Int. J. Math. And Appl., 8(4)(2020), 35–39
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Gaussian Prime Labeling of Super Subdivision of Star Graphs

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Abstract: Gaussian integers are the complex numbers of the form a + bi where $a, b \in Z$ and $i^2 = -1$ and it is denoted by $Z[i]$. A Gaussian prime labeling on G is a bijection from the vertices of G to $[\psi_n]$, the set of the first n Gaussian integers in the spiral ordering such that if $uv \in E(G)$, then $\psi(u)$ and $\psi(v)$ are relatively prime. Using the order on the Gaussian integers, we discuss the Gaussian prime labeling of super subdivision of star graphs.

MSC: 05C78.

Keywords: Gaussian Integers, Gaussian Prime Labeling, Super Subdivision of Graphs.

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1. Introduction

The graphs considered in this paper are finite and simple. The terms which are not defined here can be referred from Gallian [1] and West [2]. A labeling or valuation of a graph G is an assignment f of labels to the vertices of G that induces for each edge xy, a label depending upon the vertex labels $f(x)$ and $f(y)$. Let $G = (V(G), E(G))$ be a graph with p vertices. A bijection $f : V(G) \to \{1, 2, 3, \ldots, p\}$ is called prime labeling if for each edge $e = uv, gcd(f(u), f(v)) = 1$. A graph which admits prime labeling is called a prime graph.In the first section we introduce Gaussian integers and their properties. Prime labeling was extended to Gaussian prime labeling by defining the first n Gaussian integers. We take the spiral ordering on the Gaussian integers defined by Steven Klee, Hunter Lehmann and Andrew Park [3]. The spiral ordering allows us to linearly ordering the Gaussian integers. In the second section we discuss Gaussian prime labeling of super subdivision of star graphs.

Definition 1.1. *The star graph $K_{1,n}, n \in \mathbb{N}$ is the graph with n pendant edges incident with the vertex in K_1. The vertex having degree n in $K_{1,n}$ is called the apex vertex.*

Definition 1.2. *Let G be a graph with n vertices and m edges. A graph H is called a supersubdivision of G, if every edge uv of G is replaced by $K_{2,m}$ by identifying u and v with the two vertices in $K_{2,m}$ that form one of the two partite sets.*

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2. Gaussian Integers

The complex numbers of the form $a + bi$, where $a, b \in Z$ and $i^2 = -1$ is called the Gaussian integers and it is denoted by $Z[i]$. Any one of complex numbers $\pm 1, \pm i$ are the units in the Gaussian integers. For any Gaussian integer α, the associate is $u.\alpha$, where u is the Gaussian unit. $N(a + bi)$ denote the norm of the Gaussian integer $a + bi$ and it is given by $a^2 + b^2$. An even Gaussian integer is a Gaussian integer which is divisible by $1 + i$ and odd otherwise.

Definition 2.1. *A Gaussian integer ρ is called a prime Gaussian integer if its only divisors are $\pm 1, \pm i, \pm \rho, or \pm \rho i$.*

Definition 2.2. *Two Gaussian integers α and β are relatively prime if their only common divisors are units in $Z[i]$.*

Theorem 2.3. *In $Z[i]$, a Gaussian integer ρ is prime if and only if it is any one of the following forms:*

(1). $\rho = 1 + i, 1 - i, -1 + i, -1 - i,$

(2). $\rho = p$ or $\rho = pi$ where p is a prime in Z and $|p| \cong 3(\mod 4)$,

(3). $N(\rho)$ is a prime integer congruent to $1(\mod 4)$.

The Gaussian integers are not totally ordered. So we take the spiral ordering on n Gaussian integers defined by Hunter Lehmann and Andrew Park [3] as follows.

Definition 2.4. *The ordering of the Gaussian integers is called spiral ordering which is a recursively defined ordering of the Gaussian integers. We denote the n$^{\text{th}}$ Gaussian integer in the spiral ordering by ψ_n. The ordering is defined beginning with $\psi_1 = 1$ and continuing as:*

$$\psi_{n+1} = \begin{cases} \psi_n + i, \text{ if } \Re(\psi_n) \cong 1 \ (mod\ 2), \Re(\psi_n) > \Im(\psi_n) + 1 \\ \psi_n - 1, \text{ if } \Im(\psi_n) \cong 0 \ (mod\ 2), \Re(\psi_n) \leq \Im(\psi_n) + 1, \Re(\psi_n) > 1 \\ \psi_n + 1, \text{ if } \Im(\psi_n) \cong 1 \ (mod\ 2), \Re(\psi_n) < \Im(\psi_n) + 1 \\ \psi_n + i, \text{ if } \Im(\psi_n) \cong 0 \ (mod\ 2), \Re(\psi_n) = 1 \\ \psi_n - i, \text{ if } \Re(\psi_n) \cong 0 \ (mod\ 2), \Re(\psi_n) \geq \Im(\psi_n) + 1, \Im(\psi_n) > 0 \\ \psi_n + 1, \text{ if } \Re(\psi_n) \cong 0 \ (mod\ 2), \Im(\psi_n) = 0. \end{cases}$$

Figure [1] shows the spiral ordering of Gaussian integers.

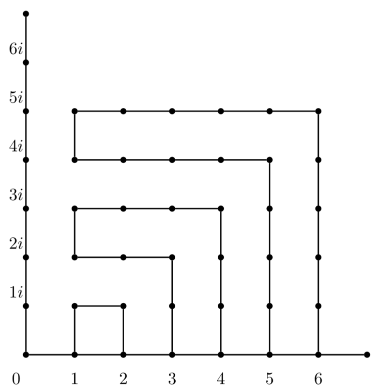

Figure 1. **Spiral ordering of Gaussian integers.**

The first 25 Gaussian integers in the spiral ordering are $1, 1+i, 2+i, 2, 3, 3+i, 3+2i, 2+2i, 1+2i, 1+3i, 2+3i, 3+3i, 4+3i, 4+2i, 4+i, 4, 5, 5+i, 5+2i, 5+3i, 5+4i, 4+4i, 3+4i, 2+4i, 1+4i$. The set of first n Gaussian integers in the spiral ordering is denoted by $[\psi_n]$.

Lemma 2.5. *Let ρ be a Gaussian integer and u be a unit. Then ρ and $\rho + u$ are relatively prime.*

Proof. Let α be a Gaussian integer such that α divides both ρ and $\rho + u$. Then α must divides $\rho + u - \rho = u$. Since u is a unit, the only Gaussian integers that divides u are the units. Then ρ and $\rho + u$ are relatively prime since α is a unit. □

Corollary 2.6. *In the spiral ordering, consecutive Gaussian integers are relatively prime.*

Lemma 2.7. *Let ρ be an odd Gaussian integer, let t be a positive integer and u be a unit. Then ρ and $\rho + u.(1+i)^t$ are relatively prime.*

Corollary 2.8. *In the spiral ordering, consecutive even Gaussian integers are relatively prime.*

Proof. The only possible difference between two even Gaussian integers in the spiral ordering are $1 + i$, 2 or one of their associates. The differences are of the form $u.(1 + i)^t$, since $2 = -i(1 + i)^2$. Therefore by using Lemma 2.7, consecutive even Gaussian integers in the spiral ordering are relatively prime. □

Corollary 2.9. *In the spiral ordering, consecutive odd Gaussian integers are relatively prime.*

Lemma 2.10. *Let α be a prime Gaussian integer and ρ be a Gaussian integer then ρ and $\rho + \alpha$ are relatively prime if and only if $\alpha \nmid \rho$.*

2.1. Gaussian Prime Labeling

The Gaussian prime labeling was defined by Steven Klee [3]. We Prove the Gaussian prime labeling of super subdivision of star graphs.

Definition 2.11 ([3])**.** *Let G be a graph on n vertices. A bijection $f : V(G) \to [\psi_n]$ is called a Gaussian prime labeling if for every edge $uv \in E(G)$, $f(u)$ and $f(v)$ are relatively prime. A graph which admits Gaussian prime labeling is called a Gaussian prime graph.*

3. Gaussian Prime Labeling Of Super Subdivision of Star Graphs

Meena and Kavitha [4] discussed the prime labeling of super subdivision of star graphs. They prove that super subdivision of a star graph $K_{1,n}$ where every edge uv of $K_{1,n}$ is replaced by $K_{2,2}$, super subdivision of a star graph $K_{1,n}$ where every edge uv of $K_{1,n}$ is replaced by $K_{2,3}$ are prime graphs. We discuss the Gaussian prime labeling of super subdivision of a star graph $K_{1,n}$ where every edge uv of $K_{1,n}$ is replaced by $K_{2,2}$, super subdivision of a star graph $K_{1,n}$ where every edge uv of $K_{1,n}$ is replaced by $K_{2,3}$ and super subdivision of a star graph $K_{1,n}$ where every edge uv of $K_{1,n}$ is replaced by $K_{2,4}$.

Theorem 3.1. *The graph S obtained by super subdivision of a star graph $K_{1,n}, n \in \mathbb{N}$ where every edge uv of $K_{1,n}$ is replaced by $K_{2,2}$, then S is a Gaussian prime graph.*

Proof. Let $x_0, x_1, x_2, \ldots, x_n$ are the vertices of the star graph $K_{1,n}$ where x_0 is the centre vertex. Let every edge x_0x_i of $K_{1,n}$ is replaced by $x_0y_ix_i$ and $x_0w_ix_i$ by joining x_0y_i, y_ix_i and x_0w_i, w_ix_i for $1 \leq i \leq n$. Then the graph S has edge set

$E(S) = \{x_0y_i, x_0w_i/1 \leq i \leq n\} \cup \{y_ix_i, w_ix_i/1 \leq i \leq n\}$.

Define a function $f : V(S) \to \{1, 2, 3, \ldots, 3n+1\}$ as follows

$$\begin{aligned}
f(x_0) &= \psi_1 \\
f(x_i) &= \psi_{3i}, \qquad 1 \leq i \leq n \\
f(y_i) &= \psi_{3i-1}, \quad 1 \leq i \leq n \\
f(w_i) &= \psi_{3i+1}, \quad 1 \leq i \leq n.
\end{aligned}$$

The Gaussian integer $\psi_1 = 1$ is relatively prime to all the Gaussian integers. The labeling on the adjacent vertices not including ψ_1 are consecutive Gaussian integers and consecutive Gaussian integers in the spiral ordering are relatively prime. Hence S is a Gaussian prime graph. □

Illustartion 3.2. *The Gaussian prime labeling of super subdivision of $K_{1,5}$ by $K_{2,2}$ is shown in figure 2.*

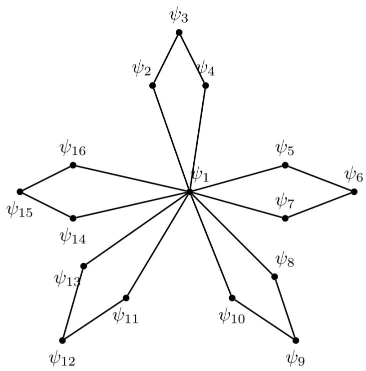

Figure 2. **Gaussian prime labeling of super subdivision of $K_{1,5}$ by $K_{2,2}$.**

Theorem 3.3. *The graph S obtained by super subdivision of a star graph $K_{1,n}, n \in \mathbb{N}$ where every edge uv of $K_{1,n}$ is replaced by $K_{2,3}$, then S is a Gaussian prime graph.*

Proof. Let $x_0, x_1, x_2, \ldots, x_n$ are the vertices of the star graph $K_{1,n}$ where x_0 is the centre vertex. Let every edge x_0x_i of $K_{1,n}$ is replaced by $x_0y_ix_i$, $x_0z_ix_i$ and $x_0w_ix_i$ by joining x_0y_i, y_ix_i, x_0z_i, z_ix_i and x_0w_i, w_ix_i for $1 \leq i \leq n$. Then the graph S has edge set $E(S) = \{x_0y_i, x_0z_i, x_0w_i/1 \leq i \leq n\} \cup \{y_ix_i, z_ix_i, w_ix_i/1 \leq i \leq n\}$.

Define a function $f : V(S) \to \{\psi_1, \psi_2, \psi_3, \ldots, \psi_{4n+1}\}$ as follows:

$$\begin{aligned}
f(x_0) &= \psi_1 \\
f(x_i) &= \psi_{4i-1}, \quad 1 \leq i \leq n \\
f(y_i) &= \psi_{4i-2}, \quad 1 \leq i \leq n \\
f(z_i) &= \psi_{4i}, \qquad 1 \leq i \leq n \\
f(w_i) &= \psi_{4i+1}, \quad 1 \leq i \leq n.
\end{aligned}$$

The Gaussian integer $\psi_1 = 1$ is relatively prime to all the Gaussian integers. The labeling on the adjacent vertices not including ψ_1 are either consecutive Gaussian integers or consecutive odd Gaussian integers. Consecutive Gaussian integers in the spiral ordering are relatively prime and consecutive odd Gaussian integers in the spiral ordering are relatively prime. Hence S is a Gaussian prime graph. □

Theorem 3.4. *The graph S obtained by super subdivision of a star graph $K_{1,n}, n \in \mathbb{N}$ where every edge uv of $K_{1,n}$ is replaced by $K_{2,4}$, then S is a Gaussian prime graph.*

Proof. Let $x_0, x_1, x_2, \ldots, x_n$ are the vertices of the star graph $K_{1,n}$ where x_0 is the centre vertex. Let every edge x_0x_i of $K_{1,n}$ is replaced by $x_0y_ix_i$, $x_0z_ix_i$, $x_0v_ix_i$ and $x_0w_ix_i$ by joining x_0y_i, y_ix_i, x_0z_i, z_ix_i x_0v_i, v_ix_i and x_0w_i, w_ix_i for $1 \leq i \leq n$. Then the graph S has edge set

$$E(S) = \{x_0y_i, x_0z_i, x_0v_i, x_0w_i/1 \leq i \leq n\} \cup \{y_ix_i, z_ix_i, v_ix_i, w_ix_i/1 \leq i \leq n\}.$$

Define a function $f : V(S) \to \{\psi_1, \psi_2, \psi_3, \ldots, \psi_{5n+1}\}$ as follows:

$$\begin{aligned}
f(x_0) &= \psi_1 \\
f(x_i) &= \psi_{5i-1}, \quad 1 \leq i \leq n \\
f(y_i) &= \psi_{5i-3}, \quad 1 \leq i \leq n \\
f(z_i) &= \psi_{5i-2}, \quad 1 \leq i \leq n \\
f(v_i) &= \psi_{5i}, \quad 1 \leq i \leq n \\
f(w_i) &= \psi_{5i+1}, \quad 1 \leq i \leq n.
\end{aligned}$$

The labeling on the adjacent vertices are consecutive Gaussian integers, consecutive even Gaussian integers or consecutive odd Gaussian integers. Consecutive Gaussian integers in the spiral ordering are relatively prime, consecutive odd Gaussian integers in the spiral ordering are relatively prime and consecutive even Gaussian integers in the spiral ordering are relatively prime. Hence S is a Gaussian prime graph. □

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