SENSITIVITY ANALYSIS BETWEEN LAGRANGE MULTIPLIERS AND CONSUMER BUDGET: UTILITY MAXIMIZATION CASE

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Abstract: In this paper sensitivity analysis between Lagrange multipliers and the total budget is discussed. The method of Lagrange multipliers is a very useful and powerful technique in multivariable calculus. In mathematical economics, utility is the vital concept that increases or decreases the overall happiness of consumers. This study tries to discuss the utility maximization policy of an organization by considering two constraints: budget constraint and coupon constraint. In this article, an attempt has been taken to achieve the best result through the application of the scientific method of optimization.

Keywords: Budget, Lagrange multipliers, sensitivity analysis, utility maximization

JEL Classification: B41, C02, C51, C61, C67, D11, F63, G15

1. Introduction

In the 21st century mathematical modeling in economics becomes an essential part to investigate optimization policy. Mathematical modeling in economics is considered as the application of mathematics in economics [Samuelson, 1947; Carter, 2001]. For the sustainable of an industry, utility maximization strategy is essential. The property of a commodity that enables to satisfy human necessities is called utility [Bentham, 1780]. It directly influences the demand and supply of the organizations [Fishburn, 1970]. The concept of utility was developed in the late 18th century by the English moral philosopher Jeremy Bentham (1748-1832) and English philosopher John Stuart Mill (1806-1873) [Bentham, 1780; Gauthier, 1975].

Two American scholars: mathematician John V. Baxley and economist John C. Moorhouse have analyzed an example of utility maximization subject to a budget constraint from a somewhat wider perspective. They have provided a mathematical formulation for nontrivial constrained optimization problem with special reference for the application in economics [Baxley & Moorhouse, 1984].

In this article, we have tried to form a mathematical formulation of economic model for maximizing utility function subject to two constrains, such as budget constraint and coupon constraint. The method of Lagrange multipliers is considered as a device for transforming a constrained problem to a higher dimensional unconstrained problem [Islam et al., 2010]. In this study, we have worked with the determinant of 6×6 Hessian matrix and 6×10 Jacobian matrix. We have operated the article with 16 variables, such as four price vectors, four types of coupon numbers, four commodity variables, two Lagrange multipliers, one total budget variable, and one total coupon variable. In the study mathematical calculations are displayed in some details.

2. Literature Review

In any research, literature review is an introductory section, which highlights previous researches in the same field [Polit & Hungler, 2013]. In 1928, two American scholars; mathematician Charles W. Cobb (1875-1949) and economist Paul H. Douglas (1892-1976), have worked on production functions [Cobb & Douglas, 1928]. Later in 1984, another two American professors; mathematician John V. Baxley and economist John C. Moorhouse, have given the utility maximization structure with sufficient mathematical techniques [Baxley & Moorhouse, 1984].

Pahlaj Moolio and his coauthors have given reasonable interpretation of the Lagrange multipliers and examined the behavior of the firm by analyzing comparative static results [Moolio et al., 2009]. Notable mathematician Jamal Nazrul Islam and his coauthors have analyzed utility maximization and other optimization problems by considering reasonable interpretation of the Lagrange multipliers [Islam et al., 2009a,b, 2011]. Qi Zhao and his coauthors have proposed for multi-product utility maximization as a general approach to the recommendation driven by economic principles [Zhao et al., 2017]. Young researcher Lia Roy and her coworkers have discussed cost minimization policy of an industry, where they have provided detail mathematical formulation [Roy et al., 2021]. Haradhan Kumar Mohajan has explored utility maximization model for Bangladeshi consumers [Mohajan, 2021a]. In a published book, he and his coauthors have discussed a series of optimization problems for the social welfare [Mohajan et al., 2013].

Devajit Mohajan and Haradhan Kumar Mohajan have discussed a profit maximization problem in an industry, where they have used four variable inputs, such as capital, labor, principal raw materials, and other inputs [Mohajan & Mohajan, 2022a]. On the other hand, Jannatul Ferdous and Haradhan Kumar Mohajan have briefly solved a profit maximization problem [Ferdous & Mohajan, 2022]. In another paper, they have calculated utility maximization policy of an organization [Mohajan & Mohajan, 2022b; Mohajan, 2022].

3. Methodology of the Study

Research is an essential and influential device to the professors to lead the academic world [Pandey & Pandey, 2015]. Methodology is a guideline that tries to describe the types of research and the types of data [Somekh & Lewin, 2005]. Research methodology is the science and philosophy behind all researches and it provides the principles for organizing, planning, designing and conducting a good research [Remenyi et al., 1998; Legesse, 2014]. It tries to create new knowledge basis on the existing knowledge [Goddard & Melville, 2001].

In this study we have used 6×6 bordered Hessian matrix and 6×10 Jacobian matrix, and we have also used four commodities b_1, b_2, b_3, and b_4, and two Lagrange multipliers λ_1 and λ_2. We have tried to provide mathematical calculations and results very clearly [Mohajan, 2017b, 2018a, 2020]. In this study we have depended on the utility maximization related mathematical secondary data

sources. The data are collected from the secondary data sources, such as from published research papers, books and handbooks of famous authors, internet, websites, etc. [Mohajan, 2017a, 2018b].

4. Objective of the Study

The leading objective of this paper is to discuss sensitivity analysis between Lagrange multipliers and total budget of consumers during the utility maximization and economic analysis. The other supplementary objectives are as follows:

- to develop the bordered Hessian and Jacobian,
- to provide the sensitivity results properly, and
- to display the mathematical calculations in some details.

5. An Economic Model

To study sensitivity analysis we consider four commodities: A_1, A_2, A_3, and A_4. Let the consumers in the society wants to purchase b_1, b_2, b_3, and b_4 amounts from these four commodities A_1, A_2, A_3, and A_4, respectively. The utility function for these four commodities can be written as [Islam et al., 2010; Mohajan & Mohajan, 2022b],

$$u(b_1, b_2, b_3, b_4) = b_1 b_2 b_3 b_4. \tag{1}$$

The budget constraint of the consumers is,

$$B = p_1 b_1 + p_2 b_2 + p_3 b_3 + p_4 b_4 \tag{2}$$

where p_1, p_2, p_3, and p_4 are the prices of per unit of commodities b_1, b_2, b_3, and b_4, respectively. Now the coupon constraint is,

$$K = k_1 b_1 + k_2 b_2 + k_3 b_3 + k_4 b_4, \tag{3}$$

where k_1, k_2, k_3, and k_4 are the coupons necessary to purchase a unit of commodity of b_1, b_2, b_3, and b_4, respectively.

Using (1), (2), and (3) we can express Lagrangian function $U(b_1, b_2, b_3, b_4, \lambda_1, \lambda_2)$ as [Baxley & Moorhouse, 1984; Ferdous & Mohajan, 2022],

$$U(b_1, b_2, b_3, b_4, \lambda_1, \lambda_2) = b_1 b_2 b_3 b_4 + \lambda_1 (B - p_1 b_1 - p_2 b_2 - p_3 b_3 - p_4 b_4)$$
$$+ \lambda_2 (K - k_1 b_1 - k_2 b_2 - k_3 b_3 - k_4 b_4). \tag{4}$$

Lagrangian function (4) is a 6-dimensional unconstrained problem that maximizes utility functions; where λ_1 and λ_2 are two Lagrange multipliers.

Now taking first and second order and cross-partial derivatives in (4) we obtain [Islam et al. 2009a,b; Mohajan & Mohajan, 2022d];

$$B_1 = p_1, \ B_2 = p_2, \ B_3 = p_3, \ B_4 = p_4.$$

$$K_1 = k_1, \ K_2 = k_2, \ K_3 = k_3, \ K_4 = k_4. \tag{5}$$

$$U_{11} = 0, \ U_{12} = U_{21} = b_3 b_4, \ U_{13} = U_{31} = b_2 b_4,$$

$$U_{14} = U_{41} = b_2 b_3, \ U_{22} = 0, U_{23} = U_{32} = b_1 b_4,$$

$$U_{24} = U_{42} = b_1 b_3, \ U_{33} = 0, U_{34} = U_{43} = b_1 b_2, \ U_{44} = 0. \tag{6}$$

Now we consider the bordered Hessian [Mohajan, 2021a; Mohajan & Mohajan, 2022c],

$$|H| = \begin{vmatrix} 0 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -K_1 & -K_2 & -K_3 & -K_4 \\ -B_1 & -K_1 & U_{11} & U_{12} & U_{13} & U_{14} \\ -B_2 & -K_2 & U_{21} & U_{22} & U_{23} & U_{24} \\ -B_3 & -K_3 & U_{31} & U_{32} & U_{33} & U_{34} \\ -B_4 & -K_4 & U_{41} & U_{42} & U_{43} & U_{44} \end{vmatrix}. \tag{7}$$

We use $p_3 = p_1$ and $p_4 = p_2$, i.e., amount of a pair of prices are same, and $k_3 = k_1$ and $k_4 = k_2$, i.e., a pair of coupon numbers are same. Now we consider that in the expansion of (7) every term contains $p_1 p_2 k_1 k_2$, then from (7) we can derive [Mohajan & Mohajan, 2022e];

$$|H| = -2 p_1 p_2 k_1 k_2 < 0. \tag{8}$$

For b_1, b_2, b_3, b_4, λ_1, and λ_2 in terms of p_1, p_2, p_3, p_4, k_1, k_2, k_3, k_4, B, and K we can calculate sixty partial derivatives, such as $\dfrac{\partial \lambda_1}{\partial p_1}$, $\dfrac{\partial \lambda_2}{\partial p_1}$, ..., $\dfrac{\partial \lambda_1}{\partial k_1}$, $\dfrac{\partial \lambda_2}{\partial k_1}$, ..., $\dfrac{\partial b_1}{\partial p_1}$, ..., $\dfrac{\partial b_1}{\partial k_1}$, ..., $\dfrac{\partial \lambda_1}{\partial B}$, ..., $\dfrac{\partial \lambda_1}{\partial K}$, etc., [Islam et al., 2011; Mohajan,

2021c]. Now we consider 6×6 Hessian and Jacobian matrix as [Mohajan, 2021b; Mohajan & Mohajan, 2022a];

$$J = H = \begin{vmatrix} 0 & 0 & -B_1 & -B_2 & -B_3 & -B_4 \\ 0 & 0 & -K_1 & -K_2 & -K_3 & -K_4 \\ -B_1 & -K_1 & U_{11} & U_{12} & U_{13} & U_{14} \\ -B_2 & -K_2 & U_{21} & U_{22} & U_{23} & U_{24} \\ -B_3 & -K_3 & U_{31} & U_{32} & U_{33} & U_{34} \\ -B_4 & -K_4 & U_{41} & U_{42} & U_{43} & U_{44} \end{vmatrix} \qquad (9)$$

which is non-singular at the optimum point $\left(b_1^*, b_2^*, b_3^*, b_4^*, \lambda_1^*, \lambda_2^*\right)$. Since the second order conditions have been satisfied, so the determinant of (9) does not vanish at the optimum, i.e., $|J| = |H|$; and we apply the implicit-function theorem.

We have total 16 variables in our study, such as λ_1, λ_2, b_1, b_2, b_3, b_4 p_1, p_2, p_3, p_4, k_1, k_2, k_3, k_4, B, and K. By the implicit function theorem, we can write [Moolio et al., 2009; Islam et al., 2010];

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \mathbf{G}\left(p_1, p_2, p_3, p_4, k_1, k_2, k_3, k_4, B, M\right). \qquad (10)$$

Now the 6×10 Jacobian matrix for $\mathbf{G}$, regarded as J_G is given by [Mohajan, 2021a; Mohajan & Mohajan, 2022a],

$$J_G = \begin{bmatrix} \dfrac{\partial \lambda_1}{\partial p_1} & \dfrac{\partial \lambda_1}{\partial p_2} & \dfrac{\partial \lambda_1}{\partial p_3} & \dfrac{\partial \lambda_1}{\partial p_4} & \dfrac{\partial \lambda_1}{\partial k_1} & \dfrac{\partial \lambda_1}{\partial k_2} & \dfrac{\partial \lambda_1}{\partial k_3} & \dfrac{\partial \lambda_1}{\partial k_4} & \dfrac{\partial \lambda_1}{\partial B} & \dfrac{\partial \lambda_1}{\partial K} \\[2mm] \dfrac{\partial \lambda_2}{\partial p_1} & \dfrac{\partial \lambda_2}{\partial p_2} & \dfrac{\partial \lambda_2}{\partial p_3} & \dfrac{\partial \lambda_2}{\partial p_4} & \dfrac{\partial \lambda_2}{\partial k_1} & \dfrac{\partial \lambda_2}{\partial k_2} & \dfrac{\partial \lambda_2}{\partial k_3} & \dfrac{\partial \lambda_2}{\partial k_4} & \dfrac{\partial \lambda_2}{\partial B} & \dfrac{\partial \lambda_2}{\partial K} \\[2mm] \dfrac{\partial b_1}{\partial p_1} & \dfrac{\partial b_1}{\partial p_2} & \dfrac{\partial b_1}{\partial p_3} & \dfrac{\partial b_1}{\partial p_4} & \dfrac{\partial b_1}{\partial k_1} & \dfrac{\partial b_1}{\partial k_2} & \dfrac{\partial b_1}{\partial k_3} & \dfrac{\partial b_1}{\partial k_4} & \dfrac{\partial b_1}{\partial B} & \dfrac{\partial b_1}{\partial K} \\[2mm] \dfrac{\partial b_2}{\partial p_1} & \dfrac{\partial b_2}{\partial p_2} & \dfrac{\partial b_2}{\partial p_3} & \dfrac{\partial b_2}{\partial p_4} & \dfrac{\partial b_2}{\partial k_1} & \dfrac{\partial b_2}{\partial k_2} & \dfrac{\partial b_2}{\partial k_3} & \dfrac{\partial b_2}{\partial k_4} & \dfrac{\partial b_2}{\partial B} & \dfrac{\partial b_2}{\partial K} \\[2mm] \dfrac{\partial b_3}{\partial p_1} & \dfrac{\partial b_3}{\partial p_2} & \dfrac{\partial b_3}{\partial p_3} & \dfrac{\partial b_3}{\partial p_4} & \dfrac{\partial b_3}{\partial k_1} & \dfrac{\partial b_3}{\partial k_2} & \dfrac{\partial b_3}{\partial k_3} & \dfrac{\partial b_3}{\partial k_4} & \dfrac{\partial b_3}{\partial B} & \dfrac{\partial b_3}{\partial K} \\[2mm] \dfrac{\partial b_4}{\partial p_1} & \dfrac{\partial b_4}{\partial p_2} & \dfrac{\partial b_4}{\partial p_3} & \dfrac{\partial b_4}{\partial p_4} & \dfrac{\partial b_4}{\partial k_1} & \dfrac{\partial b_4}{\partial k_2} & \dfrac{\partial b_4}{\partial k_3} & \dfrac{\partial b_4}{\partial k_4} & \dfrac{\partial b_4}{\partial B} & \dfrac{\partial b_4}{\partial K} \end{bmatrix}.$$

(11)

$$= -J^{-1} \begin{bmatrix} -b_1 & -b_2 & -b_3 & -b_4 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -b_1 & -b_2 & -b_3 & -b_4 & 0 & 1 \\ -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 \end{bmatrix}.$$

(12)

The inverse of Jacobian matrix is, $J^{-1} = \dfrac{1}{|J|} C^T$, where $C = \left(C_{ij}\right)$, the matrix of cofactors of J, and T indicates transpose, then (12) becomes [Mohajan, 2017a; Islam et al., 2009b, 2011],

$$
J_G = -\frac{1}{|J|} C^T \begin{bmatrix}
-b_1 & -b_2 & -b_3 & -b_4 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -b_1 & -b_2 & -b_3 & -b_4 & 0 & 1 \\
-\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & 0 \\
0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 \\
0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 \\
0 & 0 & 0 & -\lambda_1 & 0 & 0 & 0 & -\lambda_2 & 0 & 0
\end{bmatrix}.
$$

$$(13)$$

Now 6×6 transpose matrix C^T can be represented by,

$$
C^T = \begin{bmatrix}
C_{11} & C_{21} & C_{31} & C_{41} & C_{51} & C_{61} \\
C_{12} & C_{22} & C_{32} & C_{42} & C_{52} & C_{62} \\
C_{13} & C_{23} & C_{33} & C_{43} & C_{53} & C_{63} \\
C_{14} & C_{24} & C_{34} & C_{44} & C_{54} & C_{64} \\
C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{65} \\
C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}
\end{bmatrix}.
$$

$$(14)$$

Using (14) we can write (11) as a 6×10 Jacobian matrix [Mohajan & Mohajan, 2022b];

$$
J_G = -\frac{1}{|J|} \begin{bmatrix}
-b_1 C_{11} - \lambda_1 C_{31} & -b_2 C_{11} - \lambda_1 C_{41} & -b_3 C_{11} - \lambda_1 C_{51} & -b_4 C_{11} - \lambda_1 C_{61} & -b_1 C_{21} - \lambda_2 C_{31} \\
-b_1 C_{12} - \lambda_1 C_{32} & -b_2 C_{12} - \lambda_1 C_{42} & -b_3 C_{12} - \lambda_1 C_{52} & -b_4 C_{12} - \lambda_1 C_{62} & -b_1 C_{22} - \lambda_2 C_{32} \\
-b_1 C_{13} - \lambda_1 C_{33} & -b_2 C_{13} - \lambda_1 C_{43} & -b_3 C_{13} - \lambda_1 C_{53} & -b_4 C_{13} - \lambda_1 C_{63} & -b_1 C_{23} - \lambda_2 C_{33} \\
-b_1 C_{14} - \lambda_1 C_{34} & -b_2 C_{14} - \lambda_1 C_{44} & -b_3 C_{14} - \lambda_1 C_{54} & -b_4 C_{14} - \lambda_1 C_{64} & -b_1 C_{24} - \lambda_2 C_{34} \\
-b_1 C_{15} - \lambda_1 C_{35} & -b_2 C_{15} - \lambda_1 C_{45} & -b_3 C_{15} - \lambda_1 C_{55} & -b_4 C_{15} - \lambda_1 C_{65} & -b_1 C_{25} - \lambda_2 C_{35} \\
-b_1 C_{16} - \lambda_1 C_{36} & -b_2 C_{16} - \lambda_1 C_{46} & -b_3 C_{16} - \lambda_1 C_{56} & -b_4 C_{16} - \lambda_1 C_{66} & -b_1 C_{26} - \lambda_2 C_{36}
\end{bmatrix}
$$

$$
\left.
\begin{array}{ccccc}
-b_2 C_{21} - \lambda_2 C_{41} & -b_3 C_{21} - \lambda_2 C_{51} & -b_4 C_{21} - \lambda_2 C_{61} & C_{11} & C_{21} \\
-b_2 C_{22} - \lambda_2 C_{42} & -b_3 C_{32} - \lambda_2 C_{52} & -b_4 C_{22} - \lambda_2 C_{62} & C_{12} & C_{22} \\
-b_2 C_{23} - \lambda_2 C_{43} & -b_3 C_{33} - \lambda_2 C_{53} & -b_4 C_{23} - \lambda_2 C_{63} & C_{13} & C_{23} \\
-b_2 C_{24} - \lambda_2 C_{44} & -b_3 C_{34} - \lambda_2 C_{54} & -b_4 C_{24} - \lambda_2 C_{64} & C_{14} & C_{24} \\
-b_2 C_{25} - \lambda_2 C_{45} & -b_3 C_{35} - \lambda_2 C_{55} & -b_4 C_{25} - \lambda_2 C_{65} & C_{15} & C_{25} \\
-b_2 C_{26} - \lambda_2 C_{46} & -b_3 C_{36} - \lambda_2 C_{56} & -b_4 C_{26} - \lambda_2 C_{66} & C_{16} & C_{26}
\end{array}
\right] .
\tag{15}
$$

Now we analyze the nature of Lagrange multiplier λ_1 when total budget B of the consumers increases. Taking T_{19}, (i.e., term of 1st row and 9th column) from both sides of (15) we get [Islam et al., 2011; Mohajan & Mohajan, 2022e],

$$
\frac{\partial \lambda_1}{\partial B} = -\frac{1}{|J|}[C_{11}]
$$

$$
= -\frac{1}{|J|}\text{Cofactor of } C_{11}
$$

$$
= -\frac{1}{|J|}
\begin{vmatrix}
0 & -K_1 & -K_2 & -K_3 & -K_4 \\
-K_1 & U_{11} & U_{12} & U_{13} & U_{14} \\
-K_2 & U_{21} & U_{22} & U_{23} & U_{24} \\
-K_3 & U_{31} & U_{32} & U_{33} & U_{34} \\
-K_4 & U_{41} & U_{42} & U_{43} & U_{44}
\end{vmatrix}
$$

$$
= -\frac{1}{|J|}\left\{ K_1 \begin{vmatrix} -K_1 & U_{12} & U_{13} & U_{14} \\ -K_2 & U_{22} & U_{23} & U_{24} \\ -K_3 & U_{32} & U_{33} & U_{34} \\ -K_4 & U_{42} & U_{43} & U_{44} \end{vmatrix} - K_2 \begin{vmatrix} -K_1 & U_{11} & U_{13} & U_{14} \\ -K_2 & U_{21} & U_{23} & U_{24} \\ -K_3 & U_{31} & U_{33} & U_{34} \\ -K_4 & U_{41} & U_{43} & U_{44} \end{vmatrix} \right.
$$

$$
\left. + K_3 \begin{vmatrix} -K_1 & U_{11} & U_{12} & U_{14} \\ -K_2 & U_{21} & U_{22} & U_{24} \\ -K_3 & U_{31} & U_{32} & U_{34} \\ -K_4 & U_{41} & U_{42} & U_{44} \end{vmatrix} - K_4 \begin{vmatrix} -K_1 & U_{11} & U_{12} & U_{13} \\ -K_2 & U_{21} & U_{22} & U_{23} \\ -K_3 & U_{31} & U_{32} & U_{33} \\ -K_4 & U_{41} & U_{42} & U_{43} \end{vmatrix} \right\}
$$

$$
= -\frac{1}{|J|}\left[K_1 \left\{ -K_1 \begin{vmatrix} U_{22} & U_{23} & U_{24} \\ U_{32} & U_{33} & U_{34} \\ U_{42} & U_{43} & U_{44} \end{vmatrix} - U_{12} \begin{vmatrix} -K_2 & U_{23} & U_{24} \\ -K_3 & U_{33} & U_{34} \\ -K_4 & U_{43} & U_{44} \end{vmatrix} + U_{13} \begin{vmatrix} -K_2 & U_{22} & U_{24} \\ -K_3 & U_{32} & U_{34} \\ -K_4 & U_{42} & U_{44} \end{vmatrix} \right. \right.
$$

$$
\left. - U_{14} \begin{vmatrix} -K_2 & U_{22} & U_{23} \\ -K_3 & U_{32} & U_{33} \\ -K_4 & U_{42} & U_{43} \end{vmatrix} \right\} - K_2 \left\{ -K_1 \begin{vmatrix} U_{21} & U_{23} & U_{24} \\ U_{31} & U_{33} & U_{34} \\ U_{41} & U_{43} & U_{44} \end{vmatrix} + U_{13} \begin{vmatrix} -K_2 & U_{21} & U_{24} \\ -K_3 & U_{31} & U_{34} \\ -K_4 & U_{41} & U_{44} \end{vmatrix} \right.
$$

$$
\left. - U_{14} \begin{vmatrix} -K_2 & U_{21} & U_{23} \\ -K_3 & U_{31} & U_{33} \\ -K_4 & U_{41} & U_{43} \end{vmatrix} \right\} + K_3 \left\{ -K_1 \begin{vmatrix} U_{21} & U_{22} & U_{24} \\ U_{31} & U_{32} & U_{34} \\ U_{41} & U_{42} & U_{44} \end{vmatrix} + U_{12} \begin{vmatrix} -K_2 & U_{21} & U_{24} \\ -K_3 & U_{31} & U_{34} \\ -K_4 & U_{41} & U_{44} \end{vmatrix} \right.
$$

$$
\left. - U_{14} \begin{vmatrix} -K_2 & U_{21} & U_{22} \\ -K_3 & U_{31} & U_{32} \\ -K_4 & U_{41} & U_{42} \end{vmatrix} \right\} - K_4 \left\{ -K_1 \begin{vmatrix} U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \\ U_{41} & U_{42} & U_{43} \end{vmatrix} + U_{12} \begin{vmatrix} -K_2 & U_{21} & U_{23} \\ -K_3 & U_{31} & U_{33} \\ -K_4 & U_{41} & U_{43} \end{vmatrix} \right.
$$

$$
\left. \left. - U_{13} \begin{vmatrix} -K_2 & U_{21} & U_{22} \\ -K_3 & U_{31} & U_{32} \\ -K_4 & U_{41} & U_{42} \end{vmatrix} \right\} \right]
$$

176

$$= -\frac{1}{|J|}\Big\{ -K_1^2 U_{23} U_{24} U_{34} \quad -K_1^2 U_{23} U_{24} U_{34} \quad -K_1 K_2 U_{12} U_{34}^2 \quad +K_1 K_4 U_{12} U_{23} U_{34}$$

$$+K_1 K_3 U_{12} U_{24} U_{34} \quad +K_1 K_2 U_{13} U_{24} U_{34} \quad -K_1 K_3 U_{13} U_{24}^2 \quad +K_1 K_4 U_{13} U_{23} U_{24}$$

$$+K_1 K_2 U_{14} U_{23} U_{34} \quad -K_1 K_3 U_{14} U_{23} U_{24} \quad -K_1 K_4 U_{14} U_{23}^2 \quad -K_1 K_2 U_{12} U_{14} U_{34}$$

$$+K_1 K_2 U_{14} U_{23} U_{34} \quad +K_1 K_2 U_{13} U_{24} U_{34} \quad -K_2^2 U_{13} U_{14} U_{34} \quad +K_2 K_4 U_{12} U_{13} U_{34}$$

$$+K_2 K_3 U_{13} U_{14} U_{24} \quad -K_2 K_4 U_{24} U_{13}^2 \quad -K_2^2 U_{13} U_{14} U_{34} \quad +K_2 K_3 U_{12} U_{14} U_{34}$$

$$-K_2 K_3 U_{23} U_{14}^2 \quad +K_2 K_4 U_{13} U_{14} U_{23} \quad +K_1 K_3 U_{12} U_{24} U_{34} \quad -K_1 K_3 U_{13} U_{24}^2$$

$$+K_1 K_3 U_{14} U_{23} U_{24} \quad +K_2 K_3 U_{12} U_{14} U_{34} \quad -K_3 K_4 U_{34} U_{12}^2 \quad -K_3^2 U_{12} U_{14} U_{24}$$

$$+K_3 K_4 U_{12} U_{13} U_{24} \quad +K_1 K_4 U_{12} U_{23} U_{34} \quad +K_1 K_4 U_{13} U_{23} U_{24} \quad -K_1 K_4 U_{14} U_{23}^2$$

$$+K_2 K_4 U_{12} U_{13} U_{34} - K_3 K_4 U_{34} U_{12}^2$$

$$+K_3 K_4 U_{12} U_{14} U_{23} \quad -K_4^2 U_{12} U_{13} U_{23} \quad -K_2 K_4 U_{24} U_{13}^2 \quad +K_2 K_4 U_{13} U_{14} U_{23}$$

$$+K_3 K_4 U_{12} U_{13} U_{24}$$

$$-K_4^2 U_{12} U_{13} U_{23} \Big\}$$

$$= -\frac{1}{|J|}\Big\{ -2k_1^2 b_1^3 b_2 b_3 b_4 \quad -k_1 k_2 b_1^2 b_2^2 b_3 b_4 \quad +k_1 k_4 b_1^2 b_2 b_3 b_4^2 \quad +k_1 k_3 b_1^2 b_2 b_3^2 b_4$$

$$-k_1 k_2 b_1^2 b_2^2 b_3 b_4 \quad -k_1 k_3 b_1^2 b_2 b_3^2 b_4 \quad +k_1 k_4 b_1^2 b_2 b_3 b_4^2 \quad +k_1 k_2 b_1^2 b_2^2 b_3 b_4 \quad -k_1 k_3 b_1^2 b_2 b_3^2 b_4$$

$$-k_1 k_4 b_1^2 b_2 b_3 b_4^2 \quad -k_1 k_2 b_1 b_2^2 b_3^2 b_4 \quad +k_1 k_2 b_1^2 b_2^2 b_3 b_4 \quad +k_1 k_2 b_1^2 b_2^2 b_3 b_4 \quad -k_2^2 b_1 b_2^3 b_3 b_4$$

$$+k_2 k_4 b_1 b_2^2 b_3 b_4^2 + k_2 k_3 b_1 b_2^2 b_3^2 b_4 - k_2 k_4 b_1 b_2^2 b_3 b_4^2 - k_2^2 b_1 b_2^3 b_3 b_4 + k_2 k_3 b_1 b_2^2 b_3^2 b_4$$

$$-k_2 k_3 b_1 b_2^2 b_3^2 b_4 \quad +k_2 k_4 b_1 b_2^2 b_3 b_4^2 \quad +k_1 k_3 b_1^2 b_2 b_3^2 b_4 \quad -k_1 k_3 b_1^2 b_2 b_3^2 b_4 \quad +k_1 k_3 b_1^2 b_2 b_3^2 b_4$$

$$+k_2 k_3 b_1 b_2^2 b_3^2 b_4 \quad -k_3 k_4 b_1 b_2 b_3^2 b_4^2 \quad -k_3^2 b_1 b_2 b_3^3 b_4 \quad +k_3 k_4 b_1 b_2 b_3^2 b_4^2 \quad +k_1 k_4 b_1^2 b_2 b_3 b_4^2$$

$$+k_1 k_4 b_1^2 b_2 b_3 b_4^2 \quad -k_1 k_4 b_1^2 b_2 b_3 b_4^2 \quad +k_2 k_4 b_1 b_2^2 b_3 b_4^2 \quad -k_3 k_4 b_1 b_2 b_3^2 b_4^2 \quad +k_3 k_4 b_1 b_2 b_3^2 b_4^2$$

$$-k_4^2 b_1 b_2 b_3 b_4^3 - k_2 k_4 b_1 b_2^2 b_3 b_4^2 + k_2 k_4 b_1 b_2^2 b_3 b_4^2 + k_3 k_4 b_1 b_2 b_3^2 b_4^2 - k_4^2 b_1 b_2 b_3 b_4^3 \Big\}$$

$$\frac{\partial \lambda_1}{\partial B} = -\frac{1}{|J|}\left\{-2k_1^2 b_1^3 b_2 b_3 b_4 \qquad -2k_2^2 b_1 b_2^3 b_3 b_4 \qquad -k_3^2 b_1 b_2 b_3^3 b_4 \qquad -2k_4^2 b_1 b_2 b_3 b_4^3 \right.$$

$$+2k_1 k_4 b_1^2 b_2 b_3 b_4^2 + 2k_2 k_3 b_1 b_2^2 b_3^2 b_4$$

$$\left. +2k_2 k_4 b_1 b_2^2 b_3 b_4^2 + k_3 k_4 b_1 b_2 b_3^2 b_4^2 \right\}. \tag{16}$$

Using $b_1 = b_2 = b_3 = b_4 = 1$ in (16) we get,

$$\frac{\partial \lambda_1}{\partial B} = -\frac{1}{|J|}\left\{-2k_1^2 - 2k_2^2 - k_3^2 - 2k_4^2 + 2k_1 k_4 + 2k_2 k_3 + 2k_2 k_4 + 2k_3 k_4 \right\}. \tag{17}$$

Using $k_3 = k_1$ and $k_4 = k_2$ in (17) we get,

$$\frac{\partial \lambda_1}{\partial B} = (k_1 - k_2)(3k_1 - 2k_2). \tag{18}$$

If $k_1 < \frac{2}{3}k_2$ or $k_1 > k_2$ in (18) we get,

$$\frac{\partial \lambda_1}{\partial B} > 0. \tag{19}$$

Inequality (19) indicates that if the total budget of the consumers' increases, the level of marginal utility will also increase. Therefore, in this situation the consumers will collect the coupons such that, $k_1 > k_2$ or $k_1 < \frac{2}{3}k_2$. Depending on the consumers' demand, the organization should take attempts to increase the production level.

If $\frac{2}{3}k_2 < k_1 < k_2$ in (18) we get,

$$\frac{\partial \lambda_1}{\partial B} < 0. \tag{20}$$

Inequality (20) indicates that if the total budget of the consumers' increases, the level of marginal utility will decrease. Therefore, in this situation the consumers will collect the coupons such that, $\frac{2}{3}k_2 < k_1 < k_2$. Depending on the consumers' demand, the organization should take attempts to decrease the production level.

In this study we observe that, $\dfrac{\partial \lambda_1}{\partial B} \neq 0$. Therefore, from (18) we see that, $k_1 \neq k_2$ and $k_1 \neq \dfrac{2}{3} k_2$; consequently, also $k_3 \neq k_4$ and $k_3 \neq \dfrac{2}{3} k_4$ in this model.

Now we analyze the nature of Lagrange multiplier λ_2 when total budget B of the consumers increases. Taking T_{29}, (i.e., term of 2nd row and 9th column) from both sides of (15) we get [Islam et al., 2010; Mohajan & Mohajan, 2022e],

$$\frac{\partial \lambda_2}{\partial B} = -\frac{1}{|J|}\left[C_{12}\right]$$

$$= -\frac{1}{|J|}\text{Cofactor of } C_{12}$$

$$= \frac{1}{|J|}\begin{vmatrix} 0 & -K_1 & -K_2 & -K_3 & -K_4 \\ -B_1 & U_{11} & U_{12} & U_{13} & U_{14} \\ -B_2 & U_{21} & U_{22} & U_{23} & U_{24} \\ -B_3 & U_{31} & U_{32} & U_{33} & U_{34} \\ -B_4 & U_{41} & U_{42} & U_{43} & U_{44} \end{vmatrix}$$

$$= \frac{1}{|J|}\left\{ K_1 \begin{vmatrix} -B_1 & U_{12} & U_{13} & U_{14} \\ -B_2 & U_{22} & U_{23} & U_{24} \\ -B_3 & U_{32} & U_{33} & U_{34} \\ -B_4 & U_{42} & U_{43} & U_{44} \end{vmatrix} - K_2 \begin{vmatrix} -B_1 & U_{11} & U_{13} & U_{14} \\ -B_2 & U_{21} & U_{23} & U_{24} \\ -B_3 & U_{31} & U_{33} & U_{34} \\ -B_4 & U_{41} & U_{43} & U_{44} \end{vmatrix} \right.$$

$$\left. + K_3 \begin{vmatrix} -B_1 & U_{11} & U_{12} & U_{14} \\ -B_2 & U_{21} & U_{22} & U_{24} \\ -B_3 & U_{31} & U_{32} & U_{34} \\ -B_4 & U_{41} & U_{42} & U_{44} \end{vmatrix} - K_4 \begin{vmatrix} -B_1 & U_{11} & U_{12} & U_{13} \\ -B_2 & U_{21} & U_{22} & U_{23} \\ -B_3 & U_{31} & U_{32} & U_{33} \\ -B_4 & U_{41} & U_{42} & U_{43} \end{vmatrix} \right\}$$

$$= \frac{1}{|J|} \left[K_1 \left\{ -B_1 \begin{vmatrix} U_{22} & U_{23} & U_{24} \\ U_{32} & U_{33} & U_{34} \\ U_{42} & U_{43} & U_{44} \end{vmatrix} - U_{12} \begin{vmatrix} -B_2 & U_{23} & U_{24} \\ -B_3 & U_{33} & U_{34} \\ -B_4 & U_{43} & U_{44} \end{vmatrix} + U_{13} \begin{vmatrix} -B_2 & U_{22} & U_{24} \\ -B_3 & U_{32} & U_{34} \\ -B_4 & U_{42} & U_{44} \end{vmatrix} \right. \right.$$

$$\left. -U_{14} \begin{vmatrix} -B_2 & U_{22} & U_{23} \\ -B_3 & U_{32} & U_{33} \\ -B_4 & U_{42} & U_{43} \end{vmatrix} \right\}$$

$$-K_2 \left\{ -B_1 \begin{vmatrix} U_{21} & U_{23} & U_{24} \\ U_{31} & U_{33} & U_{34} \\ U_{41} & U_{43} & U_{44} \end{vmatrix} + U_{13} \begin{vmatrix} -B_2 & U_{21} & U_{24} \\ -B_3 & U_{31} & U_{34} \\ -B_4 & U_{41} & U_{44} \end{vmatrix} - U_{14} \begin{vmatrix} -B_2 & U_{21} & U_{23} \\ -B_3 & U_{31} & U_{33} \\ -B_4 & U_{41} & U_{43} \end{vmatrix} \right\}$$

$$+K_3 \left\{ -B_1 \begin{vmatrix} U_{21} & U_{22} & U_{24} \\ U_{31} & U_{32} & U_{34} \\ U_{41} & U_{42} & U_{44} \end{vmatrix} + U_{12} \begin{vmatrix} -B_2 & U_{21} & U_{24} \\ -B_3 & U_{31} & U_{34} \\ -B_4 & U_{41} & U_{44} \end{vmatrix} - U_{14} \begin{vmatrix} -B_2 & U_{21} & U_{22} \\ -B_3 & U_{31} & U_{32} \\ -B_4 & U_{41} & U_{42} \end{vmatrix} \right\}$$

$$\left. -K_4 \left\{ -B_1 \begin{vmatrix} U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \\ U_{41} & U_{42} & U_{43} \end{vmatrix} + U_{12} \begin{vmatrix} -B_2 & U_{21} & U_{23} \\ -B_3 & U_{31} & U_{33} \\ -B_4 & U_{41} & U_{43} \end{vmatrix} - U_{13} \begin{vmatrix} -B_2 & U_{21} & U_{22} \\ -B_3 & U_{31} & U_{32} \\ -B_4 & U_{41} & U_{42} \end{vmatrix} \right\} \right]$$

$$= \frac{1}{|J|} \left\{ -2B_1 K_1 U_{23} U_{24} U_{34} \quad -B_2 K_1 U_{12} U_{34}^2 \quad + B_4 K_1 U_{12} U_{23} U_{34} \quad + B_3 K_1 U_{12} U_{24} U_{34} \right.$$

$$+ B_2 K_1 U_{13} U_{24} U_{34} \quad - B_3 K_1 U_{13} U_{24}^2 \quad + B_4 K_1 U_{13} U_{23} U_{24} \quad + B_2 K_1 U_{14} U_{23} U_{34}$$

$$+ B_3 K_1 U_{14} U_{23} U_{24} \quad - B_4 K_1 U_{14} U_{23}^2 \quad - B_1 K_2 U_{12} U_{34}^2 \quad + B_1 K_2 U_{14} U_{23} U_{34}$$

$$+ B_1 K_2 U_{13} U_{24} U_{34} \quad - B_2 K_2 U_{13} U_{14} U_{34} \quad - B_4 K_2 U_{12} U_{13} U_{34} \quad + B_3 K_2 U_{13} U_{14} U_{24}$$

$$- B_4 K_2 U_{13}^2 U_{24} \quad - B_2 K_2 U_{13} U_{14} U_{34} \quad + B_3 K_2 U_{12} U_{14} U_{34} \quad - B_3 K_2 U_{14}^2 U_{23}$$

$$+ B_4 K_2 U_{13} U_{14} U_{23} \quad - B_1 K_3 U_{12} U_{24} U_{34} \quad - B_1 K_3 U_{13} U_{24}^2 \quad + B_1 K_3 U_{14} U_{23} U_{24}$$

$$+ B_2 K_3 U_{12} U_{14} U_{34} - B_4 K_3 U_{12}^2 U_{34} - B_3 K_3 U_{12} U_{14} U_{24} + B_4 K_3 U_{12} U_{13} U_{24}$$

$$+ B_1 K_4 U_{12} U_{23} U_{34} \quad + B_1 K_4 U_{13} U_{23} U_{24} \quad - B_1 K_4 U_{14} U_{23}^2 \quad + B_2 K_4 U_{12} U_{13} U_{34}$$

$$- B_3 K_4 U_{12}^2 U_{34} + B_3 K_4 U_{12} U_{14} U_{23}$$

$$- B_4 K_4 U_{12} U_{13} U_{23} \quad - B_2 K_4 U_{13}^2 U_{24} \quad + B_2 K_4 U_{13} U_{14} U_{23} \quad + B_3 K_4 U_{12} U_{13} U_{24}$$

$$- B_4 K_4 U_{12} U_{13} U_{23} \}$$

$$= -\frac{1}{|J|} \{-2 p_1 k_1 b_1^3 b_2 b_3 b_4 \quad - p_2 k_1 b_1^2 b_2^2 b_3 b_4 \quad + p_4 k_1 b_1^2 b_2 b_3 b_4^2 \quad + p_3 k_1 b_1^2 b_2 b_3^2 b_4$$

$$+ p_3 k_1 b_1^2 b_2^2 b_3 b_4 - p_3 k_1 b_1^2 b_2 b_3^2 b_4 + p_4 k_1 b_1^2 b_2 b_3 b_4^2 + p_2 k_1 b_1^2 b_2^2 b_3 b_4 + p_3 k_1 b_1^2 b_2 b_3^2 b_4$$

$$- p_4 k_1 b_1^2 b_2 b_3 b_4^2 - p_1 k_2 b_1^2 b_2^2 b_3 b_4 + 2 p_1 k_2 b_1^2 b_2^2 b_3 b_4 - p_2 k_2 b_1 b_2^3 b_3 b_4 - p_4 k_2 b_1 b_2^2 b_3 b_4^2$$

$$+ p_3 k_2 b_1 b_2^2 b_3^2 b_4 - p_4 k_2 b_1 b_2^2 b_3 b_4^2 - p_2 k_2 b_1 b_2^3 b_3 b_4 + p_3 k_2 b_1 b_2^2 b_3^2 b_4 - p_3 k_2 b_1 b_2^2 b_3^2 b_4$$

$$+ p_4 k_2 b_1 b_2^2 b_3 b_4^2 - p_1 k_3 b_1^2 b_2 b_3^2 b_4 - p_1 k_3 b_1^2 b_2 b_3^2 b_4 + p_1 k_3 b_1^2 b_2 b_3^2 b_4 + p_2 k_3 b_1 b_2^2 b_3^2 b_4$$

$$- p_4 k_3 b_1 b_2 b_3^2 b_4^2 - p_3 k_3 b_1 b_2 b_3^3 b_4 + p_4 k_3 b_1 b_2 b_3^2 b_4^2 + p_1 k_4 b_1^2 b_2 b_3 b_4^2 + p_1 k_4 b_1^2 b_2 b_3 b_4^2$$

$$- p_1 k_4 b_1^2 b_2 b_3 b_4^2 + p_2 k_4 b_1 b_2^2 b_3 b_4^2 - p_3 k_4 b_1 b_2 b_3^2 b_4^2 + p_3 k_4 b_1 b_2 b_3^2 b_4^2 - p_4 k_4 b_1 b_2 b_3 b_4^3$$

$$- p_2 k_4 b_1 b_2^2 b_3 b_4^2 + p_2 k_4 b_1 b_2^2 b_3 b_4^2 + p_3 k_4 b_1 b_2 b_3^2 b_4^2 - p_4 k_4 b_1 b_2 b_3 b_4^3 \}$$

$$\frac{\partial \lambda_2}{\partial B} = -\frac{1}{|J|} \{-2 p_1 k_1 b_1^3 b_2 b_3 b_4 \quad - 2 p_2 k_2 b_1 b_2^3 b_3 b_4 \quad - p_3 k_3 b_1 b_2 b_3^3 b_4 \quad - 2 p_4 k_4 b_1 b_2 b_3 b_4^3$$

$$+ (p_4 k_1 + p_1 k_4) b_1^2 b_2 b_3 b_4^2 + p_1 k_2 b_1^2 b_2^2 b_3 b_4 + p_3 k_1 b_1^2 b_2 b_3^2 b_4 + (p_3 k_2 + p_2 k_3) b_1 b_2^2 b_3^2 b_4$$

$$+ p_3 k_4 b_1 b_2 b_3^2 b_4^2 \}. \tag{21}$$

$$\frac{\partial \lambda_2}{\partial B} = -\frac{b_1 b_2 b_3 b_4}{|J|} \{-2 p_1 k_1 b_1^2 \ - 2 p_2 k_2 b_2^2 - p_3 k_3 b_3^2 - 2 p_4 k_4 b_4^2 + (p_4 k_1 + p_1 k_4) b_1 b_4$$

$$+ p_1 k_2 b_1 b_2 + p_3 k_1 b_1 b_3$$

$$+ (p_3 k_2 + p_2 k_3) b_2 b_3 + p_3 k_4 b_3 b_4 \}. \tag{22}$$

Now we use $p_3 = p_1$, and $p_4 = p_2$ where pair of prices are same, and $k_3 = k_1$, and $k_4 = k_2$, i.e., two types of coupon numbers are same, $|J| = |H| = -2 p_1 p_2 k_1 k_2$. Now we use $b_3 = b_1$, and $b_4 = b_2$, then (22) becomes;

$$\frac{\partial \lambda_2}{\partial B} = \frac{b_1^2 b_2^2}{2 p_1 p_2 k_1 k_2}\left\{- 2 p_1 k_1 b_1^2 - 4 p_2 k_2 b_2^2 + 2\left(p_2 k_1 + 2 p_1 k_2\right)b_1 b_2\right\}. \quad (23)$$

We put $b_1 = b_2 = 1$ then (23) becomes [Mohajan & Mohajan, 2022b];

$$\frac{\partial \lambda_2}{\partial B} = \frac{1}{2 p_1 p_2 k_1 k_2}\left\{- 2 p_1 k_1 - 4 p_2 k_2 + 2 p_2 k_1 + 4 p_1 k_2\right\}. \quad (24)$$

Now we use, $k_1 = k_2 = k$ in (24), and then we get,

$$\frac{\partial \lambda_2}{\partial B} = \frac{1}{p_1 p_2 k}\left(p_1 - p_2\right). \quad (25)$$

where $p_1, p_2, k > 0$. Now if $p_1 > p_2$ in (25) we get,

$$\frac{\partial \lambda_2}{\partial B} > 0. \quad (26)$$

Inequality (26) indicates that if the total budget of the consumers' increases, the level of marginal utility will also increase. Therefore, in this situation the consumers will find that, $p_1 > p_2$, in the commodity market. The organization should take attempts to increase the production level, depending on the consumers' demand.

Now if $p_1 < p_2$ in (25) we get,

$$\frac{\partial \lambda_2}{\partial B} < 0. \quad (27)$$

Inequality (27) indicates that if the total budget of the consumers' increases, the level of marginal utility will decrease. Therefore, in this situation the consumers will find that, $p_1 < p_2$, in the commodity market. The organization should take attempts to decrease the production level, depending on the consumers' demand.

From (25) we see that, $\frac{\partial \lambda_2}{\partial B} \neq 0$, so that, $p_1 \neq p_2$, i.e., the prices of two commodities b_1 and b_2 are not equal. It seems that these are different goods.

6. Conclusions

In this study we have tried to discuss sensitivity analysis between Lagrange multipliers and total budget during utility maximization investigation. We have applied four commodity variables and we have tried to run the mathematical calculations efficiently using two constraints: budget constraint and coupon constraint. In this study we have observed that the Lagrange multipliers are very useful both for the consumers and producers.

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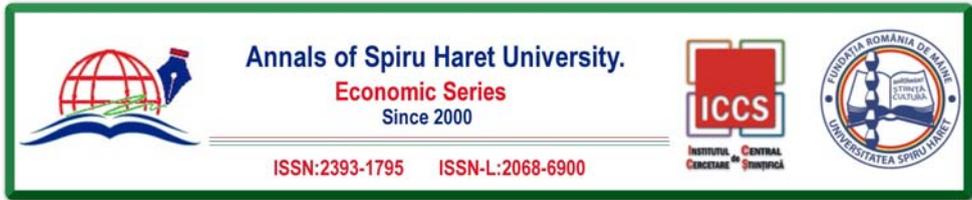
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