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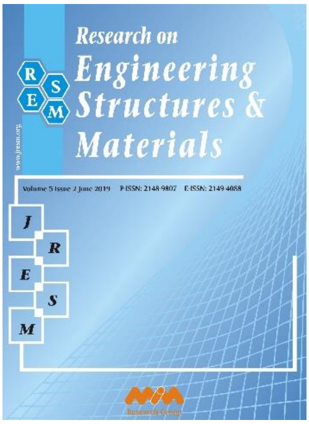

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Research Article

Dynamic and buckling of functionally graded beams based on a homogenization theory

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Abstract

In this work, the free vibration and the stability problems of functionally graded beams are analysed via the Timoshenko theory through the Navier procedure and via an appropriated finite element (FE) approach. In particular, it is shown how the definition of homogenized/generalized displacements allows to uncouple boundary conditions, obtaining a remarkable advantage in terms of computational effort. Moreover, a unified expression capable to express the buckling load for different constrain conditions is discovered. The latter may be considered the natural extension of the Euler's one derived in the century XVIII. In order to verify the reliability of the proposed method, natural frequencies, buckling loads and static displacements of differently constrained beams are numerically evaluated.

1. Introduction

Since the beginning of the actual century, there has been a considerable interest in the study of functionally graded materials (FGMs) due to their ability to satisfy the increasing demands in modern technologies. This class of composites is formed by varying percentage content of materials in any desired direction and, consequently, it owns properties that vary gradually with respect to the spatial coordinates. Compared with the traditional laminated composite materials, the FGMs have no interfaces of material property, so that the phenomena of stress concentration can be reduced greatly. The literature on the topic shows several studies on the structures in both the civil and mechanical engineering. Among these papers, for example, a three dimensional solution for the problem of transversely loaded, all around clamped, rectangular plates, within the linear, small deformation theory of elasticity, is presented by Elishakoff & Gentilini [1]. Theoretical and numerical formulations based on the third-order deformation plate are developed by Reddy [2]. Huang & Shen [3] treat the nonlinear vibration and dynamic response of FGM plates, taking into account conduction and temperature-dependent material properties. Referring to the modelling of functionally graded beams (FGBs), Reddy [4] applies couple stress theories, introduced by Eringen [5] and developed by Yang et al. [6], to describe micro-mono-dimensional structures. Most of the work mentioned have considered variation of the material properties in the thickness direction. A relatively small number of researchers consider the variation of the material properties along the axial direction [7],[8]. Moreover, Zhu & Sankar [9] consider the Euler beam theory in the special case of Young modulus varying following a polynomial law in the thickness direction. Although the presence of papers showing the application of new formulations, only a small number

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of them aims to simplify the solution procedures for the FGBs under the Euler-Bernoulli and Timoshenko hypotheses.

In Euler-Bernoulli beam theory, cross sections, perpendicular to the neutral axis before the bending, remain perpendicular to the neutral axis even after the bending. This model is suitable for slender beams and lower modes of vibration, while it is inadequate to characterize the response of short beams, due to lower shear rigidity. To overcome this draw back, the Timoshenko beam theory is largely applied. A useful review of the studies on shear deformable beams and plates can be found in the book by Wang & al. [10]. Furthermore, the work made by Li & al. [11] is also interesting. They show a unified approach for analysing static and dynamic behaviour of both Euler-Bernoulli and Timoshenko FGBs introducing a new variable linked to vertical and rotational degrees of freedom of beams. This type of technique is widely used in literature and it leads to physical speculations based on not local theories [12] and computational advantageous methods [13]. With the same philosophy, Falsone G. & La Valle G. [14] show a new kind of approach that lies on the concept of homogenization of the general beam cross-section and on the introduction of generalized quantities with the aim of simplifying differential equations governing the correspondent elastic equilibrium problem.

The goal of this work is to apply homogenized/generalized displacements, introduced in the last cited paper in static conditions, for dynamic problems and buckling problems. Moreover, a FE formulation, based on the use of the homogenized/generalized displacements, is shown for the Timoshenko FGBs. The paper is organized as follows: first the differential equations governing dynamic equilibrium of FGBs, in terms of homogenized/generalized axial and transversal displacements, are derived. Then, the above cited FE formulation is shown in the details. In section 5, the buckling problem is treated, showing how the use of the above cited displacement leads to a very simple formulation, able to give some exact solutions. Lastly, in section 6, some simple numerical examples are shown to verify the reliability of the reported approach.

2. Preliminary Concepts

In this section, some preliminary known concepts are given in order to introduce the notations and the formulations that will be used in the remaining part of the paper.

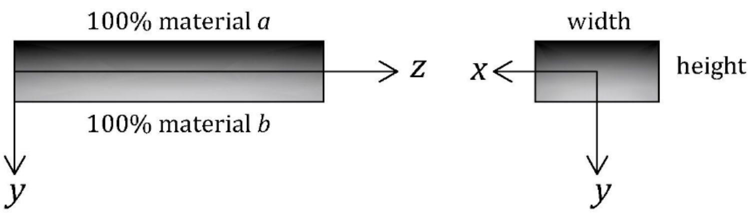

Fig. 1 Geometry of a functionally graded beam

The beams are referred to a Cartesian coordinate system (O; x, y, z) with the origin O placed in the geometrical centre of an extreme section; the z-axis coincides with the beam axis, while the y-axis coincides with the section principal axis along the thickness. The Young modulus is assumed changing along the thickness in a generic way.

2.1. Hamilton's Principle for Timoshenko FGBs.

The Timoshenko theory is based on the following assumptions in the displacement field (see [15], [16], [17], [18] and [19]):

$$u_z(x,y,z,t) = w(z,t) + y\phi(z,t) \tag{1a}$$

$$u_y(x,y,z,t) = v(z,t) \tag{1b}$$

$$u_x(x,y,z,t) = 0 \tag{1c}$$

where u_z and u_y are the punctual displacement components, while w, v and ϕ are the displacement variables generalized to the cross-section. The punctual axial and shear deformations, ε_z and γ_{zy}, and the punctual axial and shear stresses, σ_z and τ_{zy}, can be obtained by the compatibility conditions and constitutive relations that, taking into account Eq. (1a), give:

$$\varepsilon_z(x,y,z,t) = \frac{\partial u_z(x,y,z,t)}{\partial z} = \frac{\partial w(z,t)}{\partial z} + y\frac{\partial \phi(z,t)}{\partial z} \tag{2a}$$

$$\gamma_{zy}(x,y,z,t) = \frac{\partial v(z,t)}{\partial z} + \phi(z,t) \tag{2b}$$

$$\sigma_z(x,y,z,t) = E(y)\varepsilon_z(x,y,z,t) = E(y)\left[\frac{\partial w(z,t)}{\partial z} + y\frac{\partial \phi(z,t)}{\partial z}\right] \tag{2c}$$

$$\tau_{zy}(x,y,z,t) = G(y)\gamma_{zy}(x,y,z) = G(y)\left[\frac{\partial v(z,t)}{\partial z} + \phi(z,t)\right] \tag{2d}$$

$E(y)$ and $G(y)$ being the normal and transversal material modulus, respectively. Here, they are assumed to change along the y axis. The generalized internal actions, the axial one N, the transversal one T and the bending moment M, are given by:

$$N(z,t) = \int_A \sigma_z(x,y,z,t)dA = E_0\frac{\partial w(z,t)}{\partial z} + E_1\frac{\partial \phi(z,t)}{\partial z} \tag{3a}$$

$$M(z,t) = \int_A y\sigma_z(x,y,z,t)dA = E_1\frac{\partial w(z,t)}{\partial z} + E_2\frac{\partial \phi(z,t)}{\partial z} \tag{3b}$$

$$T(z,t) = \int_A \tau_{zx}(x,y,z,t)dA \approx \frac{G_0}{\chi}\left[\frac{\partial v(z,t)}{\partial z} + \phi(z,t)\right] \tag{3c}$$

where the following notation has been used:

$$E_i = \int_A y^i E(y)dA \tag{4a}$$

$$i \in \{0,1,2\}$$

$$G_0 = \int_A G(y)\,dA \tag{4b}$$

and where χ is the correction shear factor. In order to obtain the governing equations, the Hamilton's Principle is applied:

$$0 = \int_0^T (\delta K - \delta U - \delta V)dt \tag{5}$$

where δU is the virtual strain energy, δK is the virtual kinetic energy, and δV is the virtual work done by the external forces. Taking into account Eqs.(3), they assume the following form:

$$\delta U = \int_0^L \int_A \left(\sigma_z \delta\varepsilon_z + \tau_{zy}\delta\gamma_{zy}\right) dA dz \\ = \int_0^L \left\{ \begin{array}{l} E_0 \frac{\partial w}{\partial z}\frac{\partial \delta w}{\partial z} + E_1\left(\frac{\partial w}{\partial z}\frac{\partial \delta\phi}{\partial z} + \frac{\partial \phi}{\partial z}\frac{\partial \delta w}{\partial z}\right) + \\ + E_2 \frac{\partial \phi}{\partial z}\frac{\partial \delta\phi}{\partial z} + \frac{G_0}{\chi}\left[\frac{\partial v}{\partial z}\frac{\partial \delta v}{\partial z} + \frac{\partial v}{\partial z}\delta\phi + \phi\frac{\partial \delta v}{\partial z} + \phi\delta\phi\right] \end{array} \right\} dz \tag{6a}$$

$$\delta K = \int_0^L \left[m_0\left(\frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t} + \frac{\partial v}{\partial t}\frac{\partial \delta v}{\partial t}\right) + m_1\left(\frac{\partial \phi}{\partial t}\frac{\partial \delta w}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta\phi}{\partial t} + m_2\frac{\partial \phi}{\partial t}\frac{\partial \delta\phi}{\partial t}\right)\right] dz \tag{6b}$$

$$\delta V = -\int_0^L \left(q_z \delta w + q_y \delta v\right) dz \tag{6c}$$

where:

$$m_i = \int_A y^i \rho(y) dA \tag{6d}$$

q_z and q_y being the axial and transversal external loads, while $\rho(z)$ is the material mass density of the FGB. Substituting the expressions of δU, δK, δV from Eqs. (6a,c) into Eq. (5), integrating by parts with respect to both t and z and fixing the following usual dynamic boundary conditions:

$$\delta w(z,0) = \delta w(z,t_f) = 0 \tag{7a}$$

$$\delta \phi(z,0) = \delta \phi(z,t_f) = 0 \tag{7b}$$

$$\delta v(z,0) = \delta v(z,t_f) = 0 \tag{7c}$$

t_f being the final instant of the analysis temporal interval, it is possible to obtain:

$$\delta u: -E_0\left(\frac{\partial^2 w}{\partial z^2} + \frac{E_1}{E_0}\frac{\partial^2 \phi}{\partial z^2}\right) + m_0\frac{\partial^2 w}{\partial t^2} + m_1\frac{\partial^2 \phi}{\partial t^2} - q_z = 0 \tag{8a}$$

$$\delta \phi: -E_2\left(\frac{\partial^2 \phi}{\partial z^2} + \frac{E_1}{E_2}\frac{\partial^2 w}{\partial z^2}\right) + \frac{G_0}{\chi}\left(\frac{\partial v}{\partial z} + \phi\right) + m_2\frac{\partial^2 \phi}{\partial t^2} + m_1\frac{\partial^2 w}{\partial t^2} = 0 \tag{8b}$$

$$\delta v: -\frac{G_0}{\chi}\left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial \quad \phi}{\partial z}\right) - q_y + m_0\frac{\partial^2 v}{\partial t^2} = 0 \tag{8c}$$

with the boundary conditions:

$$\left[\left(E_0\frac{\partial w}{\partial z} + E_1\frac{\partial \phi}{\partial z}\right)\delta w\right]_0^L = 0 \tag{9a}$$

$$\left[\left(E_1\frac{\partial w}{\partial z} + E_2\frac{\partial \phi}{\partial z}\right)\delta \phi\right]_0^L = 0 \tag{9b}$$

$$\left[\frac{G_0}{\chi}\left(\frac{\partial v}{\partial z}+\phi\right)\delta v\right]_0^L=0 \tag{9c}$$

Eqs. (9a,b) show that the static boundary conditions are coupled. This makes not simple the application of the FE method and the solution of some relevant problems, as the analysis of the free vibration frequencies or the buckling loads.

2.2. Generalized/Homogenized Displacements

In the above cited authors' work [14], the expressions of the generalized/homogenized axial displacement, $\bar{w}$, and rotation, $\bar{\phi}$, are given as:

$$\bar{w}(z,t)=w(z,t)+\frac{E_1}{E_0}\phi(z,t)=w(z,t)+y_{GE}\phi(z,t) \tag{10a}$$

$$\bar{\phi}(z,t)=\phi(z,t)+\frac{E_1}{E_2}w(z,t)=\phi(z,t)+\frac{1}{y_{CE}}w(z,t) \tag{10b}$$

The inverse relationships are:

$$w(z,t)=\frac{\bar{w}(z,t)-y_{GE}\bar{\phi}(z,t)}{1-\frac{y_{GE}}{y_{CE}}} \tag{11a}$$

$$\phi(z,t)=\frac{\bar{\phi}(z,t)-\frac{1}{y_{CE}}\bar{w}(z,t)}{1-\frac{y_{GE}}{y_{CE}}} \tag{11b}$$

3. Hamilton's Principle In Terms Of Generalized/Homogenized Displacements

By replacing Eqs. (10a,b) and Eqs. (11a,b) into Eqs. (8a,c) and (9a,c), it is possible to obtain the governing equations of Timoshenko FGBs. They have the following expressions:

$$\delta w\colon -E_0\frac{\partial^2\bar{w}}{\partial z^2}+m_{11}\frac{\partial^2\bar{w}}{\partial t^2}+m_{12}\frac{\partial^2\bar{\phi}}{\partial t^2}-q_z=0 \tag{12a}$$

$$\delta\phi\colon -E_2\frac{\partial^2\bar{\phi}}{\partial z^2}+\frac{G_0}{\chi}\left(\frac{\partial v}{\partial z}-\Omega_{21}\bar{w}+\Omega_{22}\bar{\phi}\right)+m_{21}\frac{\partial^2\bar{w}}{\partial t^2}+m_{22}\frac{\partial^2\bar{\phi}}{\partial t^2}=0 \tag{12b}$$

$$\delta v\colon -\frac{G_0}{\chi}\left(\frac{\partial^2 v}{\partial z^2}-\Omega_{21}\frac{\partial\bar{w}}{\partial z}+\Omega_{22}\frac{\partial\bar{\phi}}{\partial z}\right)-q_y+m_0\frac{\partial^2 v}{\partial t^2}=0 \tag{12c}$$

with the following boundary conditions:

$$\left[E_0\frac{\partial\bar{w}}{\partial z}\delta w\right]_0^L=0 \tag{13a}$$

$$\left[E_2\frac{\partial\bar{\phi}}{\partial z}\delta\phi\right]_0^L=0 \tag{13b}$$

$$\left[\frac{G_0}{\chi}\left(\frac{\partial v}{\partial z}+\Omega_{22}\bar{\phi}-\Omega_{21}\bar{w}\right)\delta v\right]_0^L=0 \tag{13c}$$

in which some useful coefficients have been defined:

$$\boldsymbol{\Omega}_{11}=\frac{1}{1-\frac{y_{GE}}{y_{CE}}}=\boldsymbol{\Omega}_{22} \tag{14a}$$

$$\boldsymbol{\Omega}_{12}=y_{GE}\boldsymbol{\Omega}_{11} \tag{14b}$$

$$\boldsymbol{\Omega}_{21}=\frac{1}{y_{CE}}\boldsymbol{\Omega}_{22} \tag{14c}$$

$$m_{11}=m_0\boldsymbol{\Omega}_{11}-m_1\boldsymbol{\Omega}_{21} \tag{14d}$$

$$m_{12}=-m_0\boldsymbol{\Omega}_{12}+m_1\boldsymbol{\Omega}_{22} \tag{14e}$$

$$m_{21}=-m_2\boldsymbol{\Omega}_{21}+m_1\boldsymbol{\Omega}_{11} \tag{14f}$$

$$m_{22}=m_2\boldsymbol{\Omega}_{22}-m_1\boldsymbol{\Omega}_{12} \tag{14g}$$

It is easy to verify that the introduction of the new kinematic quantities allows to decouple the first two boundary conditions (Eqs.(13a,b)). This simplifies remarkably the application of the FE method, as will be seen in the next section

4. Dynamic and Static Applications of FE Method

4.1 Dynamic Problem

Following the usual FE procedures for the dynamic analysis of beams (see, for example [20], [21], [23] and [24]), it is useful to approximate the kinematic variables as the product of two independent functions, one of the spatial coordinate and one of the time. By applying the same assumption here, the generalized/homogenized displacements are approximated as follows:

$$\bar{w}(z,t)=\bar{W}(z)e^{-i\omega t} \tag{15a}$$

$$\bar{\phi}(z,t)=\bar{\Phi}(z)e^{-i\omega t} \tag{15b}$$

$$v(z,t)=V(z)e^{-i\omega t} \tag{15c}$$

Taking into account these relationships and fixing $q_y= q_z=0$, in order to analyse beam free vibrations, Eqs. (12a,c) become:

$$\delta w\colon -E_0\frac{d^2\bar{W}}{dz^2}-\omega^2\left(m_{11}\bar{W}+m_{12}\bar{\Phi}\right)=0 \tag{16a}$$

$$\delta\phi\colon -E_2\frac{d^2\bar{\Phi}}{dz^2}+\frac{G_0}{\chi}\left(\frac{dV}{dz}-\Omega_{21}\bar{W}+\Omega_{22}\bar{\Phi}\right)-\omega^2\left(m_{21}\bar{W}+m_{22}\bar{\Phi}\right)=0 \tag{16b}$$

$$\delta v\colon -\frac{G_0}{\chi}\left(\frac{d^2V}{dz^2}-\Omega_{21}\frac{d\bar{W}}{dz}+\Omega_{22}\frac{d\bar{\Phi}}{dz}\right)-\omega^2 m_0 V=0 \tag{16c}$$

In order to implement a FE approach, let us consider the weak formulation directly to Eqs. (16) (strong and weak formulations are crucial concepts of partial differential equations:

the weak formulation turns a differential equation to an integral one). If a generic FE of length h is referred to a local axis reference $0<\xi<h$, it is possible to write:

$$\left[-E_0\frac{d\bar{W}}{d\xi}\nu_1\right]_0^h+\int_0^h E_0\frac{d\bar{W}}{d\xi}\frac{d\nu_1}{d\xi}d\xi-\omega^2\int_0^h m_{11}\bar{W}\nu_1+m_{12}\bar{\Phi}\nu_1 d\xi=0 \tag{17a}$$

$$\begin{aligned}\left[-E_2\frac{d\bar{\Phi}}{d\xi}\nu_2\right]_0^h+\int_0^h E_2\frac{d\bar{\Phi}}{d\xi}\frac{d\nu_2}{d\xi}+\frac{G_0}{\chi}\left(\frac{dV}{dz}-\Omega_{21}\bar{W}+\Omega_{22}\bar{\Phi}\right)\nu_2 d\xi\\ -\omega^2\int_0^h m_{21}\bar{W}\nu_2+m_{22}\bar{\Phi}\nu_2 d\xi=0\end{aligned} \tag{17b}$$

$$\begin{aligned}\left[-\frac{G_0}{\chi}\left(\frac{dV}{dz}-\Omega_{21}\bar{W}+\Omega_{22}\bar{\Phi}\right)\nu_3\right]_0^h+\int_0^h\frac{G_0}{\chi}\left(\frac{dV}{dz}-\Omega_{21}\bar{W}+\Omega_{22}\bar{\Phi}\right)\frac{d\nu_3}{d\xi}d\xi\\ -\omega^2\int_0^h m_0 V\,\nu_3 d\xi=0\end{aligned} \tag{17c}$$

where ν_1, ν_2 and ν_3 are opportune weight functions. Eqs. (17a,c) allow to evidence primary and secondary variables. The degree of interpolation for the functions $\bar{W}$, $\bar{\Phi}$ and V can be generic, but the degree of interpolation of V must be an unit more than that of $\bar{\Phi}$ (we define it as Consistent Interpolation Element (CIE)). Then, it is:

$$\bar{W}(\xi)\approx\left[\underline{\Psi}^{\bar{W}}(\xi)\right]^T\underline{\bar{W}}^e\left[\underline{\Psi}^{\bar{W}}(\xi)\right]^T=\left(\Psi_1^{\bar{W}}(\xi),\Psi_2^{\bar{W}}(\xi),\ldots,\Psi_n^{\bar{W}}(\xi)\right) \tag{18a}$$

$$\bar{\Phi}(\xi)\approx\left[\underline{\Psi}^{\bar{\Phi}}(\xi)\right]^T\underline{\bar{\Phi}}^e\left[\underline{\Psi}^{\bar{\Phi}}(\xi)\right]^T=\left(\Psi_1^{\bar{\Phi}}(\xi),\Psi_2^{\bar{\Phi}}(\xi),\ldots,\Psi_n^{\bar{\Phi}}(\xi)\right) \tag{18b}$$

$$V(\xi)\approx[\underline{\Psi}^V(\xi)]^T\underline{V}\left[\underline{\Psi}^V(\xi)\right]^T=\left(\Psi_1^V(\xi),\Psi_2^V(\xi),\ldots,\Psi_{n+1}^V(\xi)\right) \tag{18c}$$

Replacing Eqs. (18a,c) into Eqs. (17a,c) and, with the aim of applying a Ritz-Galerkin FE approach, setting $\nu_1(\xi)=\Psi_i^{\bar{W}}(\xi)$ (i=1,…,n), $\nu_2(\xi)=\Psi_i^{\bar{\Phi}}(\xi)$ (i=1,…,n) and $\nu_3(\xi)=\Psi_i^V(\xi)$ (i=1,…,n+1), the following equations are obtained:

$$\begin{aligned}\left(\begin{bmatrix}\underline{\underline{K}}^{\bar{W}\bar{W}} & \underline{\underline{0}} & \underline{\underline{0}}\\ \underline{\underline{K}}^{\bar{\Phi}\bar{W}} & \underline{\underline{K}}^{\bar{\Phi}\bar{\Phi}} & \underline{\underline{K}}^{\bar{\Phi}V}\\ \underline{\underline{K}}^{V\bar{W}} & \underline{\underline{K}}^{V\bar{\Phi}} & \underline{\underline{K}}^{VV}\end{bmatrix}_e-\omega^2\begin{bmatrix}\underline{\underline{M}}^{\bar{W}\bar{W}} & \underline{\underline{M}}^{\bar{W}\bar{\Phi}} & \underline{\underline{0}}\\ \underline{\underline{M}}^{\bar{\Phi}\bar{W}} & \underline{\underline{M}}^{\bar{\Phi}\bar{\Phi}} & \underline{\underline{0}}\\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{M}}^{VV}\end{bmatrix}_e\right)\begin{bmatrix}\underline{\bar{W}}\\ \underline{\bar{\Phi}}\\ \underline{V}\end{bmatrix}_e=\begin{bmatrix}\underline{Q}_1\\ \underline{Q}_2\\ \underline{Q}_3\end{bmatrix}_e\\ \Rightarrow\ \left(\underline{\underline{K}}_e-\omega^2\underline{\underline{M}}_e\right)\underline{u}_e=\underline{Q}_e\end{aligned} \tag{19}$$

where:

$$\underline{\underline{K}}^{\bar{W}\bar{W}}=\int_0^h E_0\frac{d\underline{\Psi}^{\bar{W}}(\xi)}{d\xi}\frac{d\left[\underline{\Psi}^{\bar{W}}(\xi)\right]^T}{d\xi}d\xi \tag{20a}$$

$$\underline{\underline{K}}^{\bar{\Phi}\bar{W}}=\int_0^h-\frac{G_0}{\chi}\Omega_{21}\underline{\Psi}^{\bar{\Phi}}(\xi)\left[\underline{\Psi}^{\bar{W}}(\xi)\right]^T d\,\xi \tag{20b}$$

$$\underline{\underline{K}}^{\bar{\Phi}\bar{\Phi}}=\int_0^h E_2\frac{d\underline{\Psi}^{\bar{\Phi}}(\xi)}{d\xi}\frac{d\left[\underline{\Psi}^{\bar{\Phi}}(\xi)\right]^T}{d\xi}+\frac{G_0}{\chi}\Omega_{22}\underline{\Psi}^{\bar{\Phi}}(\xi)\left[\underline{\Psi}^{\bar{\Phi}}(\xi)\right]^T d\xi \tag{20c}$$

$$\underline{\underline{K}}^{\bar{\Phi}V} = \int_0^h \frac{G_0}{\chi} \underline{\Psi}^{\bar{\Phi}}(\xi) \frac{d[\underline{\Psi}^V(\xi)]^T}{d\xi} d\xi \tag{20d}$$

$$\underline{\underline{K}}^{V\bar{W}} = \int_0^h -\frac{G_0}{\chi} \Omega_{21} \frac{d\underline{\Psi}^V(\xi)}{d\xi} \left[\underline{\Psi}^{\bar{W}}(\xi)\right]^T d\xi \tag{20e}$$

$$\underline{\underline{K}}^{V\bar{\Phi}} = \int_0^h \frac{G_0}{\chi} \Omega_{22} \frac{d\underline{\Psi}^V(\xi)}{d\xi} \left[\underline{\Psi}^{\bar{\Phi}}(\xi)\right]^T d\xi \tag{20f}$$

$$\underline{\underline{K}}^{VV} = \int_0^h \frac{G_0}{\chi} \frac{d\underline{\Psi}^V(\xi)}{d\xi} \frac{d[\underline{\Psi}^V(\xi)]^T}{d\xi} d\xi \tag{20g}$$

$$\underline{\underline{M}}^{\bar{W}\bar{W}} = \int_0^h m_{11}\, \underline{\Psi}^{\bar{W}}(\xi) \left[\underline{\Psi}^{\bar{W}}(\xi)\right]^T d\xi \tag{20h}$$

$$\underline{\underline{M}}^{\bar{W}\bar{\Phi}} = \int_0^h m_{12} \underline{\Psi}^{\bar{W}}(\xi) \left[\underline{\Psi}^{\bar{\Phi}}(\xi)\right]^T d\xi \tag{20i}$$

$$\underline{\underline{M}}^{\bar{\Phi}\bar{W}} = \int_0^h m_{21}\, \underline{\Psi}^{\bar{\Phi}}(\xi) \left[\underline{\Psi}^{\bar{W}}(\xi)\right]^T d\xi \tag{20j}$$

$$\underline{\underline{M}}^{\bar{\Phi}\bar{\Phi}} = \int_0^h m_{22} \underline{\Psi}^{\bar{\Phi}}(\xi) \left[\underline{\Psi}^{\bar{\Phi}}(\xi)\right]^T d\xi \tag{20k}$$

$$\underline{\underline{M}}^{VV} = \int_0^h m_0\, \underline{\Psi}^V(\xi) [\underline{\Psi}^V(\xi)]^T d\xi \tag{20l}$$

and:

$$\underline{Q}_1 = [-N(0) \quad 0 \quad \dots \quad 0_{n-1} \quad N(h)]^T \tag{20m}$$

$$\underline{Q}_2 = [-M(0) \quad 0 \quad \dots \quad 0_{n-1} \quad M(h)]^T \tag{20n}$$

$$\underline{Q}_2 = [-T(0) \quad 0 \quad \dots \quad 0_n \quad T(h)]^T \tag{20o}$$

Eqs. (10) and (11) show that the generalized kinematic variables can be expressed in function of the generalized/homogenized ones and vice versa, in such a way their continuity is guaranteed. The continuity of generalized/homogenized variables is imposed by defining the location matrix $\underline{L_e}$ for each finite element. Defining $\underline{U}$ as the vector whose entrances are the nodes displacements and $\underline{Q}$ as the vector containing the internal forces at the nodes, and following the same steps of the classical FE approach, it is possible to obtain:

$$\left(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}\right)\underline{U} = \underline{Q} \tag{21}$$

where:

$$\underline{\underline{K}} = \underline{\underline{L}}_e^T \underline{\underline{K}}_e \underline{\underline{L}}_e \quad \underline{\underline{M}} = \underline{\underline{L}}_e^T \underline{\underline{M}}_e \underline{\underline{L}}_e \quad \underline{Q} = \underline{\underline{L}}_e^T \underline{Q}_e \quad \underline{U} = \underline{\underline{L}}_e^{-1} \underline{u}_e \tag{22}$$

By imposing static and kinematic boundary conditions, given Eqs. (10) and Eqs. (13), the system of equations (21) allows to derive the eigenvalues problem able to define the dynamic behaviour of a FGB.

4.2 Static Problem

Eqs. (12), simplified for the static case, take the form:

$$-E_0\frac{d^2\bar{w}}{dz^2}-q_z=0 \tag{23a}$$

$$-E_2\frac{d^2\bar{\phi}}{dz^2}+\frac{G_0}{\chi}\left(\frac{dv}{dz}-\Omega_{21}\bar{w}+\Omega_{22}\bar{\phi}\right)=0 \tag{23b}$$

$$-\frac{G_0}{\chi}\left(\frac{d^2v}{dz^2}-\Omega_{21}\frac{d\bar{w}}{dz}+\Omega_{22}\frac{d\bar{\phi}}{dz}\right)-q_y=0 \tag{23c}$$

Through the same steps considered previously, the solving equation is obtained as:

$$\underline{\underline{K}}\underline{U}=\underline{F}+\underline{Q} \tag{24}$$

where $\underline{\underline{K}}$, $\underline{U}$, $\underline{Q}$ are equal to those which appear in Eq. (21); $\underline{F}$ is defined as follows:

$$\underline{F}=\underline{\underline{L}}_e^T\underline{F}_e \tag{25}$$

and:

$$\underline{F}_e=\begin{bmatrix}\underline{F}_1\\ \underline{0}\\ \underline{F}_3\end{bmatrix} \tag{26a}$$

$$\underline{F}_1=\int_0^h\underline{\Psi}^{\bar{W}}(\xi)q_z(\xi)d\xi \tag{26b}$$

$$\underline{F}_3=\int_0^h\underline{\Psi}^{V}(\xi)q_y(\xi)d\xi \tag{26c}$$

5. Analytical Solutions for Buckling Analysis

In this section the generalized/homogenized displacements are used with the aim of finding analytical solutions for the buckling problem of Timoshenko FGBs. It is noteworthy that some of the analytical solutions set out herein are a novelty in literature; this despite a lot of paper which deal with this problem already exist, for example, [24], [25], [26], [27], [28], and [29].

5.1. Buckling Load of a Clamped-Free Beam

In order to investigate the buckling of a generic column subjected to a compressive force P, Eqs. (12) become:

$$-E_0\frac{d^2\bar{w}}{dz^2}=0 \tag{27a}$$

$$-E_2\frac{d^2\bar{\phi}}{dz^2}+\frac{G_0}{\chi}\left(\frac{dv}{dz}+\Omega_{22}\bar{\phi}-\Omega_{21}\bar{w}\right)=0 \tag{27b}$$

$$-\frac{G_0}{\chi}\left(\frac{d^2v}{dz^2}+\mathbf{\Omega}_{22}\frac{d\bar{\phi}}{dz}-\mathbf{\Omega}_{21}\frac{d\bar{w}}{dz}\right)+P\frac{d^2v}{dz^2}=0 \tag{27c}$$

with the boundary conditions:

$$\left[E_0\frac{d\bar{w}}{dz}\delta w\right]=0 \tag{28a}$$

$$\left[\left(\frac{G_0}{\chi}\left(\frac{dv}{dz}+\Omega_{22}\bar{\phi}-\Omega_{21}\bar{w}\right)-P\frac{dv}{dz}\right)\delta v\right]_0^L=0 \tag{28b}$$

$$\left[E_2\frac{d\bar{\phi}}{dz}\delta\phi\right]_0^L=0 \tag{28c}$$

Noting that for a clamped-free beam the generalized transversal stress is equal to zero in the free end, $T(L)$=0, it is possible to assume $T(z)$=0 for all the beam. Furthermore, combining Eq.(27a), Eq.(28a) and Eqs.(11), $\bar{w}(z)$=0 is obtained. Hence, Eq. (28b) allows to give the generalized/ homogenised rotation $\bar{\phi}$ as a function of the generalized transversal displacement v:

$$\frac{G_0}{\chi}\left(\frac{dv(z)}{dz}+\mathbf{\Omega}_{22}\bar{\phi}(z)\right)-P\frac{dv(z)}{dz}=0\Rightarrow\bar{\phi}(z)=\frac{\left(P-\frac{G_0}{\chi}\right)}{\frac{G_0}{\chi}\mathbf{\Omega}_{22}}\frac{dv(z)}{dz} \tag{29}$$

Substituting Eq.(29) into Eq.(27b) and deriving with respect to z, the differential equations governing the buckling problem of a Timoshenko FGB is derived:

$$\frac{d^4v(z)}{dz^4}+\alpha^2\frac{d^2v}{dz^2}=0 \qquad \alpha=\sqrt{\frac{P\Omega_{22}/E_2}{1-P/G_0}} \tag{30}$$

This result implies that the buckling load P_n for this kind of beam is given by:

$$P_n=\frac{P_E}{\Omega_{22}+\frac{\chi P_E}{G_0}} \Leftarrow P_E=\frac{n^2\pi^2E_2}{(2L)^2} \tag{31}$$

In order to clarify the transition process between Eqs. (30) and (31), we need to remember that the general solution of Eq. (30) is:

$$v(z)=A\cos\alpha z+Bsin\alpha z+Cz+D \tag{32}$$

By imposing the boundary conditions $v(0)$=0, $v(L)$=0, $dv/dz(0)$=0 (see Eqs. (11a,b-29) with $w(0)$=0 since the beam is supposed clamped in z=0), $d^2v/dz^2(L)$=0 (see Eqs. (13b-29)); we get the following system in the unknown variables A, B, C and D:

$$\begin{cases} A+D=0 \\ A\cos(\alpha L)+B\sin(\alpha L)+CL+D=0 \\ B+C=0 \\ A\cos(\alpha L)+B\sin(\alpha L)=0 \end{cases} \tag{33}$$

System (33) gives no trivial solution if and only if cos(αL)=0: it follows αL=nπ/2 then Eq. (31) is achieved.

5.2. Buckling Load for a Simply Supported-Simply Supported Beam

In this case, Eqs.(27) and Eqs.(28) continue to hold. Taking into account Eq.(27a), Eq.(28a) and Eqs.(11), it is possible to obtain $\bar{w}$(z)=$y_{GE}\bar{\phi}(0)$. Therefore, deriving Eq. (27b) and adding to it Eq.(27c), it is easy to show that Eq.(30) still holds. This implies that the buckling load is given by:

$$P_n = \frac{P_E}{\Omega_{22} + \dfrac{\chi P_E}{G_0}} \quad \Leftarrow \quad P_E = \frac{n^2\pi^2 E_2}{L^2} \tag{34}$$

Similar steps as the one just made in Subsection 5.1 needs to be performed.

6. Numerical Applications

From here on out, the following acronyms are introduced: S-S for simply supported-simply supported; C-C for clamped-clamped; C-F for clamped-free and C-S for clamped-simply supported.

6.1. Natural Frequencies for S-S Beams

Let us consider a S-S Timoshenko FG beam with a rectangular cross of width b, height h, length L, shear correction factor χ:

$$h=5\times 17.6\times 10^{-6}\,\text{m} \quad b=2h \quad L=20h \quad \chi=1.2 \tag{35}$$

The beam is characterized by an elastic modulus and a density variable along the thickness with an exponential law:

$$E(y) = (E_b - E_a)\left(\frac{y}{h}+\frac{1}{2}\right)^N + E_a \tag{36a}$$

$$\rho(y) = (\rho_b - \rho_a)\left(\frac{y}{h}+\frac{1}{2}\right)^N + \rho_a \tag{36b}$$

where E_a=14400 MPa, E_b=1440 MPa, ρ_a=122 kg/mm, ρ_b=12.2 kg/mm. The exponent N defines the composition of the section and, here, it is assumed equal to five different values. The elastic tangential module is given by the usual expression:

$$G(y) = \frac{E(y)}{2(1+\upsilon)} \tag{37}$$

ν being the classical Poisson coefficient, here chosen equal to 0.38. The results obtained by the FE approach previously introduced (numerically approximated) are compared with the natural frequencies obtained in accordance with the procedure exposed by Li [11] (theoretical and numerical approximated). Ten finite elements having the same length have been used; moreover, polynomial interpolating functions of order one for $\overline{w}$ and $\overline{\phi}$ and of order two for v have been applied (This level of refinement is maintained for all numerical applications exposed in this section). The accordance of the values reported in Table 1 ensures the correctness of the methodology applied. Clearly, these FE results could be improved by a p-refinement or a h-refinement.

Table 1. First three natural frequencies (expressed in s^{-1}) for a S-S beam with different values of N

N	ω_1 Li	ω_1 FEM	ω_2 Li	ω_2 FEM	ω_3 Li	ω_3 FEM
0.2	0.85	0.85	3.36	3.41	7.41	7.67
0.5	0.81	0.82	3.22	3.27	7.11	7.37
1	0.77	0.78	3.06	3.11	6.77	7.02
5	0.82	0.82	3.25	3.30	7.17	7.43
100	0.93	0.92	3.67	3.67	8.08	8.24

6.2. Natural Frequencies for C-C and C-S Beams

The same mechanical and geometrical characteristics assumed in the previous subsection are taken into account. For the beams here considered, no analytical solution exists in the literature. Consequently, the FE method exposed in section 4 has been applied for any examined boundary conditions. Natural frequencies derived for a C-S Timoshenko beam are collected in Table 2, while those derived for a S-S beam are given in Table 1. Table 2 shows the results obtained for the C-C beam. Lastly, the natural frequencies linked to a C-F beam are reported in Table 3.

Table 2. First three natural frequencies (expressed in s^{-1}) for C-C and C-S beams with different values of N

	C-C beam			C-S beam		
N	ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
0.2	1.98	5.62	11.44	1.33	4.32	9.03
0.5	1.90	5.41	11.03	1.27	4.15	8.69
1	1.81	5.15	10.54	1.21	3.95	8.29
5	1.92	5.45	11.11	1.29	4.19	8.76
100	2.17	6.12	12.36	1.46	4.73	9.82

Table 3. First three natural frequencies (expressed in s^{-1}) for the C-F beam with different N

	C-F beam		
N	ω_1	ω_2	ω_3
0.2	0.30	1.90	5.33
0.5	0.29	1.83	5.12
1	0.28	1.73	4.87
5	0.29	1.84	5.16
100	0.33	2.09	5.82

Eqs. (36a,b) describe the change of material along the cross section of the analysed beam; these laws are functions of a parameter N which can be considered an indicator of the mixture of the two components that constitute the sample in exam. In detail, when N tends to infinity then the cross section tends to be homogeneous with the properties of the stiffest component (E_a=14400 MPa, ρ_a=122 kg/mm); on the other hand, when N tends to zero then the beam tends to be homogenous with the properties of the weakest one (E_b=1440 MPa, ρ_b=12.2 kg/mm). In light of the above, it is not surprising that in Table 1, 2 and 3 the natural frequencies increase with growing *N.*

6.3. Buckling Loads for S-S and C-F Beams

Eqs.(31) and Eq.(34) are here applied to the same above considered beams. In this case, the use of the homogenized/generalized displacements makes possible to have exact relations expressing the buckling load of these Timoshenko FGBs. Table 4 shows the numerical results obtained evaluating Eqs. (31) and (34): indeed, buckling loads increases with growing *N* since the sample becomes progressively more stiff (same remarks are true for natural frequencies (see Subsections 6.1 and 6.2)

6.4. Static Bending C-C and C-S Beams

In this application, the same mechanical and geometrical characteristics used previously are maintained. The static external action is represented by a vertical uniformly distributed load with intensity q_y= 10 N/mm. The coefficient *N* appearing in Eqs.(41) is assumed equal to 10. Fig.2 and Fig.3 show the transversal displacements obtained with the FE formulation, here presented, compared with the analytical ones.

Table 4. First three buckling loads (expressed in newton (N)) for S-S and the C-F beam with different values of *N*

	S-S beam			C-F beam		
N	P_1	P_2	P_3	P_1	P_2	P_3
0.2	0.06	0.22	0.47	0.014	0.06	0.12
0.5	0.10	0.39	0.85	0.025	0.10	0.22
1	0.19	0.76	1.67	0.05	0.19	0.43
5	0.28	1.08	2.34	0.07	0.28	0.61
100	0.36	1.42	3.09	0.09	0.36	0.81

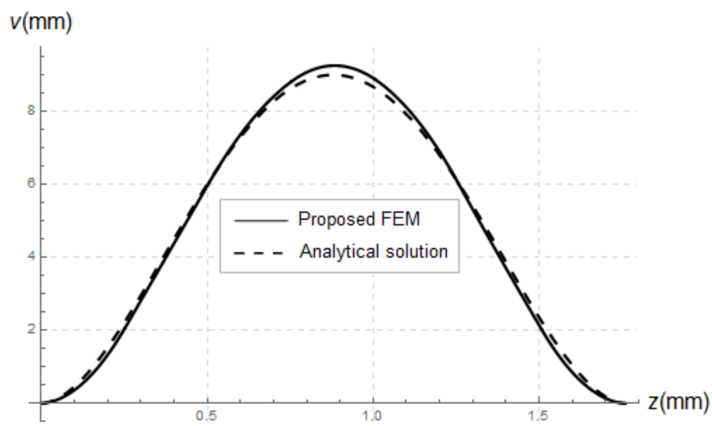

Fig.2 Static transversal deflection (expressed in mm) under constant load for C-C beam with *N*=10

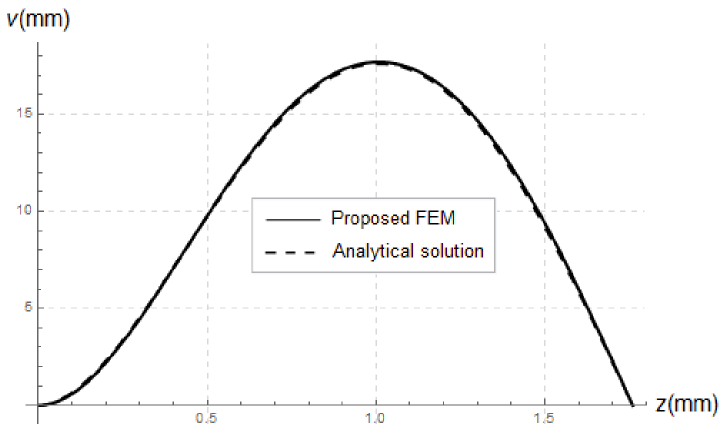

Fig.3 Static transversal deflection (expressed in mm) under constant load for C-S beam with N=10

Fig.2-3 show a higher rate of convergence in the case in which natural (or static) boundary conditions need to be imposed: in the same condition, the accuracy of the numerical solution obtained for the C-S beam (Fig. 3) is better than the one obtained in the C-C case (Fig. 3). This aspect is strictly related to the new approach introduced: the generalized-homogenized displacements aim to create natural boundary conditions equivalent but simpler than the classical ones valid for FG beams. Indeed, the analytical solution is derived following the same procedure shown in the previous paper of the authors [14].

7. Conclusions

In the present work, a new theoretical formulation based on some homogenized/generalized displacements has been introduced in order to solve bending, free vibration and buckling problems of Timoshenko FGBs, i.e. mono dimensional elements with constitutive properties that vary gradually along the thickness. The proposed formulation allows the definition of a new FE approach able to uncouple the required boundary conditions and it involves the use of the Navier method for solving dynamic problems: according to the latter, the kinematic variables are approximated as the product of two independent functions. The homogenized/generalized displacements are appropriately defined with the aim of uncoupling the expression of axial, shear and moment stresses generalized to the cross section in the case of FGMs so as to achieve a greater procedural saving. Numerical results are compared, when possible, to the analytical ones obtained by following Li's model and a good match is shown. Different conditions of constraint are analysed to verify the reliability and the physical correspondence of the proposed approach: natural frequencies related to S-S, C-C, C-S, C-F beams and buckling loads for S-S, C-F are compared for different values of an opportune parameter which rules the material composition of the cross section. In accordance with physical intuition, natural frequencies and buckling loads increase with increasing the mean Young's modulus and the mean shear modulus of the cross section. Just for the sake of clarity, transversal numerical displacements are derived only for C-C and C-S beams under static condition of load and they are compared with the exact ones. The performed numerical applications have revealed that the proposed procedure is highly competitive with the most used in literature: moreover, a higher rate of convergence in the case in which natural (or static) boundary conditions need to be imposed is shown. Finally, a closed expression capable of expressing buckling loads linked to FGBs is discovered: it does not yet exist in literature and it could be considered the natural extension of that valid for homogenous mono-dimensional structures.

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