POTENTIAL, ELECTRIC FIELD AND CAPACITANCE PER UNIT LENGTH OF A THIN LINE CONDUCTOR IN AN ANGLE BETWEEN TWO GROUND HALF PLANES

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Abstract: In this paper we have derived a simple formula for the potential of an infinitely thin charged line conductor between two ground half planes at an arbitrary angle. Based on this formula the conductor capacitance per unit length is determined, followed by illustrative examples.

1. INTRODUCTION

It is well known [1] that the problem of determining the potential and electric field of an infinitely thin charged line conductor in an angle between two grounded half planes can be solved analytically by the method of images only if the angle has discrete values $\alpha=\pi/n$, n=1,2,3... However, for large values of n, the calculations are rather cumbersome, since the number of images that take into account the induced electrical charges on the grounded half planes is $2n$-1.

Given an arbitrary value of the angle between the grounded half planes, a suitable method for solving the problem is separation of variables in Laplace's equation, leading to solutions for the potential in the form of an infinite series [2], or in the form of an integral over an infinite interval [3].

The approach from [2] was adopted in this paper, but the potential is obtained in a closed form, by summing up the infinite series. Based on this formula, the conductor capacitance per unit length is determined.

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2. POTENTIAL OF AN INFINITELY THIN LINE CONDUCTOR IN AN ANGLE BETWEEN TWO GROUNDED HALF PLANES

The problem's geometry is presented in Fig. 1. The half planes kept at the potential $V=0$ make the angle α, while an infinitely thin line conductor carrying charge q' stretches parallel to them. Its position is defined by the polar coordinates R and β.

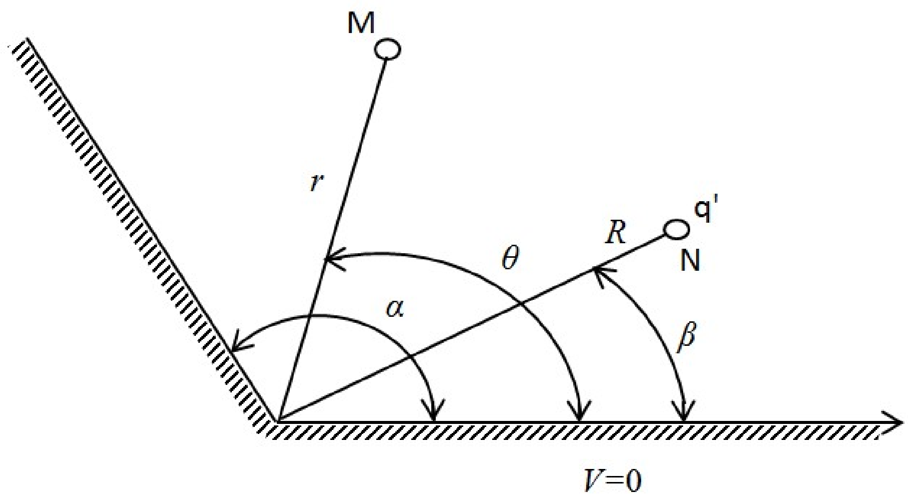

Fig. 1. The problem's geometry.

The potential in the domain between the half planes must satisfy Laplace's equation (except at the point N, where it is infinite)

$$\Delta V\left(r,\theta\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}=0 \tag{1}$$

and the boundary conditions

$$V=0 \text{ za } \theta=0 \text{ i } \theta=\alpha. \tag{2}$$

Separating the variables in (1) and taking into account (2) the potential can be found in the form of an infinite series [2]

$$V\left(r,\theta\right)=\frac{q'}{\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{R}\right)^{\frac{n\pi}{\alpha}}\sin\frac{n\pi\beta}{\alpha}\sin\frac{n\pi\theta}{\alpha},\ r<R \tag{3}$$

If $r > R$, then r/R on the right-hand side of (3) should be replaced by R/r. It is readily verified that (3) satisfies (1) - (2).

The infinite series on the right-hand side of (3) can be summed up in a closed form by using the formula [4][1)]

$$\sum_{n=1}^{\infty}\frac{A^n}{n}\sin nx\sin ny=\frac{1}{4}\ln\frac{4A\sin^2\frac{x+y}{2}+(A-1)^2}{4A\sin^2\frac{x-y}{2}+(A-1)^2},\quad |A|<1 \tag{4}$$

thereby yielding

$$V(r,\theta)=\frac{q'}{4\pi\varepsilon_0}\cdot\ln\frac{4\left(\frac{r}{R}\right)^{\pi/\alpha}\sin^2\frac{\pi}{2\alpha}(\beta+\theta)+\left[\left(\frac{r}{R}\right)^{\pi/\alpha}-1\right]^2}{4\left(\frac{r}{R}\right)^{\pi/\alpha}\sin^2\frac{\pi}{2\alpha}(\beta-\theta)+\left[\left(\frac{r}{R}\right)^{\pi/\alpha}-1\right]^2},\ r<R \tag{5}$$

For $r > R$, r and R in (4) should interchange places. However, it can be elementary verified that this interchange leaves (4) invariant, i.e. unchanged. Hence, (4) is also valid for $r > R$.

Fig 2. presents the normalized potential $V(r/R)\cdot 4\pi\varepsilon_0/q'$, for $\alpha=2\pi/3$, $\beta=\pi/6$ and different values of the angle θ. The curves are obtained by using (5).
It can be observed from Fig. 2 that, for any value of the angle θ, the maximal value of the potential is obtained for $r/R=1$.

3. ELECTRIC FIELD

The components of the electric field $\vec{E}$ = - grad$V$ can be obtained from (4) by using the formula for gradient in the cylindrical coordinate system [1]

$$E_r(r,\theta)=-\frac{\partial V}{\partial r}, \tag{6a}$$

$$E_\theta(r,\theta)=-\frac{1}{r}\frac{\partial V}{\partial\theta}. \tag{6b}$$

Alternatively, for calculating the electric field by (6), (3) can be used instead of (5).

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1) Details of a derivation of this formula are given in the Appendix.

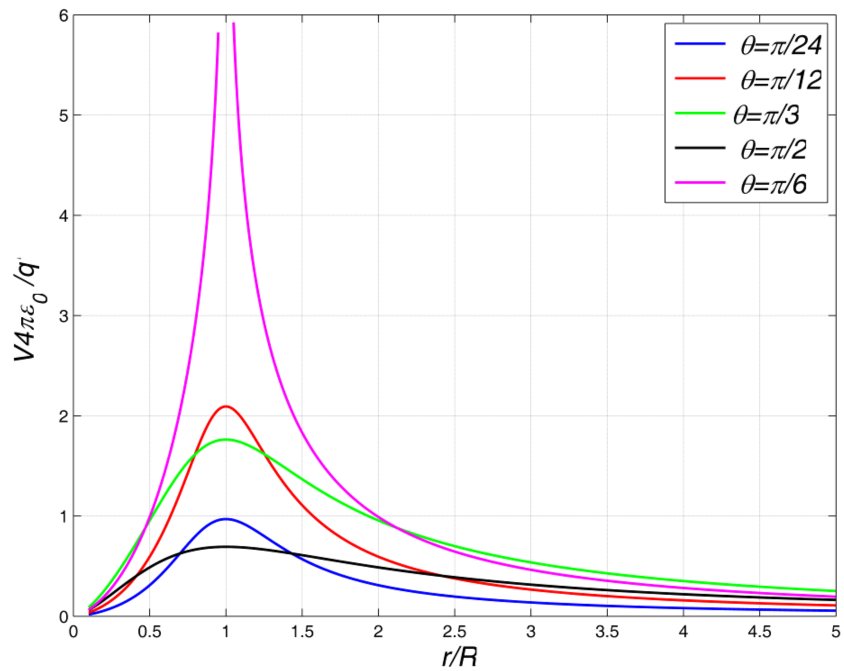

Fig. 2. Distribution of the normalized potential for various values of angle *θ*.

4. SURFACE CHARGE DENSITY ON THE HALF PLANES

It is known [1] that the charge density on the surface of a conductor is related to the normal component (only existing) of the electric field

$$\sigma = \varepsilon_o \cdot \vec{E} \cdot \vec{n} = \pm \varepsilon_o \cdot E_n$$

where $\vec{n}$ is the unit normal pointed from the conductor surface outward. From this formula we can find, by using (6b), the surface charge densities on the half planes $\theta=0$ and $\theta=\alpha$

$$\sigma(r,0) = \varepsilon_o \cdot E_\theta(r,0) = -\varepsilon_o \cdot \frac{1}{r}\frac{\partial V}{\partial \theta}\bigg|_{\theta=0} = -\frac{q'\sin\frac{\pi\beta}{\alpha}}{\alpha R} \cdot \frac{\left(\frac{r}{R}\right)^{\frac{\pi}{\alpha}-1}}{1-2\left(\frac{r}{R}\right)^{\frac{\pi}{\alpha}}\cos\frac{\pi\beta}{\alpha}+\left(\frac{r}{R}\right)^{2\frac{\pi}{\alpha}}}$$

and

$$\sigma(r,\alpha) = -\varepsilon_o \cdot E_\theta(r,\alpha) = +\varepsilon_o \cdot \frac{1}{r}\frac{\partial V}{\partial \theta}\bigg|_{\theta=\alpha} = -\frac{q'\sin\frac{\pi\beta}{\alpha}}{\alpha R} \cdot \frac{\left(\frac{r}{R}\right)^{\frac{\pi}{\alpha}-1}}{1+2\left(\frac{r}{R}\right)^{\frac{\pi}{\alpha}}\cos\frac{\pi\beta}{\alpha}+\left(\frac{r}{R}\right)^{2\frac{\pi}{\alpha}}}.$$

Fig 3. shows the normalized surface charge density $-\sigma R/q'$ on the half planes $\theta=0$ and $\theta=\alpha$. It was taken $\alpha=\pi/2$ and $\beta=\pi/6$.

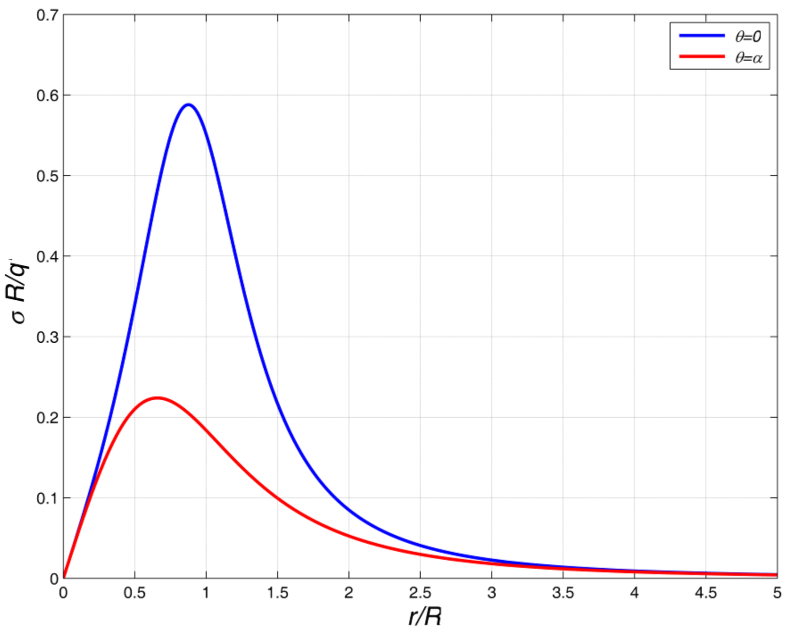

Fig. 3. Normalized charge distribution on the half planes $\theta=0$ and $\theta=\alpha$.

Elementary integrations of the surface charge densities over the half planes give the induced charges per unit length

$$q'_{ind}(\theta=0) = -q'\left(1-\frac{\beta}{\alpha}\right),$$

$$q'_{ind}(\theta=\alpha) = -q'\left(\frac{\beta}{\alpha}\right).$$

The total induced charge per unit length is:

$$q'_{ind} = q'_{ind}\left(\theta = 0\right) + q'_{ind}\left(\theta = \alpha\right) = -q'.$$

5. CAPACITANCE PER UNIT LENGTH OF CONDUCTOR

In order to determine the conductor capacitance per unit length it is necessary to assume a certain thickness for the conductor. Suppose the conductor diameter is a ($a<<R$). Then, the potential of the conductor is obtained from (5) by putting $r=R-a$ and $\theta=\beta$

$$V \equiv V(R-a,\beta) = \frac{q'}{4\pi\varepsilon_0}\cdot\ln\frac{4\left(\frac{R-a}{R}\right)^{\pi/\alpha}\sin^2\frac{\pi\beta}{\alpha}+\left[\left(\frac{R-a}{R}\right)^{\pi/\alpha}-1\right]^2}{\left[\left(\frac{R-a}{R}\right)^{\pi/\alpha}-1\right]^2}.$$

Now, the conductor capacitance per unit length is

$$C' = \frac{q'}{V} = \frac{4\pi\varepsilon_0}{\ln\frac{4\left(\frac{R-a}{R}\right)^{\pi/\alpha}\sin^2\frac{\pi\beta}{\alpha}+\left[\left(\frac{R-a}{R}\right)^{\pi/\alpha}-1\right]^2}{\left[\left(\frac{R-a}{R}\right)^{\pi/\alpha}-1\right]^2}} \tag{7}$$

Since $a<<R$, the following approximation holds

$$\left(\frac{R-a}{R}\right)^{\pi/\alpha} = \left(1-\frac{a}{R}\right)^{\pi/\alpha} \approx 1-\frac{\pi a}{\alpha R}.$$

Therefore, (7) is simplified to

$$C' \approx \frac{4\pi\varepsilon_0}{\ln\frac{4\sin^2\frac{\pi\beta}{\alpha}+\left(\frac{\pi a}{\alpha R}\right)^2}{\left(\frac{\pi a}{\alpha R}\right)^2}}.$$

Finally, if β is not too close to 0 or α, the second term in the numerator of the logarithm can be neglected, thereby giving

$$C' \approx \frac{2\pi\varepsilon_o}{\ln\left(\frac{2\alpha R}{\pi a}\sin\frac{\pi\beta}{\alpha}\right)}. \quad (8)$$

6. EXAMPLES OF CALCULATION OF FIELD AND CAPACITANCE PER UNIT LENGTH

Fig. 4 presents the normalized electric field $E(r/R)\cdot 4\pi\varepsilon_0 R/q'$ on the half planes θ=0 and $\theta=\alpha$. The curves were obtained by using (5) and (6b). It was taken α=2π/3, β=π/6.

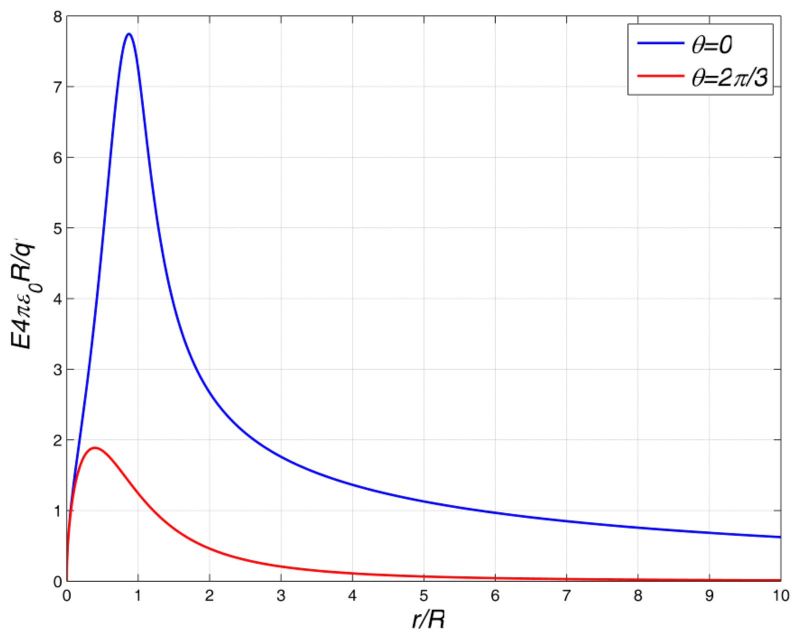

Fig. 4. Normalized electric field on the half planes θ=0 and $\theta=\alpha$.

The capacitance per unit length of the conductor for several special cases (Figs. 5-7) (that can also be treated by the method of images [1]) is determined by (8).

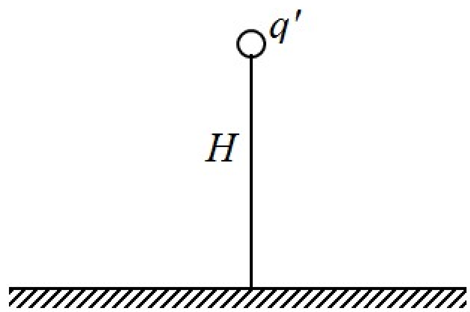

Fig. 5.

For the case in Fig. 5, where $\alpha=\pi$, $H=R\sin\beta$, the capacitance per unit length is

$$C' \approx \frac{2\pi\varepsilon_0}{\ln\frac{2H}{a}}.$$

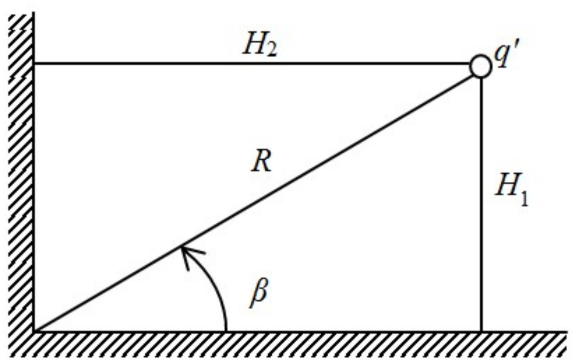

Fig. 6.

For the case shown in Fig. 6, where $\alpha=\pi/2$, $H_1=R\sin\beta$, $H_2=R\cos\beta$

$$C' \approx \frac{2\pi\varepsilon_0}{\ln\left(\frac{2H_1\cdot H_2}{a\cdot\sqrt{H_1^2+H_2^2}}\right)}.$$

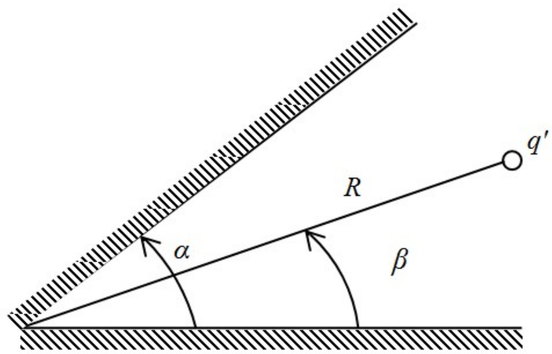

Fig. 7.

Finally, for the case in Fig. 7, where $\alpha=\pi/3$, and, if, for instance, $\beta=\pi/6$, we obtain

$$C' \approx \frac{2\pi\varepsilon_0}{\ln\left(\frac{2R}{3a}\right)}.$$

7. CONCLUSIONS

A simple formulas for the potential and capacitance per unit length of a thin line conductor in an angle between two grounded half planes are derived in this paper. Unlike the method of images that can be applied only for certain discrete values of the angle between the half planes, the described method applies for an arbitrary angle. The further research will comprise an important practical case of a system of thin conductors (lines).

APPENDIX

In this appendix we sketch a derivation of formula (4). By using the indentity $\sin nx \cdot \sin ny = \frac{1}{2}\left[\cos n(x\text{-}y) - \cos n(x+y)\right]$ the summation in (4) is reduced to two summations of the same form

$$F\left(A,B\right) = \sum_{n=1}^{\infty} \frac{1}{n} A^n \cos nB \,, \ \left|A\right| < 1 \tag{A1}$$

Differentation of (A1) with respect to B gives a simpler summation

$$\frac{\partial F}{\partial B} = -\sum_{n=1}^{\infty} A^n \sin nB \tag{A2}$$

which can be summed as follows

$$\sum_{n=1}^{\infty} A^n \sin nB = \operatorname{Im}\sum_{n=1}^{\infty} z^n \,, \quad z = Ae^{iB} \tag{A3}$$

The summation on the right-hand side of (A3) is an infinite geometrical progression

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z} = \frac{Ae^{iB}}{1-Ae^{iB}} \,, \quad |z| \equiv |A| < 1$$

whence, from (A3) we find

$$\sum_{n=1}^{\infty} A^n \sin nB = \operatorname{Im}\frac{Ae^{iB}}{1-Ae^{iB}} = \frac{A\sin \mathrm{B}}{1-2A\cos B + A^2}$$

and, according to (A2)

$$\frac{\partial F}{\partial B} = -\frac{A\sin \mathrm{B}}{1-2A\cos B + A^2} \,. \tag{A4}$$

Finally, an integration of (A4) with respect to B gives the summation in (A1)

$$\sum_{n=1}^{\infty} \frac{1}{n} A^n \cos nB \equiv F(\mathrm{A},\mathrm{B}) = -\int \frac{A\sin B}{1-2A\cos B + A^2}\, dB = -\frac{1}{2}\ln\left(1-2A\cos B + A^2\right).$$

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