Capacitance of a thin line conductor inside a grounded square shield

Vladan Durković, Dragan Filipović, ***

Keywords: *capacitance, grounded square shield, thin line conductor*

Abstract: In this paper, by using separation of variables in Laplace's equation, we derive a formula for the capacitance of a thin line conductor positioned inside a grounded square shield at one of its cross-section symmetry axes. The formula, which is in the form of a slow convergent series, is transformed into the sum of two series, one of which can be summed in a closed form, and the other one is a very rapidly convergent series and may be replaced accurately enough by its first term. As a result a very simple formula for the capacitance is obtained. Based on this formula, we give some numerical results for the capacitance versus the conductor position inside the shield. In particular, when the conductor is at the center of the shield our results are compared with the results obtained from an aviabile empirical formula for different values of the shield size.

1. INTRODUCTION

Various analytical methods are available for determination of the potential function which is necessary for finding the capacitance of a thin line conductor inside a grounded shield of sufficiently regular shape. Among them, the most frequently used are separation of variables in Laplace's equation, the method of images and the method of conformal mappings (in the two dimensional case) [1] – [3].

In this paper we use separation of variables in two-dimensional Laplace's equation to derive a simple formula for the capacitance of a thin line conductor inside a grounded square shield, the conductor being positioned at one of the shield cross-section symmetry axes.

---

* Vladan Durković is with the Faculty of Electrical Engineering, University of Montengro, 81000 Podgorica, Montenegro. He is a PHD candidate at School of Electrical Engineering, University of Belgrade, Serbia (vladan.d@ucg.ac.me).

** Dragan Filipovć is with the Faculty of Electrical Engineering, University of Montengro, 81000 Podgorica, Montenegro (e-mail: draganf@ucg.ac.me).

A similar method is used in [4] where simple formulas for the capacitance of a two-wire line symmetrically positioned inside a rectangular shield were derived.

2. DETERMINATION OF THE POTENTIAL INSIDE THE SHIELD

Fig. 1 shows the cross-section of the structure under investigation. A thin line conductor of radius R, with charge density q', is at an arbitrary position along the x-axis, determined by x_o, inside a grounded square shield of side a (a>>R). We also assume that the conductor is not too close to the shield walls, i.e $a/2-x_o$>>R.

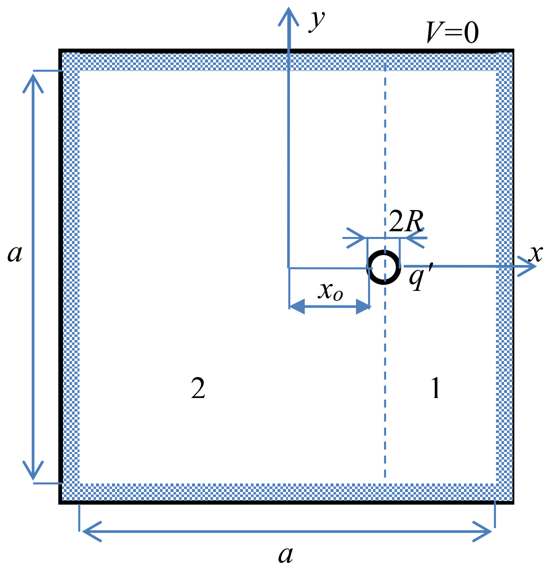

Fig.1. Thin line conductor inside grounded square shield.

Let us divide the interior of the shield into two subdomains 1: $x_o \le x \le a/2$, $|y| \le b/2$ and 2: $-a/2 \le x \le x_o$, $|y| \le b/2$. By using separation of variables in two-dimensional Laplace's equation we can write the potentials in subdomains 1 and 2 in the following forms

$$V_1 = \sum_{n=1}^{\infty} A_{2n-1} \operatorname{sh}\left[(2n-1)\frac{\pi}{a}\left(\frac{a}{2}-x\right)\right]\cos\left[(2n-1)\frac{\pi y}{a}\right] \tag{1}$$

$$V_2 = \sum_{n=1}^{\infty} B_{2n-1} \operatorname{sh}\left[(2n-1)\frac{\pi}{a}\left(\frac{a}{2}+x\right)\right]\cos\left[(2n-1)\frac{\pi y}{a}\right] \tag{2}$$

where A_{2n-1} and B_{2n-1} are unknown coefficients. It is evident from (1) and (2) that the boundary conditions in the shield walls are met (V_1=0 for $x=a/2$ and $y=\pm a/2$, V_2=0 for

$x= -a/2$ and $y=\pm a/2$), and that V_1 and V_2 are even function in y, as then should be by symmetry. The potential must be a continuous function at the subdomain interface $x=x_o$, so the condition

$$V_1 = V_2 \text{ for } x=x_o \tag{3}$$

must be imposed. From (3), by using (1) and (2), we obtain

$$B_{2n-1} = A_{2n-1} \frac{\mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}-x_o\right)}{\mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}+x_o\right)}$$

so that

$$V_2 = \sum_{n=1}^{\infty} A_{2n-1} \frac{\mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}-x_o\right)}{\mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}+x_o\right)} \mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}+x\right)\cos\left(2n-1\right)\frac{\pi y}{a} \tag{4}$$

To determine the unknown coefficients A_{2n-1} in (1) and (4), we have at our disposal the boundary condition for the normal components $E_x=-\partial V/\partial x$ of the electric field at the interface $x=x_o$

$$\left.\left(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial x}\right)\right|_{x=x_o} = \frac{q'}{\varepsilon_o}\delta\left(y\right) \tag{5}$$

where the Dirac δ – function accounts for the conductor line charge. By using (1) and (4), eqn. (5) reduces to

$$\sum_{n=1}^{\infty} A_{2n-1}\left(2n-1\right)\frac{\pi}{a} \frac{\mathrm{sh}\left(2n-1\right)\pi}{\mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}+x_o\right)} \cos\left(2n-1\right)\frac{\pi y}{a} = \frac{q'}{\varepsilon_o}\delta\left(y\right) \tag{6}$$

Now, we multiply (6) by $\cos(2n-1)\pi y/a$, and integrate with respect to y from $-a/2$ to $+a/2$. This allows to find the unknown coefficients A_{2n-1} in (1) and (4)

$$A_{2n-1} = \frac{2q'\cdot \mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}+x_o\right)}{\pi\varepsilon_o\left(2n-1\right)\mathrm{sh}\left(2n-1\right)\pi} \tag{7}$$

In deriving (7) we used orthogonality of the cosine function and the fact that $\int_{-a/2}^{a/2}\delta\left(y\right)f\left(y\right)dy = f\left(0\right)$. Finally, potential V_1 is determined by substitutng (7) into (1)

$$V_1 = \frac{2q'}{\pi\varepsilon_o}\sum_{n=1}^{\infty}\frac{\mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}+x_o\right)}{\left(2n-1\right)\mathrm{sh}\left(2n-1\right)\pi}\cdot \mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}-x\right)\cos\left(2n-1\right)\frac{\pi y}{a} \tag{8}$$

3. DETERMINATION OF THE CONDUCTOR CAPACITANCE

To find the conductor potential we choose the point $x=x_o+R$, $y=0$ on the conductor surface and substitute its coordinates into (8), which yields

$$V_{cond} = \frac{2q'}{\pi\varepsilon_o}\sum_{n=1}^{\infty}\frac{\mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}+x_o\right)}{\left(2n-1\right)\mathrm{sh}\left(2n-1\right)\pi}\cdot \mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}-x_o-R\right) \tag{9}$$

hence, the capacitance is

$$C' = \frac{q'}{V_{cond}} = \frac{\pi\varepsilon_o}{2\sum_{n=1}^{\infty}a_{2n-1}} \tag{10}$$

where

$$a_{2n-1} = \frac{\mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}+x_o\right)}{\left(2n-1\right)\mathrm{sh}\left(2n-1\right)\pi}\mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}-x_o-R\right) \tag{11}$$

The series in (10), defined by (11), diverges for R=0. So, since R is small, its convergence will be slow. We can accelerate convergence in the following way. We may write

$$\sum_{n=1}^{\infty} a_{2n-1} = \sum_{n=1}^{\infty}\left(a_{2n-1} - a_{2n-1}^{\infty}\right) + \sum_{n=1}^{\infty} a_{2n-1}^{\infty} \tag{12}$$

where a^{∞}_{2n-1} is the asymptotic value of a_{2n-1} when $n\to\infty$. Since sh$x \sim 0.5\mathrm{e}^{x}$ when $x\to\infty$, this asymptotic value is

$$a_{2n-1}^{\infty} = \frac{1}{2}\frac{\mathrm{e}^{-(2n-1)\frac{\pi R}{a}}}{2n-1} \tag{13}$$

and the series $\sum_{n=1}^{\infty} a_{2n-1}^{\infty}$ can be summed in a closed form [5]

$$\sum_{n=1}^{\infty} a_{2n-1}^{\infty} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-(2n-1)\frac{\pi R}{a}}}{2n-1} = \frac{1}{4}\ln\frac{1+\mathrm{e}^{-\frac{\pi R}{a}}}{1-\mathrm{e}^{-\frac{\pi R}{a}}} \approx \frac{1}{4}\ln\frac{2a}{\pi R} \tag{14}$$

where we used approximations $1+\mathrm{e}^{-\frac{\pi R}{a}} \approx 2$, $1-\mathrm{e}^{-\frac{\pi R}{a}} \approx \pi R/a$, valid for small R.
By using (11), (13) and (14), eqn. (12) can be rewritten as

$$\sum_{n=1}^{\infty} a_{2n-1} = \sum_{n=1}^{\infty}\left(\frac{\mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}+x_o\right)\mathrm{sh}\left(2n-1\right)\frac{\pi}{a}\left(\frac{a}{2}-x_o-R\right)}{\left(2n-1\right)\mathrm{sh}\left(2n-1\right)\pi} - \frac{1}{2}\frac{\mathrm{e}^{-(2n-1)\frac{\pi R}{a}}}{2n-1}\right) + \tag{15}$$

$$+\frac{1}{4}\ln\frac{2a}{\pi R}$$

The essence of the performed transformation is that the series the right hand side of (15) converges very rapidly; in fact it can be replaced, with high accuracy, by its first term. Therefore,

$$\sum_{n=1}^{\infty} a_{2n-1} \approx \frac{\operatorname{sh}\frac{\pi}{a}\left(\frac{a}{2}+x_o\right)\operatorname{sh}\frac{\pi}{a}\left(\frac{a}{2}-x_o-R\right)}{\operatorname{sh}\pi} - \frac{1}{2}e^{-\frac{\pi R}{a}} + \frac{1}{4}\ln\frac{2a}{\pi R} \approx$$
$$\approx \frac{\operatorname{sh}\frac{\pi}{a}\left(\frac{a}{2}+x_o\right)\operatorname{sh}\frac{\pi}{a}\left(\frac{a}{2}-x_o\right)}{\operatorname{sh}\pi} - \frac{1}{2} + \frac{1}{4}\ln\frac{2a}{\pi R} \tag{16}$$

where we used approximations $\operatorname{sh}\frac{\pi}{a}\left(\frac{a}{2}-x_o-R\right) \approx \operatorname{sh}\frac{\pi}{a}\left(\frac{a}{2}-x_o\right)$ and $\mathrm{e}^{-\frac{\pi R}{a}} \approx 1$ which are justified by smallness of R.

Finally, from (10) and (16) we obtain a simple formula for the capacitance

$$C' = \frac{\pi\varepsilon_o}{2\dfrac{\operatorname{sh}\frac{\pi}{a}\left(\frac{a}{2}+x_o\right)\operatorname{sh}\frac{\pi}{a}\left(\frac{a}{2}-x_o\right)}{\operatorname{sh}\pi} - 1 + \dfrac{1}{2}\ln\dfrac{2a}{\pi R}} \tag{17}$$

In particular, for x_o=0,

$$C' = \frac{\pi\varepsilon_o}{\operatorname{th}\frac{\pi}{2} - 1 + \frac{1}{2}\ln\frac{2a}{\pi R}} \tag{18}$$

which was obtained in [5]. We note that the logarithmic term in (17) - (18) gives the main contribution to the capacitance; the rest in the denominators serves as a correction factor.

4. NUMERICAL RESULTS

Fig. 2 shows the capacitance calculated from (17) versus the conductor position. As expected, the capacitance increases as the conductor gets closer to the shield.

In the special case when the conductor is at the center of the shield we can check accuracy of our approximate formula (18) by using an approximate formula for the characteristic impedance of the line constituted by the conductor and the shield [6]

$$Z_c = 138\log_{10}\frac{a}{2R} + 3.54\ [\Omega]$$

which gives the capacitance [7]

$$C' = \frac{1}{c \cdot Z_C} = \frac{1}{c \cdot \left(138 \log_{10} \frac{a}{2R} + 3.54\right)} \qquad (19)$$

where c=3·10^8 m/s is speed of light.

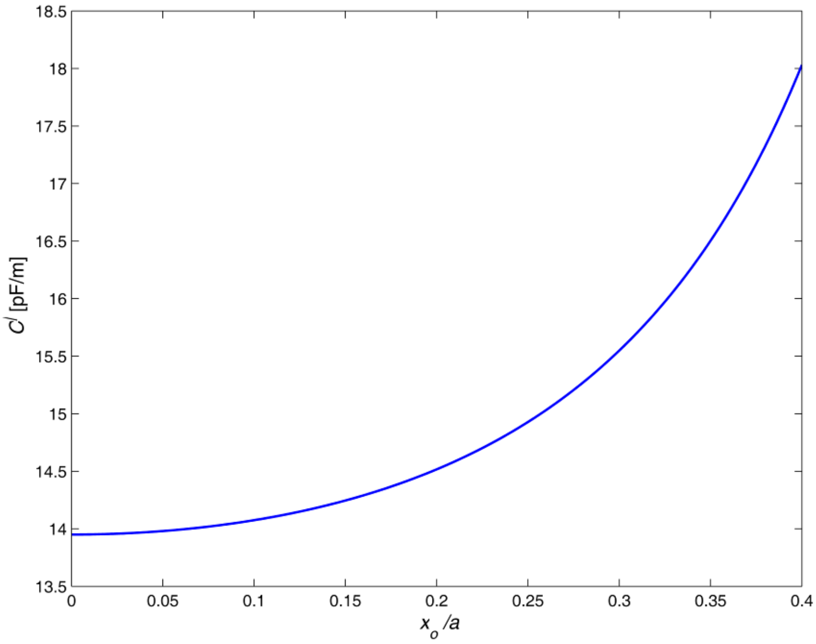

Fig. 2. Conductor capacitance versus its position.

Table I shows a comparison of the capacitance values obtained by our approximate formula (18) and the corresponding values obtained from (19), for various *a/R* ratios. As can be seen, the procentual error does not exceed 0.4%.

Table I
Capacitance of the centrally spaced conductor

a/R	formula (18) [pF/m]	formula (19) [pF/m]	error (%)
100	13.9501	14.0057	0.3975
150	12.6626	12.7082	0.3586
200	11.8844	11.9244	0.3351
250	11.3436	11.3799	0.3187
300	10.9370	10.9707	0.3065
350	10.6153	10.6469	0.2967
400	10.3516	10.3816	0.2888
450	10.1296	10.1582	0.2821
500	9.9389	9.9664	0.2763

5. CONCLUSION

In this paper we derived a simple formula for the capacitance of a thin line conductor spaced inside a grounded shield, along one of the shield cross-section symmetry axes. In a particular case, when the conductor is at the center of the shield, we checked our results against the ones from an empirical formula, found in handbook literature. The exhibited error is partically negligible.

REFERENCES

[1] W. R. Smythe, *Static and Dynamic Electricity*, 3rd ed., McGraw-Hill, New York, 1968.
[2] J. D. Jackson, *Classical Electrodynamics*, John Wiley, New York 1962.
[3] W. K. Panofsky, M. Phillips, *Classical Electricity and Magnetism*, 2nd ed., Addison – Wesley, Reading, Massachusetts, 1962.
[4] D. Filipović, V. Durković, "Capacitance of the two-wire line symmetrically spaced inside a rectangular shield" 5th International conference on electrical, electronic and computing engineering, IcETRAN 2018, Palić, Serbia.
[5] D. Filipović, V. Durković, "Capacitance of a thin line conductor spaced at the center of a grounded square shield", XXIII International IT conference, Žabljak, Montenegro, 2018 (in Serbian).
[6] *Reference Data for Radio Engineers*, 5th ed., Howard W. Sams & Co, Inc., Indianopolis - Kansas City - New York, 1968.
[7] B. Popović, *Elektromagnetics,* Građevinska knjiga, Belgrade, 1989 (in Serbian).