^2k^2l^2\} \\
& +\frac{1}{|J|}\frac{A^3\chi_1^{3i}\chi_2^{3j}\chi_3^{3k}\chi_4^{3l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4^2}\frac{A\chi_1^i\chi_2^j\chi_3^k\chi_4^l\Phi}{B}\{-b_1\chi_1\chi_4ij^2k^2l \quad +b_1\chi_1\chi_4ij^2k(k-1)l \\
& -b_1\chi_1\chi_4ij^2k^2l \quad +b_1\chi_1\chi_4ij(j-1)k^2l \quad -b_1\chi_1\chi_4ij(j-1)k(k-1)l \quad +b_1\chi_1\chi_4ij^2k^2l \\
& +b_2\chi_2\chi_4i(i-1)jk^2l \qquad -b_2\chi_2\chi_4i(i-1)jk(k-1)l \qquad +b_2\chi_2\chi_4i^2jk(k-1)l
\end{aligned}
$$

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$$-b_2\chi_2\chi_4 i^2 jk^2 l \ +b_2\chi_2\chi_4 i^2 jk(k-1)l \ -b_2\chi_2\chi_4 i^2 jk^2 l \ -b_3\chi_3\chi_4 i(i-1)j(j-1)kl$$
$$+b_3\chi_3\chi_4 i(i-1)j^2 kl \quad -b_3\chi_3\chi_4 i^2 j^2 kl \quad +b_3\chi_3\chi_4 i^2 j^2 kl \quad -b_3\chi_3\chi_4 i^2 j^2 kl$$
$$+b_3\chi_3\chi_4 i^2 j(j-1)kl \quad +b_4\chi_4^2 i(i-1)j(j-1)k(k-1) \quad -b_4\chi_4^2 i(i-1)j^2k^2$$
$$+b_4\chi_4^2 i^2 j^2 k^2 -b_4\chi_4^2 i^2 j^2 k(k-1)+b_4\chi_4^2 i^2 j^2 k^2 -b_4\chi_4^2 i^2 j(j-1)k^2\}$$
$$=\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4}\{(i-1)(j-1)(k-1)(l-1) -(i-1)(j-1)kl \ +2(i-1)jkl$$
$$-(i-1)jk(l-1) \quad -(i-1)j(k-1)l \quad -ij(k-1)(l-1) \quad -3ijkl \quad +2ij(k-1)l$$
$$+2ijk(l-1) \quad -i(j-1)k(l-1) \quad -i(j-1)(k-1)l \quad +2i(j-1)kl\}$$
$$+\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4}\{-2ijk+2ij(k-1) \quad +i(j-1)k \quad -i(j-1)(k-1)$$
$$+(i-1)jk-(i-1)j(k-1)-(i-1)(j-1)k+(i-1)(j-1)(k-1)\}$$
$$=\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4}(1-\Phi)-\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4}$$
$$\frac{\partial\eta}{\partial b_4}=-\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}\Phi}{\chi_1^2\chi_2^2\chi_3^2\chi_4}<0. \qquad (16)$$

From (16) we have realized that when per unit cost of irregular inputs increases, the level of marginal production of the firm decreases. The firm purchase less irregular inputs for the industrial production. This seems that irregular inputs have no substitutes.

Now we analyze the economic effects on Lagrange multiplier η when the budget size of the firm increases. Taking T_{15} (i.e., term of 1st row and 5th column) from both sides of (12) we get [Islam et al., 2010b; Mohajan, 2021b,c; Mohajan & Mohajan, 2023],

$$\frac{\partial\eta}{\partial B}=-\frac{1}{|J|}[C_{11}]$$
$$=-\frac{1}{|J|}\text{Cofactor of } C_{11}$$

$$= -\frac{1}{|J|}\begin{vmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{vmatrix}$$

$$= -\frac{1}{|J|}\left\{ Z_{11}\begin{vmatrix} Z_{22} & Z_{23} & Z_{24} \\ Z_{32} & Z_{33} & Z_{34} \\ Z_{42} & Z_{43} & Z_{44} \end{vmatrix} - Z_{12}\begin{vmatrix} Z_{21} & Z_{23} & Z_{24} \\ Z_{31} & Z_{33} & Z_{34} \\ Z_{41} & Z_{43} & Z_{44} \end{vmatrix} + Z_{13}\begin{vmatrix} Z_{21} & Z_{22} & Z_{24} \\ Z_{31} & Z_{32} & Z_{34} \\ Z_{41} & Z_{42} & Z_{44} \end{vmatrix} \right.$$

$$\left. - Z_{14}\begin{vmatrix} Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \\ Z_{41} & Z_{42} & Z_{43} \end{vmatrix} \right\}$$

$$= -\frac{1}{|J|}\left[Z_{11}\{Z_{22}(Z_{33}Z_{44} - Z_{43}Z_{34}) + Z_{23}(Z_{42}Z_{34} - Z_{32}Z_{44}) + Z_{24}(Z_{32}Z_{43} - Z_{42}Z_{33})\} \right.$$

$$- Z_{12}\{Z_{21}(Z_{33}Z_{44} - Z_{43}Z_{34}) + Z_{23}(Z_{41}Z_{34} - Z_{31}Z_{44}) + Z_{24}(Z_{31}Z_{43} - Z_{41}Z_{33})\}$$
$$+ Z_{13}\{Z_{21}(Z_{32}Z_{44} - Z_{42}Z_{34}) + Z_{22}(Z_{41}Z_{34} - Z_{31}Z_{44}) + Z_{24}(Z_{31}Z_{42} - Z_{41}Z_{32})\}$$
$$\left. - Z_{14}\{Z_{21}(Z_{32}Z_{43} - Z_{42}Z_{33}) + Z_{22}(Z_{41}Z_{33} - Z_{31}Z_{43}) + Z_{23}(Z_{31}Z_{42} - Z_{41}Z_{32})\}\right]$$

$$= -\frac{1}{|J|}\frac{A^4\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4^2}\{i(i-1)j(j-1)k(k-1)l(l-1) \quad - i(i-1)j(j-1)k^2l^2$$
$$+ i(i-1)j^2k^2l^2 \quad - i(i-1)j^2k^2l(l-1) \quad + i(i-1)j^2k^2l^2 \quad - i(i-1)j^2k(k-1)l^2$$
$$- i^2j^2k(k-1)l(l-1) \quad + i^2j^2k^2l^2 \quad - i^2j^2k^2l^2 \quad + i^2j^2k^2l(l-1) \quad - i^2j^2k^2l^2$$
$$+ i^2j^2k(k-1)l^2 + i^2j^2k^2l(l-1) - i^2j^2k^2l^2 + i^2j(j-1)k^2l^2 - i^2j(j-1)k^2l(l-1)$$
$$+ i^2j^2k^2l^2 \quad - i^2j^2k^2l^2 \quad - i^2j^2k^2l^2 \quad + i^2j^2k(k-1)l^2 \quad - i^2j(j-1)k(k-1)l^2$$
$$+ i^2j(j-1)k^2l^2 - i^2j^2k^2l^2 + i^2j^2k^2l^2\}$$

$$= -\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4^2}\{(i-1)(j-1)(k-1)(l-1) \quad -(i-1)(j-1)kl$$
$$+2(i-1)jkl \quad -(i-1)jk(l-1) \quad -(i-1)j(k-1)l \quad -ij(k-1)(l-1) \quad -3ijkl$$
$$+2ij(k-1)l + 2ijk(l-1) - i(j-1)k(l-1) - i(j-1)(k-1)l + 2i(j-1)kl\}$$

$$\frac{\partial \eta}{\partial B} = \frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4}(\Phi - 1). \quad (17)$$

If $\Phi > 1$, i.e., for increasing returns to scale, from (17) we get,

$$\frac{\partial \eta}{\partial B} = \frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4}(\Phi - 1) > 0. \quad (18)$$

From (18) we can say that when budget of the firm increases, the level of marginal product also increases. This seems that budget is essential for this firm, and it will increase its budget for profit maximization.

If $\Phi < 1$, i.e., for decreasing returns to scale, from (17) we get,

$$\frac{\partial \eta}{\partial B} = -\frac{1}{|J|}\frac{A^4 ijkl\chi_1^{4i}\chi_2^{4j}\chi_3^{4k}\chi_4^{4l}}{\chi_1^2\chi_2^2\chi_3^2\chi_4}(1 - \Phi) < 0. \quad (19)$$

From (19) we can say that when budget of the firm increases, the level of marginal product decreases. This seems that the production of this firm will decrease for decreasing returns to scale [Mohajan & Mohajan, 2022f; 2023].

If $\Phi = 1$, i.e., for constant returns to scale, from (17) we get,

$$\frac{\partial \eta}{\partial B} = 0. \quad (20)$$

From (20) we can say that when budget of the firm increases, there is no change in the level of marginal product. In this situation the firm suffers critical condition in its future production, and it cannot take major attempts for sustainability; and consequently, the increasing budget will not be benefited for this firm.

8. Conclusions

In this study we have worked on mathematical economic activities to see the economic effects when the costs of various inputs increase, and also the budget of the firm increases. In the global economic arena, every firm expects to move through the profit maximization policy. Sensitivity analysis is always helpful for the sustainable atmosphere of the firms to take attempts for the future production. The method of Lagrange multiplier is an important device in mathematical economic models. In this study we have used this tool to make the 4-dimensional constrained problem to a higher dimensional unconstrained problem, where Cobb-Douglas productions function is considered as our profit function.

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