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On Spectra of D-Eccentricity Matrix of Some Graphs

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Abstract: For any two vertices u and v of a graph G, $d(u,v)$ is the length of the shortest path between the vertices u and v. D. Reddy Babu and P.L.N. Varma introduced the concept of D-distance. D-distance considers the degree of all vertices present in a path while defining its length. In this paper, D-eccentricity spectra of D-eccentricity matrix of some class of graphs are computed.

MSC: 0512, 05C50

Keywords: Distance, D-distance, Eccentricity matrix, Spectrum.

1. Introduction

The theory of Linear Algebra, in particular theory of matrices is a powerful tool to study the spectral properties of the graph spectra and in turn matrix properties of the graph can be recognized from the spectrum of its matrix.

By a graph G, we mean non-trivial, finite and undirected graph without multiple edges and loops.

In graph G, the usual distance $d(u,v)$ is the length of the minimum path connecting the vertices u and v of G.

The D-distance $d^D(u,v)$ between two vertices of a connected graph G is defined as

$$d^D(u,v) = \min\left\{d(u,v) + deg(u) + deg(v) + \sum deg(w)\right\}$$

where sum runs over all the intermediate vertices w in the path and minimum is taken over all $u-v$ paths in G [1].

The D-eccentricity of any vertex v, $e^D(v)$ is defined as the maximum D-distance from v to any other vertex, that is $e^D(v) = \max\left\{d^D(u,v) : u \in V(G)\right\}$, where $V(G)$ is the vertex set of graph G [1].

Let $\beta_1 \geq \beta_2 \geq \beta_3 \geq \ldots \geq \beta_r$ denote different eigenvalues of the matrix $D_\varepsilon(G)$. Since, this matrix is real symmetric, all the D_ε eigen values are real D_ε spectrum is denoted by $spec\, D_\varepsilon$ and defined as,

$$spec\, D_\varepsilon = \begin{Bmatrix} \beta_1 & \beta_2 & \beta_3 & \ldots & \beta_r \\ m_1 & m_2 & m_3 & \ldots & m_r \end{Bmatrix}$$

Where m_i is the algebraic multiplicity of the eigenvalues β_i, for $1 \leq i \leq r$.

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1.1. Definitions, notations and preliminary results

For a square matrix A of order n with real entries $det(A)$, $det(\lambda I - A)$ and $spec(A)$ denote the determinant, characteristic polynomial and spectrum of A respectively.

$J_{n\times n}$ or J_n denotes the $n \times n$ matrix with all entries as 1 and I_n denotes $n \times n$ identity matrix.

Lemma 1.1 ([5]). *If matrix A is an $n \times n$ matrix partitioned as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where A_{11}, A_{22} are square matrices. If A_{11} is non singular matrix then,* $\det(A) = \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12})$. *Also, if A_{22} is non singular matrix then,* $\det(A) = \det(A_{22})\det(A_{12} - A_{12}A_{22}^{-1}A_{21})$.

Lemma 1.2 ([5]). *Let B is square matrix of order n. If each column sum of B is equal to one of the eigenvalues (say α) of B, then*

$$J_{1\times n}(\lambda I - B)^{-1}J_{n\times 1} = \frac{n}{n-\alpha}.$$

Lemma 1.3 ([5]). *Let $B = \begin{bmatrix} B_0 & B_1 \\ B_1 & B_0 \end{bmatrix}$ be a symmetric 2×2 block matrix with B_0 and B_1 are square matrices of the same order. Then spectrum of B is the union of spectra $(B_0 + B_1)$ and spectra $(B_0 - B_1)$.*

Lemma 1.4 ([3]). *Let A and B be square matrices of order n. If $spec(A) = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\}$ and $spec(B) = \{\mu_1, \mu_2, \mu_3, \ldots, \mu_n\}$ then, $spec(A \otimes B) = \{\lambda_i\mu_j;\ i = 1, 2, 3, \ldots, n,\ j = 1, 2, 3, \ldots, n\}$, where $\otimes$ denotes tensor product.*

Definition 1.5 ([4]). *A star graph on n vertices is denoted by $K_{1,n-1}$.*

Definition 1.6 ([4]). *The n-barbell graph $B_{n,n}$ is a graph obtained by connecting two copies of K_n by a bridge.*

Definition 1.7 ([7]). *The corona $G \circ H$ of G and H is defined as the graph obtained by taking one copy of G and n disjoint copies of H, say $H_1, H_2, \ldots, H_n$ and joining the vertex v_i of G to every vertex in H_i, the i^{th} copy of H .*

In this article, motivated by the definition of eccentricity matrix $\varepsilon(G)$ of a connected graph G and spectra of eccentricity matrix of some class of graphs [6, 8], we define D-eccentricity matrix $D_\varepsilon(G)$ and find D-eccentricity spectra $spec\, D_\varepsilon(G)$ of some class of graphs.

2. Spectra of D-Eccentricity Matrix of Some Class of Graphs

For a graph G of order n, if $u_1, u_2, u_3, \ldots, u_n \in V(G)$, D-eccentricity matrix is defined by,

$$D_\varepsilon(G) = \begin{cases} d_{ij}^D & if\, d_{ij}^D = \min\{e^D(u_i),\, e^D(u_j)\} \\ 0 & if\, d_{ij}^D < \min\{e^D(u_i),\, e^D(u_j)\} \end{cases}$$

The D_ε spectrum of a graph consists of D_ε eigenvalues of D-eccentricity matrix.

Theorem 2.1. *Let $K_{1,n-1}$ be a star graph of n vertices then*

$$\det\left(D_\varepsilon(K_{1,n-1})\right) = (n+3)^{n-2}(-1)^{n-1}(n+1)^2(n-1)$$

and

$$spec\, D_\varepsilon(K_{1,n-1}) = \begin{Bmatrix} \frac{(n+3)(n-2)\pm\sqrt{(n+3)^2(n-2)^2+4(n-1)(n+1)^2}}{2} & -(n+3) \\ 1 & n-2 \end{Bmatrix}$$

Proof. Let $K_{1,n-1}$ be a star graph of n vertices $\{v_1, v_2, v_3, \dots v_n\}$, where v_1 is the vertex of degree (n-1). Then,

$$D_\varepsilon(K_{1,n-1}) = \begin{bmatrix} 0 & (n+1)J_{1\times(n-1)} \\ (n+1)J_{(n-1)\times 1} & (n+3)(J_{n-1} - I_{n-1}) \end{bmatrix}.$$

Since, $(n+3)(J_{n-1} - I_{n-1})$ is a non singular matrix, by Lemma 1.1, we have

$$\begin{aligned}
\det\left(D_\varepsilon(K_{1,n-1})\right) &= \det\left\{(n+3)\left[J_{n-1} - I_{n-1}\right]\right\} \det\left[0 - (n+1)\, J_{1\times(n-1)} \left\{\left((n+3)\left(J_{n-1} - I_{n-1}\right)\right)^{-1} (n+1)\, J_{(n-1)\times 1}\right\}\right] \\
&= (n+3)^{n-2}(-1)^{n-2}\,(n-2)\,(n+1)^2 \det\left[J_{1\times(n-1)}(I_{n-1} - J_{n-1})^{-1} J_{(n-1)\times 1}\right] \\
&= (n+3)^{n-2}(-1)^{n-2}\,(n-2)\,(n+1)^2 \left[\frac{n-1}{1-(n-1)}\right] \\
&= (n+3)^{n-2}(-1)^{n-1}(n+1)^2\,(n-1)\,.
\end{aligned}$$

The characteristic polynomial of $D_\varepsilon(K_{1,n-1})$ is,

$$\det\left[D_\varepsilon(K_{1,n-1} - \lambda I_n)\right] = \det \begin{bmatrix} -\lambda & (n+1)J_{1\times(n-1)} \\ (n+1)J_{(n-1)\times 1} & (n+3)(J_{n-1} - I_{n-1}) - \lambda I_{n-1} \end{bmatrix}$$

By Lemma 1.1,

$$\begin{aligned}
\det\left[D_\varepsilon(K_{1,n-1} - \lambda I_n)\right] &= (-\lambda)\det\left[(n+3)(J_{n-1} - I_{n-1}) - \lambda I_{n-1} - (n+1)J_{1\times(n-1)}(-\lambda)^{-1}(n+1)J_{(n-1)\times 1}\right] \\
&= (-\lambda)\det\left[(n+3)(J_{n-1} - I_{n-1}) - \lambda I_{n-1} + \frac{(n+1)^2}{\lambda} J_{n-1}\right] \\
&= (-\lambda)\det\left[\left\{(n+3) + \frac{(n+1)^2}{\lambda}\right\} J_{n-1} - \left\{(n+3) + \lambda\right\} I_{n-1}\right] \\
&= (-\lambda)\left[(n-1)\left\{(n+3) + \frac{(n+1)^2}{\lambda}\right\} - \left\{(n+3) + \lambda\right\}\right]\left[-(n+3) - \lambda\right]^{n-2} \\
&= \left[\lambda - \left\{\frac{(n-2)(n+3) \pm \sqrt{(n+3)^2(n-2)^2 + 4(n-1)(n+1)^2}}{2}\right\}\right]\left[-(n+3) - \lambda\right]^{n-2}
\end{aligned}$$

Therefore

$$spec\, D_\varepsilon(K_{1,n-1}) = \begin{Bmatrix} \frac{(n+3)(n-2) \pm \sqrt{(n+3)^2(n-2)^2 + 4(n-1)(n+1)^2}}{2} & -(n+3) \\ 1 & n-2 \end{Bmatrix}$$

□

Corollary 2.2. *If $n \geq 3$ then the least eigenvalue of $D_\varepsilon(K_{1,n-1})$ is $-(n+3)$.*

Proof. Suppose it is not so, then

$$\begin{aligned}
\frac{(n-2)(n+3) - \sqrt{(n-2)^2(n+3)^2 + 4(n-1)(n+1)^2}}{2} &< -(n+3) \\
(n-2)(n+3) + 2(n+3) &< \sqrt{(n-2)^2(n+3)^2 + 4(n-1)(n+1)^2} \\
n(n+3) &< \sqrt{(n-2)^2(n+3)^2 + 4(n-1)(n+1)^2}
\end{aligned}$$

This implies, $(n+3)^2 < (n+1)^2$. This is not possible, hence $-(n+3)$ is the least eigenvalue of $D_\varepsilon(K_{1,n-1})$. □

Example 2.3. *For the Star graph $K_{1,3}$ of Figure 1, D eccentricity matrix is*

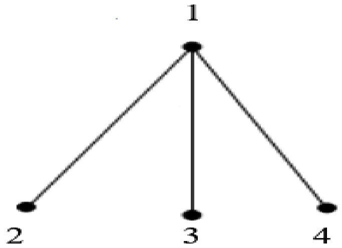

Figure 1: Star graph $K_{1,3}$

$$D_\varepsilon(K_{1,3}) = \begin{bmatrix} 0 & 5 & 5 & 5 \\ 5 & 0 & 7 & 7 \\ 5 & 7 & 0 & 7 \\ 5 & 7 & 7 & 0 \end{bmatrix}$$

$$det D_\varepsilon(K_{1,3}) = -3675$$

$$spec D_\varepsilon(K_{1,3}) = -7, -7, -4.1355, 18.1355.$$

The following Lemma 2.4 is proved for the sake of completeness, which is about spectrum of a kind of block matrix.

Lemma 2.4. *Let A be a $(n+1)\times(n+1)$ matrix of the form* $A = \begin{bmatrix} 0 & aJ_{1\times n} \\ aJ_{n\times 1} & bJ_n \end{bmatrix}$, *then* $spec(A) = \begin{Bmatrix} 0 & \frac{bn\pm\sqrt{b^2n^2+4a^2n}}{2} \\ n-1 & 1 \end{Bmatrix}$, *where $a, b > 0$.*

Proof. $\det[\lambda I_{n+1} - A] = \det \begin{bmatrix} \lambda & -aJ_{1\times n} \\ -aJ_{n\times 1} & I_n - bJ_n \end{bmatrix}$. By Lemma 1.1 and Lemma 1.2

$$\begin{aligned} \det[\lambda I_{n+1} - A] &= \det[\lambda I_n - bI_n] . \det\left[\lambda - a^2(\lambda I_n - bJ_n)^{-1}J_{1\times n}\right] \\ &= \lambda^{n-1}(\lambda - bn) \det\left[\lambda - \frac{a^2n}{\lambda - bn}\right] \\ &= \lambda^{n-1}\left[\lambda^2 - bn\lambda - a^2n\right] \end{aligned}$$

□

We use the Lemma 2.4 to prove the following theorem.

Theorem 2.5. *Let $B_{n,n}$ be the n-barbell graph then,*

$$spec D_\varepsilon(B_{n,n}) = \begin{Bmatrix} 0 & \frac{(4n+1)(n-1)\pm\sqrt{(4n+1)^2(n-1)^2+4(3n+1)^2(n-1)}}{2} & -\frac{(4n+1)(n-1)\pm\sqrt{(4n+1)^2(n-1)^2+4(3n+1)^2(n-1)}}{2} \\ 2(n-2) & 1 & 1 \end{Bmatrix}$$

Proof. Let K_n be the complete graph on n vertices with vertex set $\{v_1, v_2, v_{3,.......,} v_n\}$ and let us consider a copy of K_n with vertex set $\{w_1, w_{2,..........,} w_n\}$. Let $B_{n,n}$ be the barbell graph obtained by joining the vertices of v_1 and w_1 in the two copies of K_n. Then the D-eccentricity matrix of $B_{n,n}$ is given by

$$\mathrm{D}\varepsilon\left(B_{n,n}\right) = \begin{bmatrix} 0_{n\times n} & A_{n\times n} \\ A_{n\times n} & 0_{n\times n} \end{bmatrix}$$

Where

$$A_{n\times n} = \begin{bmatrix} 0 & (3n+1)\,J_{1\times n-1} \\ (3n+1)J_{1\times n-1} & (4n+1)\,J_{n-1}. \end{bmatrix}.$$

Putting $a = 3n+1$ and $b = 4n+1$ in Lemma 2.4 we get,

$$spec\,(A) = \left\{ \begin{matrix} 0 & \frac{(4n+1)(n-1)\pm\sqrt{(4n+1)^2(n-1)^2+4(3n+1)^2(n-1)}}{2} \\ n-2 & 1 \end{matrix} \right\}$$

for $a = 3n+1$ and $b = 4n+1$. By Lemma 1.3, the spectrum of $\mathrm{D}\varepsilon\left(B_{n,n}\right)$ is the union of eigenvalues A and $-A$. Hence,

$$specD_{\varepsilon}(B_{n,n}) = \left\{ \begin{matrix} 0 & \frac{(4n+1)(n-1)\pm\sqrt{(4n+1)^2(n-1)^2+4(3n+1)^2(n-1)}}{2} & -\frac{(4n+1)(n-1)\pm\sqrt{(4n+1)^2(n-1)^2+4(3n+1)^2(n-1)}}{2} \\ 2(n-2) & 1 & 1 \end{matrix} \right\}.$$

□

Example 2.6. *For the Barbell graph $G = B_{3\times 3}$ of Figure 2, D- eccentricity matrix is*

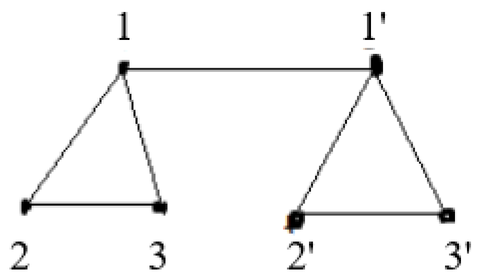

Figure 2: Barbell graph $G = B_{3\times 3}$

$$D\varepsilon\,(G) = \begin{bmatrix} 0 & 0 & 0 & 0 & 10 & 10 \\ 0 & 0 & 0 & 10 & 13 & 13 \\ 0 & 0 & 0 & 10 & 13 & 13 \\ 0 & 10 & 10 & 0 & 0 & 0 \\ 10 & 13 & 13 & 0 & 0 & 0 \\ 10 & 13 & 13 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} O_{3\times 3} & A_{3\times 3} \\ A_{3\times 3} & O_{3\times 3} \end{bmatrix}$$

For,

$$A_{3\times 3} = \begin{bmatrix} 0 & 10J_{1\times 2} \\ 10J_{2\times 1} & 13J_2 \end{bmatrix}$$

$$spec\,D_{\varepsilon}(G) = \left\{ \begin{matrix} 0 & -32.2094 & 32.2094 & 6.2094 & -6.2094 \\ 2 & 1 & 1 & 1 & 1 \end{matrix} \right\}.$$

Before, proceeding to next theorem, we use this definition.

Definition 2.7. *Cocktail party graph is a regular graph on $2n$ vertices with degree $2n-2$.*

Theorem 2.8. *Let CP_k be the cocktail party graph on $k = 2n$ vertices, $n \geq 2$ then,*

$$spec\, D_\varepsilon(CP_k) = \begin{Bmatrix} 2+3(2n-2) & -[2+3(2n-2)] \\ n & n \end{Bmatrix}.$$

Proof. Let CP_k be the cocktail party graph on $k = 2n$ vertices, $n \geq 2$ then, the eccentricity matrix of CP_k is

$$D_\varepsilon(CP_k) = \begin{bmatrix} O_{n\times n} & 2+3(2n-2)I_{n\times n} \\ 2+3(2n-2)I_{n\times n} & O_{n\times n} \end{bmatrix}$$

Therefore, by Lemma 1.3

$$spec\, D_\varepsilon(CP_k) = \begin{Bmatrix} 2+3(2n-2) & -[2+3(2n-2)] \\ n & n \end{Bmatrix}.$$

□

Example 2.9. *For the Cocktail party graph $G = CP_2$ of Figure 3,*

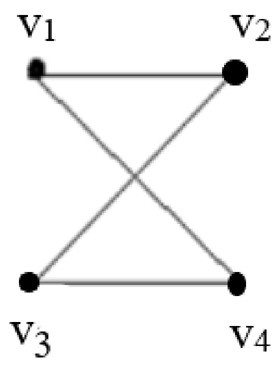

Figure 3: $G =$ Cocktail Party Graph (CP_2)

$$D\varepsilon\,(G) = \begin{bmatrix} 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 8 \\ 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \end{bmatrix}$$

$$spec\, D\varepsilon\,(G) = \begin{Bmatrix} 8 & -8 \\ 2 & 2 \end{Bmatrix}$$

We use this definition to proceed to next theorem,

Definition 2.10. *Suppose CS_k is a Crown graph with k vertices where $k = 2n$. Then the vertex set of CS_k is partitioned into two subsets V_1 and V_2 such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$.*

Theorem 2.11. *Let CS_k is a Crown graph on $k = 2n$ vertices for $n > 2$ then,*

$$spec\, D\varepsilon\,(CS_k) = \begin{Bmatrix} 3+4\,(n-1) & -[3+4\,(n-1)] \\ n & n \end{Bmatrix}.$$

Proof. Let $V = \{v_1,v_2,v_3,...v_n\}$ and $W = \{w_1,w_2,...w_n\}$ be two subsets of CS_k and all vertices of v_1 are correlated to each vertex of v_2 except paired ones. The eccentricity matrix of CS_k is

$$\mathrm{D}\varepsilon\,(CS_k) = \begin{bmatrix} 0_{n\times n} & 3+4\,(n-1)\,I_{n\times n} \\ 3+4\,(n-1)\,I_{n\times n} & 0_{n\times n} \end{bmatrix}.$$

By Lemma 1.3

$$\mathrm{spec}\ \mathrm{D}\varepsilon\,(CS_k) = \begin{Bmatrix} 3+4\,(n-1) & -[3+4\,(n-1)] \\ n & n \end{Bmatrix}.$$

□

Example 2.12. *For the Crown graph $G = CS_3$ of Figure 4,*

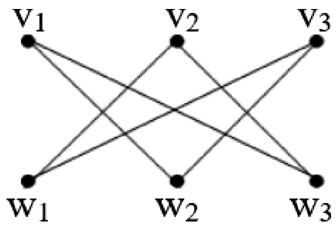

Figure 4: G = Crown graph CS_3

$$D_\varepsilon(G) = \begin{bmatrix} 0 & 0 & 0 & 11 & 0 & 0 \\ 0 & 0 & 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 11 \\ 11 & 0 & 0 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11 & 0 & 0 & 0 \end{bmatrix}$$

$$spec\, D_\varepsilon(G) = \begin{Bmatrix} 11 & -11 \\ 3 & 3 \end{Bmatrix}$$

Theorem 2.13. *Let $K_{n_1,n_2,n_3,...n_k}$ be complete k-partite graph such that $\sum_{i=1}^{k} n_i = n$; and $n_i \geq 2$ and $k \leq n-1$. Then,*

$$spec\, D_\varepsilon(K_{n_1,n_2,n_3,...n_k}) = \begin{Bmatrix} -2+3(n-n_1) & 2+3(n-n_1)\,\{n_1-1\} & 2+3(n-n_2)\,\{n_2-1\} & \dots & 2+3(n-n_k)\,\{n_k-1\} \\ (n-k) & 1 & 1 & \dots & 1 \end{Bmatrix}$$

that is

$$\begin{Bmatrix} -[2+3(n-n_1)] & 2+3(n-n_1)\,\{n_1-1\} \\ n-k & k \end{Bmatrix}$$

where $n_1 = n_2 = n_3 = \ldots = n_k = n_1$.

Proof. $D\varepsilon\left(K_{n_1,n_2,n_3\ldots\ldots\ldots n_k}\right)$

$$= \begin{bmatrix} [2+3(n-n_1)]\{J_{n_1}-I_{n_1}\} & 0 & 0 & \cdots & 0 \\ 0 & [2+3(n-n_2)]\{J_{n_2}-I_{n_2}\} & 0 & \cdots & 0 \\ 0 & 0 & [2+3(n-n_3)]\{J_{n_3}-I_{n_3}\} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \cdots & [2+3(n-n_k)]\{J_{n_k}-I_{n_k}\} \end{bmatrix}$$

Hence, spectrum of $D\varepsilon\left(K_{n_1,n_2,n_3\ldots\ldots\ldots n_k}\right)$ is the union of eigenvalues of

$$[2+3(n-n_2)]\{J_{n_1}-I_{n_1}\}, [2+3(n-n_2)]\{J_{n_2}-I_{n_2}\}, \ldots [2+3(n-n_k)]\{J_{n_k}-I_{n_k}\}.$$

□

Example 2.14. *For the complete 3-partite graph*

$$D\varepsilon(G) = \begin{bmatrix} 0 & 20 & 20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 20 & 0 & 20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 20 & 20 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 20 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20 & 20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 20 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 & 0 & 20 \\ 0 & 0 & 0 & 0 & 0 & 0 & 20 & 20 & 0 \end{bmatrix}$$

$$spec\ D\varepsilon(G) = \begin{Bmatrix} -20 & 40 \\ 6 & 3 \end{Bmatrix}.$$

Theorem 2.15. *Let K_n be the complete graph on n-vertices and P_2 be a path on two vertices. Then*

$$spec\ D\varepsilon(K_n O P_2) = \begin{Bmatrix} 0 & -\lambda_1 & -\lambda_2 & \lambda_1(n-1) & \lambda_2(n-1) \\ n & n-1 & n-1 & 1 & 1 \end{Bmatrix}.$$

Here λ_1 and λ_2 are the roots of $\lambda^2 - 2b\lambda - 2a^2 = 0$, where $a = 2n+6$ and $b = 2n+9$.

Proof. Let K_n be the complete graph on n-vertices and P_2 be a path on vertices. Then, the graph $K_n \circ P_2$ consists of vertices of the complete graph K_n which are labeled as the index set $\{v_1, v_2, v_{3,\ldots\ldots\ldots}, v_n\}$ and disjoint copies of P_2. Each vertex of K_n is joined to both the vertices of P_2. The D-eccentricity matrix of $K_n \circ P_2$ is given by $D_\varepsilon(K_n \circ P_2) = A \otimes B$, where $A = \begin{bmatrix} 0 & (2n+6)\,J_{1\times 2} \\ (2n+6)J_{2\times 1} & (2n+9)\,J_2 \end{bmatrix}$ and $B = J_n - I_n$. By Lemma 2.4

$$spec(A) = \begin{Bmatrix} 0 & \frac{(2n+9)2 \pm \sqrt{(2n+9)^2 2^2 (n-1)^2 + 4.2(2n+6)^2}}{2} \\ 1 & 1 \end{Bmatrix}$$

$$spec(B) = \begin{Bmatrix} -1 & (n-1) \\ (n-1) & 1 \end{Bmatrix}$$

Therefore, by Lemma 1.4,

$$spec\,\mathrm{D}\varepsilon\,(A \otimes B) = \begin{Bmatrix} 0 & -\lambda_1 & -\lambda_2 & -\lambda_1(n-1) & -\lambda_2(n-1) \\ n & n-1 & n-1 & 1 & 1 \end{Bmatrix}$$

Hence,

$$spec\,\mathrm{D}\varepsilon\,(K_n \circ P_2) = \begin{Bmatrix} 0 & -\lambda_1 & -\lambda_2 & -\lambda_1(n-1) & -\lambda_2(n-1) \\ n & n-1 & n-1 & 1 & 1 \end{Bmatrix}$$

□

Example 2.16. *For the graph $G \cong K_3 \circ P_2$ of Figure 5,*

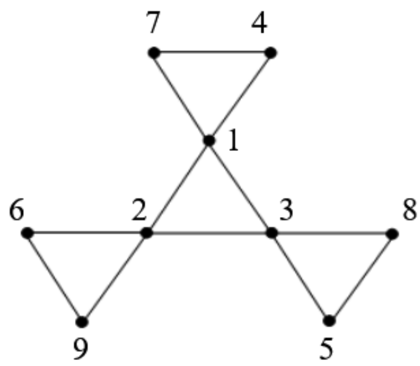

Figure 5: $G \cong K_3 \circ P_2$

$$D\varepsilon\,(G) = \begin{bmatrix} 0 & 0 & 0 & 0 & 12 & 12 & 0 & 12 & 12 \\ 0 & 0 & 0 & 12 & 0 & 12 & 12 & 0 & 12 \\ 0 & 0 & 0 & 12 & 12 & 0 & 12 & 12 & 0 \\ 0 & 12 & 12 & 0 & 15 & 15 & 0 & 15 & 15 \\ 12 & 0 & 12 & 15 & 0 & 15 & 15 & 0 & 15 \\ 12 & 12 & 0 & 15 & 15 & 0 & 15 & 15 & 0 \\ 0 & 12 & 12 & 0 & 15 & 15 & 0 & 15 & 15 \\ 12 & 0 & 12 & 15 & 0 & 15 & 15 & 0 & 15 \\ 12 & 12 & 0 & 15 & 15 & 0 & 15 & 15 & 0 \end{bmatrix}$$

$$spec\,D_\varepsilon(G) = \begin{Bmatrix} 0 & -37.6495 & 7.6495 & -15.2990 & 75.2990 \\ 3 & 2 & 2 & 1 & 1 \end{Bmatrix}$$

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