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Markovian Queueing System with Bulk Arrival and Three Types of Essential and Optional Services

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Abstract: This paper analyses the single server bulk arrival Markovian queueing system where customers arrive according to the general bulk rule and follows the Poisson distribution. Customers are served with three types of services, one is essential service and two of them are optional services. Server is giving service to the customers according to Exponential distribution with the bulk service rule. After providing service to the bulk, server goes for a vacation according to the exponentially distribution. Server returns from the vacation when there are sufficient number of customers is available for receiving the service. Again server go back for a vacation if arrived number of customers is not enough to get the essential service. The server returns when arrived bulk is sufficient. After receiving the essential service, customers have the choice that they take first essential service, second essential service or take both. We have also include the impatience behaviour of customers i.e. the bulk of customers also balks from the system if the server is busy to giving service to other customers or server is on the vacation. We have computed the time dependent probability generating functions with the help of corresponding steady state results. Also we determine the number of customers waiting in the queue for this system.

Keywords: Bulk Arrival, Essential Service, Optional Service, Markovian Queueing System.
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1. Introduction

In congestion situations, it has been observed that analyzing the queueing models with phase service are beneficial to our daily life activates. In general queueing models, all arriving customers take service which is to be completed in single phase but In many real time situations the service may be complete a few arriving customers need the optional services along with essential service for their satisfaction. Our aim of this paper is to present the overview of queueing models with phase service and its applications along with real life queueing problems such as computer and communication networks, inventory and distribution systems, performance evaluation and dimensioning of production and manufacturing systems and so on. Manufacturing systems are also useful for defining a series of work stations. In phase service queueing system, all customers are served by the server in different type of phases; out of these some are essential phases and some of the optional phases of service. Essential phases of service are provided by the server to all the customers come to service station, in other optional services provided by the server to the customers depending upon their demand.

2. Related Work

During the last few years, most of the researchers worked on the phase service queueing models. Many developments vary from time to time in this research area and further research. F. Nuts [6] gave the theory of general bulk service rule first

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in which customer arrive according to the Poisson process and served in batches according to the general distribution with general bulk service rule. Kumar and Arumuganathan [3] examined the queueing model with single server batch arrival retrial queue, active breakdowns and two types of repair facility, first is essential service and second is optional service. They have obtained performances measures with some particular cases and numerical analysis. M. I. Afthab [4] worked on a single server, bulk service queue with general arrival pattern and multiple working vacation. They derived the steady state Probability distribution at pre arrival epoch and arbitrary epoch, mean queue length. Benard [5] worked on the single server Markovian queuing model with first come first server and infinite population by using sensitivity analysis. K. C. Madan [7] worked on the *M/G/1* queueing system with first essential service and second optional service. They assumed that customers arrive with the Poisson distribution and essential service rate of customers follow a general distribution and that of the second optional service rate follows the exponential distribution. They have determine the time-dependent probability generating functions by using the Laplace transforms, mean queue length ,the mean waiting time. J. Medhi [8] worked on the *M/G/1* queueing system with the first essential service and second optional service follows the general distribution. Wang and Li [9] analyzed the queueing system with second multi-optional service, bulk-arrival single server with the homogenous Poisson process. They determined the transient and the steady-state solutions for both queueing using a supplementary variable method. Ke [10] have given a brief summary of the most recent research works on vacation queueing systems in the past 10 years. Bagyam [11] have worked on the bulk arrival queueing system in which two phase service essential and optional, retrial and general distribution is considered. They have determine the steady state distributions of the server state and the number of customers in the orbit by using the supplementary variable technique. S.R. Chakravarthy [12] has Studied on the Markovian arrival queueing system with single server, secondary optional service and first come- first-served service discipline. With a certain probability a customer may require an optional secondary service. The model is examined as a QBD-process with the help of matrix-analytic methods and examples are also explained. Jain [15] worked on the maximum entropy principle (MEP) to explore the steady state behaviour of the bulk arrival retrial queuing system. The concepts of Bernoulli vacation schedule and second optional service. After completing the first essential service, server may go for vacation with probability p if there is not sufficient number of customers for receiving service. They have determined the expected number of customers and expected waiting time of customers in the retrial group supplementary variables and using generating function technique and derived various system performance measures.

3. Model Description

We consider Markovian queueing system with bulk arrival and bulk service. Customer arrival rate follows the compound Poisson distribution with mean λ. Here we have considered the bulk queue rule, server will give the service to minimum number of m_1 customers and maximum number of n_1 customers. Also there are one essential service and two optional services and follows the exponential distribution. All arriving customers require the essential service and some customers may further demand the optional service. After completing the essential service, the customer may leave the system with probability p or they may wait for the optional service with probability q, $p + q = 1$. The server can only provide either the essential or the optional service for one customer at a time and that service is independent of the arrival of customers. Customers arrive for receiving the service but server is busy then they wait for some time till the server gets free because server can provide the service to one batch of customers in one time.

We assume the following to describe the queueing model of our study

(a). Customers arrive at the system in batches of random size follow the compound Poisson process.

(b). The service of customers is based on the bulk service rate $[(m_1, n_1)\ \ rule]$, we have considered the one essential service and two optional service. Arrivals take the essential service and may choose the first optional service or second optional service. When customers take the first essential service then he may choose the first optional service or second optional service.

(c). Here we assume that the customer is served by the general bulk rule $[(m_1, n_1)rule]$, where 'm_1' is minimum number of customers and 'n_1' is maximum number of customers taking the service at any instant of time.

(d). Let $\lambda\tau_i$ be the probability that a batch of i customers arrive at the system during the short interval of time $(t, t+dt)$ is the mean arrival rate of batches. and using $\sum\limits_{i=1}^{\infty} \tau_i = 1$, $0 \leq \tau_i \leq 1$, $\lambda > 0$. He may choose the first optional service with the probability η or leave the service system with probability $1-\eta$. If he chooses the second optional service then the probability will be ξ or leave the service system with probability $1-\xi$.

(e). Here we assume that random variable of service time Y_j $(j = 1, 2, 3)$ of the j$^{\text{th}}$ kind of service follow the general probability law with the distribution function $f_i(y_i)$. $f_i(y_i)$ is the probability density function and $f(y_i)$ is the r$^{\text{th}}$ moment of service time $j = 1, 2, 3$ (first essential service, second and third are optional service).

(f). Let $N_i(x)$ be the conditional probability of type j service during the period $(x, x+dx)$, given that elapsed service time is x, so that

$$N_i\left(x\right) = \frac{f_i(x)}{1 - F_j(x)}, \quad j = 1, 2, 3 \tag{1}$$

$$F_j\left(y_i\right) = N_j\left(y_i\right) e^{-\int\limits_0^{s_j} N_j(x)dx}, \quad j = 1, 2, 3 \tag{2}$$

(g). After completion of continuous service to the batches of random size (m_1, n_1), server will go for the compulsory vacation.

(h). Server's vacation will start after the completion of service to a batch, the duration of vacation period is assumed to be exponential with mean vacation time $= 1/\zeta$

(i). On returning from the vacation the server immediately starts the essential service if there is a batch of variable size (m_1, n_1) or he waits idle in the system.

(j). We assume that $(1-b_1)$ $(0 \leq b_1 \leq 1)$ is the probability that an arriving batch balks during the period when the server is busy (but available on the system) and $(1-b_2)$ $(0 \leq b_2 \leq 1)$ is the probability that an arriving batch balks during the period when the server is on vacation.

(k). It is also assumed that inter arrival times of customers, service times of each kind of service and vacation times of the server, independent of each other.

4. Definitions and Notations

(a). $P_{n,j}\left(x,t\right)$: It is the probability that at a time t, the server is active providing and there are $n (n \geq 0)$ customers in the queue. Excluding a batch of $k (m_1 \leq k \leq n_1)$ customers in the type j $(j = 1, 2, 3)$ service and the elapsed time of this customer is x.

(b). $\int_0^{\infty} P_{n,j}\left(x,t\right) dx$: Denotes the probability that there are n customers in the queue excluding a batch of $k\left(m \leq k \leq n_1\right)$ customers in type j service $j = 1, 2, 3$ irrespective of elapsed service time x.

(c). $R(t)$: Probability that at a time, there are less than 'm' customers in the system and the server is idle but available in the system.

(d). $W_n(t)$: Probability that at a time, there are n $(n \geq 0)$ customers in the queue and the server is on vacation.

5. Equation Governing the System

According to the mathematical model mentioned above, the system has following set of differential-difference equations

$$\frac{\partial}{\partial x}P_{n,1}(x,t)+\frac{\partial}{\partial t}P_{n,1}(x,t)=-\lambda P_{n,1}(x,t)-\sum_{n=m_1}^{n_1}\mu_1(x)P_{n,1}(x,t)+b_1\lambda\sum_{r=1}^{n}f_rP_{n-r,1}(x,t)+\lambda(1-b_1)P_{n,1}(x,t) \tag{3}$$

$$\frac{\partial}{\partial x}P_{0,1}(x,t)+\frac{\partial}{\partial t}P_{0,1}(x,t)=-\lambda P_{0,1}(x,t)-\sum_{n=m_1}^{n_1}\mu_1(x)P_{0,1}(x,t)+\lambda(1-b_1)P_{0,1} \tag{4}$$

$$\frac{\partial}{\partial x}P_{n,2}(x,t)+\frac{\partial}{\partial t}P_{n,2}(x,t)=-\lambda P_{n,2}(x,t)-\sum_{n=m_1}^{n_1}\mu_2(x)P_{n,2}(x,t)+b_1\lambda\sum_{r=1}^{n}f_rP_{n-r,2}(x,t)+\lambda(1-b_1)P_{n,2}(x,t) \tag{5}$$

$$\frac{\partial}{\partial x}P_{0,2}(x,t)+\frac{\partial}{\partial t}P_{0,2}(x,t)=-\lambda P_{0,2}(x,t)-\mu_2(x)P_{0,2}(x,t)+\lambda(1-b_1)P_{0,2}(x,t) \tag{6}$$

$$\frac{\partial}{\partial x}P_{n,3}(x,t)+\frac{\partial}{\partial t}P_{n,3}(x,t)=-\lambda P_{n,3}(x,t)-\sum_{n=m_1}^{n_1}\mu_3(x)P_{n,3}(x,t)+b_1\lambda\sum_{r=1}^{n}f_rP_{n-r,3}(x,t)+\lambda(1-b_1)P_{n,3}(x,t) \tag{7}$$

$$\frac{\partial}{\partial x}P_{0,3}(x,t)+\frac{\partial}{\partial t}P_{0,3}(x,t)=-\lambda P_{0,3}(x,t)-\mu_3(x)P_{0,3}(x,t)+\lambda(1-b_1)P_{0,3}(x,t) \tag{8}$$

$$\begin{aligned}\frac{d}{dt}W_n(t)+(\lambda+\zeta)V_n(t)&=\lambda b_2\sum_{r=1}^{n}\rho_rW_{n-r}(t)+\lambda(1-b_2)W_n(t)+(1-\eta)(1-\xi)\int_0^\infty P_{n,1}(x,t)\mu_1(x)\,dx\\&+\int_0^\infty P_{n,2}(x,t)\mu_2(x)\,dx+\int_0^\infty P_{n,3}(x,t)\mu_3(x)\,dx\end{aligned} \tag{9}$$

$$\begin{aligned}\frac{d}{dt}W_0(t)+(\lambda+\zeta)V_0(t)&=\lambda(1-b_2)V_0(t)+(1-\eta)(1-\xi)\int_0^\infty P_{0,1}(x,t)\mu_1(x)\,dx+\int_0^\infty P_{0,2}(x,t)\mu_2(x)\,dx\\&+\int_0^\infty P_{0,3}(x,t)\mu_3(x)\,dx\end{aligned} \tag{10}$$

$$\frac{d}{dt}R(t)+\lambda R(t)=\zeta V_0(t)+\lambda(1-b_1)R(t) \tag{11}$$

Equations (3) to (11) are to be solved subject to the following boundary conditions.

$$P_{n,1}(0,t)=\zeta W_{n+n_1}(t) \tag{12}$$

$$P_{n,1}(0,t)=\zeta\sum_{b=m_1}^{n_1}W_b(t)+\lambda b_1R(t) \tag{13}$$

$$P_{n,2}(0,t)=\eta\int_0^\infty P_{n,1}(x,t)\mu_1(x)\,dx \tag{14}$$

$$P_{n,3}(0,t)=\xi\int_0^\infty P_{n,1}(x,t)\mu_1(x)\,dx \tag{15}$$

According to the initial condition, server is idle because number of customers are less than m_1

$$W_n(0)=0,\ W_0(0)=0,\ R(0)=1,\ P_{n,j}(0)=0\ \ for\ \ n=0,1,2\ldots\ \ and\ \jmath=1,2,3 \tag{16}$$

6. Probability Generating Functions

We have define the following probability generating functions

$$\left.\begin{aligned}
P_j\left(x,z,t\right) &= \sum_{n=0}^{\infty} P_{n,j}\left(x,t\right)z^n, \quad j=1,2,3\\
P_j\left(z,t\right) &= \sum_{n=0}^{\infty} P_{n,j}\left(t\right)z^n, \qquad j=1,2,3\\
W\left(z,t\right) &= \sum_{n=0}^{\infty} W_n\left(t\right)z^n\\
\tau\left(z\right) &= \sum_{n=1}^{\infty} \tau_n z^n
\end{aligned}\right\} \tag{17}$$

Taking Laplace transformation of equations (3) to (11) and using equation (17), we have

$$\frac{\partial}{\partial x}\overline{P}_{n,1}\left(x,s\right)+\left(s+\lambda+\mu_1\left(x\right)\right)\overline{P}_{n,1}\left(x,s\right)=\lambda b_1\sum_{r=1}^{n}\rho_r\overline{P}_{n-r,1}(x,s)+\lambda\left(1-b_1\right)\overline{P}_{n,1}(x,s) \tag{18}$$

$$\frac{\partial}{\partial x}\overline{P}_{0,1}\left(x,s\right)+\left(s+\lambda+\sum_{n=m_1}^{n_1}\mu_1\left(x\right)\right)\overline{P}_{0,1}\left(x,s\right)=\lambda\left(1-b_1\right)\overline{P}_{0,1}(x,s) \tag{19}$$

$$\frac{\partial}{\partial x}\overline{P}_{n,2}\left(x,s\right)+\left(s+\lambda+\sum_{n=m_1}^{n_1}\mu_2\left(x\right)\right)\overline{P}_{n,2}\left(x,s\right)=\lambda b_1\sum_{r=1}^{n}\rho_r\overline{P}_{n-r,2}(x,s)+\lambda\left(1-b_1\right)\overline{P}_{n,1}(x,s) \tag{20}$$

$$\frac{\partial}{\partial x}\overline{P}_{0,2}\left(x,s\right)+\left(s+\lambda+\mu_2\left(x\right)\right)\overline{P}_{0,2}\left(x,s\right)=\lambda\left(1-b_1\right)\overline{P}_{0,2}(x,s) \tag{21}$$

$$\frac{\partial}{\partial x}\overline{P}_{n,s}+\left(s+\lambda+\sum_{m_1}^{n_1}\mu_3\left(x\right)\right)\overline{P}_{n,3}\left(x,s\right)=\lambda b_1\sum_{r=1}^{n}\rho_r P_{n-r,3}\left(x,s\right)+\lambda(1-b_1)P_{n,3}(x,s) \tag{21A}$$

$$\frac{\partial}{\partial x}\overline{P}_{0,3}(x,s)+\left(s+\lambda+\sum_{m_1}^{n_1}\mu_2\left(x\right)\right)\overline{P}_{0,3}\left(x,s\right)=\lambda(1-b_1)\overline{P}_{n,3}(x,s) \tag{21B}$$

$$\begin{aligned}
\left(S+\lambda+\zeta\right)\overline{W}_n\left(s\right) &= \lambda b_2\sum_{r=1}^{n}\rho_r\overline{W}_{n-r}(s)+\lambda\left(1-b_2\right)\overline{W}_n\left(s\right)+\left(1-\eta\right)\left(1-\xi\right)\int_0^{\infty}\overline{P}_{n,1}\left(x,s\right)\mu_1\left(x\right)dx\\
&\quad+\int_0^{\infty}\overline{P}_{n,2}\left(x,s\right)\mu_2\left(x\right)dx+\int_0^{\infty}\overline{P}_{n,3}\left(x,s\right)\mu_3\left(x\right)dx
\end{aligned} \tag{22}$$

$$\begin{aligned}
\left(S+\lambda+\zeta\right)\overline{W}_0\left(s\right) &= \lambda\left(1-b_2\right)\overline{W}_0\left(s\right)+\left(1-\eta\right)\left(1-\xi\right)\int_0^{\infty}\overline{P}_{0,1}\left(x,s\right)\mu_1\left(x\right)dx+\int_0^{\infty}\overline{P}_{0,2}\left(x,s\right)\mu_2\left(x\right)dx\\
&\quad+\int_0^{\infty}\overline{P}_{0,3}\left(x,s\right)\mu_3\left(x\right)dx
\end{aligned} \tag{23}$$

$$\left(s+\lambda\right)R\left(s\right)+\lambda R\left(s\right)=1+\zeta W_0\left(s\right)+\lambda\left(1-b_1\right)R\left(s\right) \tag{24}$$

Now, also taking the Laplace of boundary conditions

$$\overline{P}_{n,1}\left(0,s\right)=\zeta\overline{w}_{n+n_1}(s) \tag{25}$$

$$\overline{P}_{n,1}\left(0,s\right)=\zeta\sum_{b=m_1}^{n_1}\overline{w}_b(s)+\lambda b_1\overline{R}(s) \tag{26}$$

$$\overline{P}_{n,2}\left(0,s\right)=\eta\int_0^{\infty}P_{n,1}\left(x,s\right)\mu_1\left(x\right)dx \tag{27}$$

$$\overline{P}_{n,3}\left(0,s\right)=\xi\int_0^{\infty}P_{n,1}\left(x,s\right)\mu_1\left(x\right)dx \tag{28}$$

(29)

Now multiply the equation (18) by z^n and summing over n from 1 to ∞ and adding equation (19) and using the generating function we have obtained,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\partial}{\partial x}\overline{P}_{n,1}(x,s)z^n+\sum_{n=1}^{\infty}(s+\lambda+\sum_{n=m_1}^{n_1}\mu_1(x))\overline{P}_{n,1}(x,s)\,z^n+\frac{\partial}{\partial x}\overline{P}_{0,1}(x,s)+(s+\lambda+\mu_1(x))\,\overline{P}_{0,1}(x,s)\\=\sum_{n=1}^{\infty}\lambda b_1\sum_{r=1}^{n}\rho_r\overline{P}_{n-r,1}(x,s)+\lambda\,(1-b_1)\,P_{n,1}(x,s)+\lambda\,(1-b_1)\,P_{0,1}(x,s)\frac{\partial}{\partial x}\overline{P}_1(x,z,s)\\+\left\{s+\lambda b_1\,(1-f\,(z))+\sum_{n=m_1}^{n_1}\mu_1\,(x)\right\}\overline{P}_1\,(x,z,s)=0\end{aligned}\tag{30}$$

Performing similar operations on (20), (21), (22) and (23) we get

$$\frac{\partial}{\partial x}\overline{P}_2\,(x,z,s)+\left\{s+\lambda b_1\,(1-\rho\,(z))+\sum_{n=m_1}^{n_1}\mu_2\,(x)\right\}\overline{P}_2\,(x,z,s)=0\tag{31}$$

$$\frac{\partial}{\partial x}\overline{P}_3\,(x,z,s)+\left\{s+\lambda b_1\,(1-\rho\,(z))+\sum_{n=m_1}^{n_1}\mu_2\,(x)\right\}\overline{P}_2\,(x,z,s)=0\tag{31A}$$

$$\begin{aligned}[\zeta+s+\lambda b_2(1-\rho(z)]\,\overline{W}\,(z,s)=(1-\eta)\,(1-\xi)\int_0^{\infty}P_1\,(x,z,s)\,\mu_1\,(x)\,dx\\+\int_0^{\infty}P_2\,(x,z,s)\,\mu_2\,(x)\,dx+\int_0^{\infty}P_3\,(x,z,s)\,\mu_3\,(x)\,dx\end{aligned}\tag{32}$$

Now assume that

$$A_1=s+\lambda b_1(1-\rho\,(z))\tag{33}$$

$$A_2=s+\lambda b_2(1-\rho\,(z))\tag{34}$$

Now using this resulting above equations (30), (31) and (32)

$$\frac{\partial}{\partial x}\overline{P}_1\,(x,z,s)+\left(A_1+\sum_{n=m}^{n_1}\mu_1\,(x)\right)\overline{P}_1\,(x,z,s)=0\tag{35}$$

$$\frac{\partial}{\partial x}\overline{P}_2\,(x,z,s)+\left(A_1+\sum_{n=m}^{n_1}\mu_2\,(x)\right)\overline{P}_2\,(x,z,s)=0\tag{36}$$

$$\frac{\partial}{\partial x}\overline{P}_3\,(x,z,s)+\left(A_1+\sum_{n=m}^{n_1}\mu_3\,(x)\right)\overline{P}_3\,(x,z,s)=0\tag{37}$$

$$\begin{aligned}(\zeta+A_2)\overline{W}\,(z,s)=(1-\eta)\,(1-\xi)\int_0^{\infty}\overline{P}_1\,(x,z,s)\,\mu_1\,(x)\,dx\\+\int_0^{\infty}\overline{P}_2\,(x,z,s)\,\mu_2\,(x)\,dx+\int_0^{\infty}\overline{P}_3\,(x,z,s)\,\mu_3\,(x)\,dx\end{aligned}\tag{38}$$

Now multiply the equation (25) by z^{n+n_1} and summing over n from 0 to ∞ and adding equation (26) multiplying by z^n and using the generation function results, we get

$$P_1\,(0,z,s)=\left[\lambda \mathrm{b}_1-(s+\lambda b_1)\,z^{-n_1}\right]R\,(s)+\zeta z^{-n_1}W\,(z,s)+z^{-n_1}+\zeta\sum_{b=m}^{n_1-1}W_b(s)+\zeta W_0(s)z^{-n_1}\tag{39}$$

Multiply the equation (27) by z^n and summing over n from 0 to ∞ and using the generating function results

$$P_2\,(0,z,s)=\ \eta\int_0^{\infty}P_1(x,z,s)\mu_1\,(x)\,dx\tag{40}$$

Similarly

$$P_3\,(0,z,s)=\ \xi\int_0^{\infty}P_1(x,z,s)\mu_2\,(x)\,dx\tag{41}$$

Now by integrating equations (35), (36) and (37) between the limits 0 to ∞ and obtain

$$P_1(x,z,s) = P_1(0,z,s)\,e^{-A_1+\int\limits_0^x \sum\limits_{n=m_1}^{n_1} \mu_1(x)dx} \tag{42}$$

$$P_2(x,z,s) = P_2(0,z,s)\,e^{-A_1+\int\limits_0^x \sum\limits_{n=m_1}^{n_1} \mu_2(x)dx} \tag{43}$$

$$P_3(x,z,s) = P_3(0,z,s)\,e^{-A_1+\int\limits_0^x \sum\limits_{n=m_1}^{n_1} \mu_3(x)dx} \tag{44}$$

Now by integrating equation (42), (43) and (44) between the limits 0 to x and obtain

$$\int P_1(x,z,s)\,dx = P_1(0,z,s)\left[e^{-A_1-\int\limits_0^x \sum\limits_{n=m_1}^{n_1} \mu_1(x)dx} - \frac{F_1(A_1)}{A_1}\right] \tag{45}$$

$$\int P_2(x,z,s)\,dx = P_2(0,z,s)\left[e^{-A_1-\int\limits_0^x \sum\limits_{n=m_1}^{n_1} \mu_2(x)dx} - \frac{F_2(A_1)}{A_1}\right] \tag{46}$$

$$\int P_3(x,z,s)\,dx = P_3(0,z,s)\left[e^{-A_2-\int\limits_0^x \sum\limits_{n=m_1}^{n_1} \mu_3(x)dx} - \frac{F_3(A_1)}{A_1}\right] \tag{47}$$

Now integrating by parts with respect to x, we get

$$\int_0^\infty P_1(x,z,s)\,\mu_1(x)\,dx = P_1(0,z,s)\,G_1(A_1) \tag{48}$$

$$\int_0^\infty P_2(x,z,s)\,\mu_2(x)\,dx = P_2(0,z,s)\,G_2(A_1) \tag{49}$$

$$\int_0^\infty P_3(x,z,s)\,\mu_3(x)\,dx = P_3(0,z,s)\,G_3(A_1) \tag{50}$$

$$P_2(0,z,s) = \eta\left[\left(\lambda b_1-(s+\lambda b_1)z^{-n_1}\right)R(s)+\zeta z^{-n_1}w(z,s)+\zeta\sum_{b=m}^{n_1-1}w_b(s)+\zeta w_0(s)\,z^{-n_1}\right]G_1(A_1) \tag{51}$$

$$P_3(0,z,s) = \xi\left[\left(\lambda b_1-(s+\lambda b_1)z^{-n_1}\right)R(s)+\zeta z^{-n_1}w(z,s)+\zeta\sum_{b=m}^{n_1-1}w_b(s)+\zeta w_0(s)\,z^{-n_1}\right]G_2(A_1) \tag{52}$$

From equation (38), (48), (49) and (50), we get

$$(\zeta+A_2)\,\overline{w}(z,s) = (1-\eta)(1-\xi)\,P_1(0,z,s)\,G_1(A_1)+P_2(0,z,s)\,G_2(A_1)+P_3(0,z,s)\,G_3(A_1) \tag{53}$$

$$w(z,s) = \frac{\left[\left(\lambda b_1-(s+\lambda b_1)z^{-n_1}\right)R(s)+\zeta\sum\limits_{b=m_1}^{n_1}w_b(s)+\zeta w_0(s)\,z^{-n_1}\right]\times\left[(1-\eta)(1-\zeta)\,G_1(A_1)+\eta G_1(A_1)\,G_2(A_1)+\xi G_2(A_2)\,G_3(A_1)\right]}{(\zeta+A_2)-\zeta z^{-n_1}\times\left[(1-\eta)(1-\zeta)\,G_1(A_1)+\eta G_1(A_1)\,G_2(A_1)+\xi G_2(A_1)\,G_3(A_1)\right]} \tag{54}$$

$P_1(0,z,s)\,G_1(A_1)$, $P_2(0,z,s)\,G_2(A_2)$ and $P_3(0,z,s)$ in equation (45), (46) and (47) and using the definition of probability density function, we get

$$P_1(z,s) = \left[\left(\lambda b_1-(s+\lambda b_1)z^{-n_1}\right)R(s)+\zeta z^{-n_1}w(z,s)+z^{-n_1}+\zeta\sum_{b=m}^{n_1-1}w_b(s)+\zeta w_0(s)\,z^{-n_1}\right]\left[e^{-A_1x-\int\limits_0^x \sum\limits_{n=m_1}^{n_1} \mu_3(x)dx} - \frac{F_1(A_1)}{A_1}\right] \tag{55}$$

$$P_2(z,s) = \left[\eta\left[\left(\lambda b_1-(s+\lambda b_1)z^{-n_1}\right)R(s)+\zeta z^{-n_1}w(z,s)+\zeta\sum_{b=m}^{n_1}w_b(s)+\zeta w_0(s)\,z^{-n_1}\right]G_1(A_1)\right]$$

$$\left[e^{-A_2 x-\int_0^x \sum_{n=m_1}^{n_1} \mu_2(x)dx} - \frac{F_1(A_1)}{A_1}\right] \tag{56}$$

$$P_3\left(z,s\right)=\left[\xi\left[\left(\lambda b_1-\left(s+\lambda b_1\right)z^{-n_1}\right)R\left(s\right)+\zeta z^{-n_1}w\left(z,s\right)+\zeta\sum_{b=m}^{n_1}w_b\left(s\right)+\zeta w_0\left(s\right)z^{-n_1}\right]G_2(A_1)\right]$$
$$\left[e^{-A_3 x-\int_0^x \sum_{n=m_1}^{n_1} \mu_3(x)dx} - \frac{F_3(A_1)}{A_1}\right] \tag{57}$$

Equation (54) defines the probability generating function with n_1 unknowns, now by using Rouche's theorem of complex variable, it is given that denominators of equation (54) has n_1 zeros inside the cantour $z = 1$. Since *w(z,s)* is analysis inside the unit circle $|z| = 1$. Therefore the numerator of the right hand side of this equation must be vanish at these points which rise to a set of n_1 linear unknowns. Therefore $w(z, s)$, $P_1\left(z,s\right)$, $P_2\left(z,s\right)$ and $P_3\left(z,s\right)$ can be unequally determined.

7. The Steady State Analysis

To determine the steady state results we leave the argument t, and To do this we use the time dependent analysis at this point. Therefore the corresponding steady state results can be obtained by using the well known probability.

$$\log_{s\to 0}\overline{s}f(s)=\log_{t\to\infty}f(t) \tag{58}$$

Now from equation (54), (55), (56) and (57)

$$w\left(z\right)=\left[\left(\lambda b_1-\lambda b_1 z^{-n_1}\right)R+\zeta\sum_{b=m_1}^{n_1}W_b+\zeta W_0 z^{-n_1}\right]$$
$$\left[\left(1-\eta\right)\left(1-\zeta\right)G_1\left(y_1(z)\right)+\eta G_1\left(y_1(z)\right)G_2\left(y_1(z)\right)+\xi G_2\left(y_1(z)\right)G_3\left(y_1(z)\right)\right] \tag{59}$$

$$P_1\left(z\right)=\left[\left(\lambda b_1-\lambda b_1 z^{-n_1}\right)R+\zeta z^{-n_1}w\left(z\right)+z^{-n_1}+\zeta\sum_{b=m}^{n_1}w_b+\zeta w_0 z^{-n_1}\right]\times\left[e^{-y_1(z)-\int_0^x \mu_1(x)dx}-\frac{F_1(y_1(z))}{y_1(z)}\right] \tag{60}$$

$$P_2\left(z\right)=\eta\left(\left(\lambda b_1-\lambda b_1 z^{-n_1}\right)R+\zeta z^{-n_1}w\left(z\right)+z^{-n_1}+\zeta\sum_{b=m}^{n_1}w_b+w_0 z^{-n_1}\right)F_2\left(y_1\left(z\right)\right)$$
$$\left[e^{-y_1(z)-\int_0^x \sum_{n=m}^{n_1} \mu_2(x)dx}-\frac{F_2(y_2(z))}{y_2(z)}\right] \tag{61}$$

$$P_3\left(z\right)=\xi\left[\left(\left(\lambda b_1-\lambda b_1 z^{-n_1}\right)R+\zeta z^{-n_1}w\left(z\right)+z^{-n_1}+\zeta\sum_{b=m}^{n_1}w_b+\zeta w_0 z^{-n_1}\right)F_3\left(y_1\left(z\right)\right)\right]$$
$$\left[e^{-y_3(z)-\int_0^x \sum_{n=m}^{n_1} \mu_3(x)dx}-\frac{F_3(y_1(z))}{y_3(z)}\right] \tag{62}$$

Let us assume that $B_q(z)$ is the probability generating function therefore, we can write

$$B_q\left(z\right)=P_1\left(z\right)+P_2\left(z\right)+P_3\left(z\right)+w\left(z\right) \tag{63}$$

Now we put $z = 1$ in equation (63), we get

$$B_q\left(1\right)=\left\{\left(\lambda b_1+\lambda b_1 n_1\right)R-\zeta n_1 w\left(1\right)+\zeta w'\left(1\right)-n_1-n_1\zeta w_0\right\}\left(e_1^*+e_2^*+e_3^*\right)$$
$$+\left\{\left(\lambda b_1-\lambda b_1\right)R+\zeta w_0\left(1\right)+1+\zeta\sum_{b=m_1}^{n_1}w_b+\zeta w_0\right\}\left[q_1^*+q_2^*+q_3^*+q_4^*+q_5^*\right]+w'\left(1\right)$$

Where:

$$
\begin{aligned}
e_1^* &= \frac{e^{-y_1(1)-\int_0^x \mu_1(x)dx} - f_1\left(y_1\left(1\right)\right)}{y_1'\left(1\right)},\\
e_2^* &= \frac{e^{-y_1(1)-\int_0^x \sum_{n=m_1}^{n_1} \mu_2(x)dx} - f_2\left(y_2\left(1\right)\right)}{y_2'\left(1\right)},\\
e_3^* &= \frac{e^{-y_3(1)-\int_0^x \sum_{n=m_1}^{n_1} \mu_3(x)dx} - f_3(y_3\left(1\right))}{y_3'(1)},\\
q_1^* &= \frac{\left\{e^{-y_1(1)-\int_0^x \mu_1(x)dx}\right\}\{y_1'(1) - f_1(y_1'\left(1\right))\}}{y_1'(1)},\\
q_2^* &= \frac{\left\{e^{-y_1(1)-\int_0^x \sum_{m_1}^{n_1} \mu_2(x)dx}\right\}\{y_1'\left(1\right) - f_2\left(y_2'\left(1\right)\right)\} f_2(y_1\left(1\right))}{y_2'(1)}\\
q_3^* &= \frac{\left\{e^{-y_3(1)-\int_0^x \sum_{m_1}^{n_1} \mu_3(x)dx}\right\}\{y_3'\left(1\right) - f_3\left(y_3'\left(1\right)\right)\} f_2(y_2\left(1\right))}{y_3'(1)}\\
q_4^* &= \left\{f_2\left(y_2'\left(1\right)\right)\right\}\left\{e^{-y_1(1)-\int_0^x \sum_{m_1}^{n_1} \mu_2(x)dx} - f_2(y_2\left(1\right)\right\}\\
q_5^* &= \left\{f_3\left(y_3'\left(1\right)\right)\right\}\left\{e^{-y_3(1)-\int_0^x \sum_{m_1}^{n_1} \mu_2(x)dx} - f_3(y_3\left(1\right)\right\}
\end{aligned}
$$

Number of customers waiting in the queue for the essential service will be

$$L_q = \frac{d}{dz}B_q\left(z\right)\Big|_{z=1} = \frac{d}{dz}B_q\left(1\right)$$

We suppose that $B_q\left(z\right)$ can be written as $B_q\left(z\right) = \dfrac{\alpha(z)}{\beta(z)}$

$$
\begin{aligned}
\frac{d}{dz}B_q\left(z\right)\Big|_{z=1} = &\left(\frac{d}{dz}\left[e_1^* + e_2^* + e_3^*\right]\right)\left(\left(\lambda b_1 + \lambda b_1 n_1 e^{n_1}\right)R - \zeta n_1 w\left(z\right) + \zeta w'\left(z\right) - n_1 e^{n_1} - n_1 \zeta w_0 e^{-n_1}\right)\\
&+ \left(e_1^* + e_2^* + e_3^*\right)\frac{d}{dz}\left(\lambda b_1 + \lambda b_1 n_1 e^{-n_1} - \zeta n_1 w\left(z\right) + \zeta w'\left(z\right) - n_1 e^{-n_1} - n_1 \zeta w_0 e^{-n_1}\right)\\
&+ \frac{d}{dz}\left(q_1^* + q_2^* + q_3^* + q_4^*\right)\left[\left(\lambda b_1 - \lambda b_1\right)R - \zeta w_0\left(z\right) + z^{-n_1} + \zeta\sum_{b=m_1}^{n_1} w_b + \zeta w_0\right]\\
&+ \left(q_1^* + q_2^* + q_3^* + q_4^*\right)\left[-\zeta w_0'\left(z\right) - n_1 z^{-n_1-11}\right]
\end{aligned}
\tag{64}
$$

For finding the value of R, we will use normalized condition will be

$$R = 1 - B_q\left(1\right) \tag{65}$$

The above equation (65) represents the utilization factor for the mathematical model.

8. Conclusion

In this paper, we studied the Markovian queueing system with bulk arrival and three types of bulk service, first is essential service, second and third are optional services as per customer choice. This work is helpful to reduce the customers waiting time for the service station where customer deals with one or more service facilities. We have determined the server busy period, system utilization factor and waiting time factor by using the study state equations.

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