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Common Fixed Point Theorems for Mappings Satisfying Common Limit Range Property

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Abstract: In this paper, we define generalized weakly contractions in G-metric spaces with which we prove two common fixed point theorems for self maps satisfying common limit range property .The obtained results extend and improve some results in the literature. Further, we deduce some consequences of the results.

Keywords: G-metric space, fixed point, common limit range property, weakly compatible mappings.

1. Introduction

In an attempt to generalize the notion of a metric space, several authors proposed different generalizations. In this direction, Mustafa and Sims [5] introduced G-metric space in 2006 as a generalization of a metric space and proved some fixed point theorems. Afterwards, several authors studied many fixed point and common fixed point results for self-mappings in G-metric spaces under certain contractive conditions [6, 7].

On the other hand, in 2002, Aamri and Moutawakil [1] introduced the notion of E.A. property in metric spaces which contains the class of compatible and noncompatible maps. In 2011, Sintunavarat [8] introduced the concept of common limit range property for a pair of self mappings in metric spaces and studied some common fixed point theorems under this notion. Recently, the concept of common limit range property was extended to two pairs of self mappings by Imdad [3].

The purpose of this paper is to define two generalized weakly contractions in G-metric spaces and to establish some common fixed point theorems for self maps satisfying common limit range property.

2. Preliminaries

In this section, we present some definitions and results which will be used in this paper.

Definition 2.1 ([5]). *Let X be a non empty set and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:*

$$G(x, y, z) = 0 \quad if \quad x = y = z.$$

$$G(x, x, y) > 0 \quad forall \quad x, y \in X, \quad with \quad x \neq y.$$

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$$G(x,x,y) = G(x,y,z) \quad \forall \quad x,y,z \in X, \quad with \quad y \neq z.$$

$$G(x,y,z) = G(x,z,y) = G(y,z,x) = ...(symmetry\ in\ all\ three\ variables)$$

$$G(x,y,z) = G(x,a,a) + G(a,y,z), \quad \forall \quad x,y,z,a \in X \quad (rectangular\ inequality).$$

Then the function G is called a G-metric on X and the pair (X,G) is called a G-metric space.

Example 2.2 ([5]). *Let (X,d) be a usual metric space. Then (X,G) is G-metric space, where $G(x,y,z) = d(x,y)+d(y,z)+d(x,z)$, for all $x,y,z \in X$.*

Definition 2.3 ([6]). *A G-metric space (X,G) is said to be symmetric if $G(x,y,y) = G(y,x,x)$, for all $x,y \in X$.*

Proposition 2.4 ([6]). *Every G-metric space (X,G) defines a metric space (X,d_G), where d_G defined by $d_G(x,y) = G(x,y,y) + G(y,x,x)$, for all $x,y \in X$.*

Proposition 2.5 ([5]). *Let (X,G) be a G-metric space. Then for any $x,y,z,a \in X$, the following hold:*

$$G(x,y,z) = 0 \quad then \quad x = y = z.$$

$$G(x,y,z) = G(x,x,y) + G(x,x,z).$$

$$G(x,y,y) = 2G(y,x,x).$$

$$G(x,y,z) = G(x,a,z) + G(a,y,z).$$

$$G(x,y,z) = \frac{2}{3}\{G(x,a,a) + G(y,a,a) + G(z,a,a)\}.$$

Definition 2.6 ([4]). *A mapping $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if ψ is continuous and non-decreasing and $\psi(t) = 0$ if and only if $t = 0$.*

Definition 2.7 ([8]). *Two self-mappings E and F of a metric space (X,d) are said to be weakly compatible if $Efu = FEu$ whenever Eu=Fu, for some $u \in X$.*

Definition 2.8 ([1]). *Two self-mappings E and F of a metric space (X,d) are said to satisfy E.A.property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = t$, for some $t \in X$.*

Definition 2.9 ([8]). *Two self-maps E and F of a metric space (X,d) are said to satisfy the common limit in the range of F property, denoted by CLR_F property, if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = Fu$ for some $u \in X$.*

Definition 2.10 ([3]). *Two pairs (E,F) and (f,g) of self-maps of a metric space (X,d) are said to satisfy the common limit range property with respect to the mappings F and g, denoted by CLR_{Fg} property,if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = \lim_{n\to\infty} fy_n = \lim_{n\to\infty} gy_n = u$ for some $u \in FX \cap gX$.*

In 2011, B.S.Choudhury [2] defined the following contraction in metric spaces and proved a fixed point theorem.

Definition 2.11 ([2]). *A self mapping $T : X \to X$, where (X,d) is a metric space, is said to be a generalized weakly contractive mapping if $\psi(d(Tx,Ty)) = \psi(M(x,y)) - \emptyset(\max\{d(x,y), d(y,Ty)\})$ for all $x,y \in X$, where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}(d(x,Ty) + d(y,Tx))\}$, ψ is an altering distance function and $\emptyset : [0,\infty) \to [0,\infty)$ is continuous and non-decreasing function such that $\emptyset(t) = 0$ if and only if $t = 0$.*

3. Main Results

We begin this section with the following definition.

Definition 3.1. *Two self maps T and f of a G-metric space (X,G) said to satisfy a generalized weakly contractive condition if*

$$\mu\left(G(Tx,Ty,Tz)\right) = \mu(L\left(x,y,z\right)) - \varphi\left(M(x,y,z)\right) \tag{1}$$

for all $x,y,z \in X$, where μ is an altering distance function and $\varphi : [0,\infty) \to [0,\infty)$ is lower semi continuous and non-decreasing function such that $\varphi\left(t\right) = 0$ if and only if $t = 0$ and

$$\begin{aligned} L\left(x,y,z\right) = \max&\left\{G\left(fx,fy,fz\right), G\left(fx,Tx,Tx\right), G\left(fy,Ty,Ty\right), G\left(fz,Tz,Tz\right),\right.\\ &\left.\frac{1}{3}\left\{G\left(fx,Ty,Ty\right) + G\left(fy,Tz,Tz\right) + G\left(fz,Tx,Tx\right)\right\},\right\} \\ \textit{and}\quad M\left(x,y,z\right) = \max&\left\{G\left(fx,fy,fz\right), G\left(fy,Ty,Ty\right), G\left(fz,Tz,Tz\right)\right\}. \end{aligned}$$

Theorem 3.2. *Let T and f be self maps of a G-metric space satisfying (1), CLR_f property and (T,f) is weakly compatible. Then the mappings T and f have a unique common fixed point in X.*

Proof. Since T and f satisfy CLR_f property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = ft$ for some t in X. Now from (1) with $x = x_n, y = z = t$, we get

$$\mu(G\left(Tx_n,Tt,Tt\right) = \mu(L\left(x_n,t,t\right)) - \varphi\left(M(x_n,t,t\right)) \tag{2}$$

where

$$\begin{aligned} L\left(x_n,t,t\right) = \max&\{G\left(fx_n,ft,ft\right), G\left(fx_n,Tt,Tt\right), G\left(ft,Tt,Tt\right), G\left(ft,Tt,Tt\right),\\ &\frac{1}{3}\left\{G\left(fx_n,Tt,Tt\right) + G\left(ft,Tt,Tt\right) + G\left(ft,Tx_n,Tx_n\right)\right\},\} \\ \textit{and}\quad M\left(x_n,t,t\right) = \max&\left\{G\left(fx_n,ft,ft\right), G\left(ft,Tt,Tt\right), G\left(ft,Tt,Tt\right)\right\} \end{aligned}$$

So

$$\begin{aligned} \lim_{n\to\infty} L\left(x_n,t,t\right) &= \max\{G\left(ft,ft,ft\right), G\left(ft,Tt,Tt\right), G\left(ft,Tt,Tt\right), G\left(ft,Tt,Tt\right),\\ &\qquad \frac{1}{3}\left\{G\left(ft,Tt,Tt\right) + G\left(ft,Tt,Tt\right) + G\left(ft,ft,ft\right)\right\}\} \\ &= \max\{0, G\left(ft,Tt,Tt\right), G\left(ft,Tt,Tt\right), G\left(ft,Tt,Tt\right), \frac{1}{3}\left\{G\left(ft,Tt,Tt\right) + G\left(ft,Tt,Tt\right) + 0\right\}\} \\ &= G\left(ft,Tt,Tt\right) \quad \textit{and} \\ \lim_{n\to\infty} M\left(x_n,t,t\right) &= \max\left\{G\left(ft,ft,ft\right), G\left(ft,Tt,Tt\right), G\left(ft,Tt,Tt\right)\right\} \\ &= G\left(ft,Tt,Tt\right) \end{aligned}$$

On letting $n \to \infty$ in (2), we get $\mu(G\left(ft,Tt,Tt\right)) = \mu(G\left(ft,Tt,Tt\right)) - \varphi(G\left(ft,Tt,Tt\right))$ which implies that $\varphi\left(G\left(ft,Tt,Tt\right)\right) = 0$, that is, $ft = Tt = p$ (say), showing that t is a coincidence point of T and f. Since T and f are weakly compatible, we have $Tft = fTt$ and so $Tp = fp$.

We now claim that $Tp = fp = p$. On using (1) with $x = p$, $y = z = t$,

$$\mu(G(Tp, Tt, Tt)) = \mu(L(p, t, t)) - \varphi(M(p, t, t)) \;\; or \;\; \mu(G(Tp, p, p)) = \mu(L(p, t, t)) - \varphi(M(p, t, t)),$$

where

$$\begin{aligned} L(p, t, t) &= \max\{G(fp, ft, ft), G(fp, Tp, Tp), G(ft, Tt, Tt), G(ft, Tt, Tt), \frac{1}{3}\{G(fp, Tt, Tt) + G(ft, Tt, Tt) + G(ft, Tp, Tp)\}\}, \\ &= \max\{G(Tp, p, p), 0, 0, 0, \frac{1}{3}\{G(Tp, p, p) + 0 + G(p, Tp, Tp)\} \\ &= G(Tp, p, p) \end{aligned}$$

and

$$\begin{aligned} M(p, t, t) &= \max\{G(fp, ft, ft), G(ft, Tt, Tt), G(ft, Tt, Tt)\} \\ &= \max\{G(Tp, p, p), 0, 0\} \\ &= G(Tp, p, p) \end{aligned}$$

Hence $\mu(G(Tp, p, p)) = \mu(G(Tp, p, p)) - \varphi(G(Tp, p, p))$ which implies that $\varphi(G(Tp, p, p)) = 0$ and so $G(Tp, p, p) = 0$, that is, $Tp = p$. Therefore $Tp = fp = p$, showing that p is a common fixed point of T and f.

Uniqueness: Let q ($\neq p$) be another common fixed point of T and f. Then, we have $Tq = fq = q$. Now from (1) with $x = p$, $y = z = q$, we obtain

$$\begin{aligned} \mu(G(Tp, Tq, Tq)) &= \mu(L(p, q, q)) - \varphi(M(p, q, q)) \;\; or \\ \mu(G(p, q, q)) &= \mu(L(p, q, q)) - \varphi(M(p, q, q)) \end{aligned}$$

where

$$\begin{aligned} L(p, q, q) &= \max\{G(fp, fq, fq), G(fp, Tp, Tp), G(fq, Tq, Tq), G(fq, Tq, Tq), \\ &\qquad \frac{1}{3}\{G(fp, Tq, Tq) + G(fq, Tq, Tq) + G(fq, Tp, Tp)\}, \\ &= \max\{G(p, q, q), G(p, p, p), G(q, q, q), G(q, q, q), \frac{1}{3}\{G(p, q, q) + G(q, q, q) + G(q, p, p)\}, \\ &= G(p, q, q) \quad \text{and} \\ M(p, q, q) &= \max\{G(fp, fq, fq), G(fq, Tq, Tq), G(fq, Tq, Tq)\} \\ &= \max\{G(p, q, q), G(q, q, q), G(q, q, q)\} \\ &= G(p, q, q) \end{aligned}$$

Thus $\mu(G(p, q, q)) = \mu(G(p, q, q)) - \varphi(G(p, q, q))$ which implies that $\varphi(G(p, q, q)) = 0$, and hence $G(p, q, q) = 0$, that is, $p = q$. This completes the proof. □

Corollary 3.3. *Let (T, f) be a pair of weakly compatible self mappings of a G-metric space (X, G) satisfy the conditions (1) and E.A. property. If the range of f is a closed subspace of X, then the mappings T and f have a unique common fixed point in X.*

Proof. Since T and f satisfy E.A. property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} fx_n = u$ for some u in X. Also, since fX is a closed subspace of X, we can a t in X such that $u = ft$. So T and f satisfy CLR_f property. Hence by the Theorem 3.2, the mappings T and f have a unique common fixed point in X. □

Next, we prove a common fixed point theorem for four self maps .

Theorem 3.4. *Let P, Q, f and g be self mappings of a symmetric G-metric space (X, G) satisfy*

$$\mu(G(Px, Qy, Qz)) = \mu(L(x, y, z)) - f(L(x, y, z)) \quad \text{for all } x, y, z \in X \tag{3}$$

where μ is an altering distance function and $f : [0, \infty) \to [0, \infty)$ is lower semi continuous and non-decreasing function such that $f(t) = 0$ if and only if $t = 0$ and

$$L(x, y, z) = \max\{G(fx, gy, gz), G(fx, Px, Px), G(gy, Qy, Qz), \frac{1}{3}\{G(fx, Qy, Qz) + G(gy, Px, Px)\}.$$

Further, if (P, f) and (Q, g) are weakly compatible and satisfy CLR_{fg} property, then the mappings P, Q, f and g have a unique common fixed point in X.

Proof. Since (P, f) and (Q, g) satisfy CLR_{fg} property, we can find two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} Qy_n = \lim_{n\to\infty} gy_n = t$ for some $t \in fX \cap gX$. Now, since $t \in fx$, there is a point u in X such that $t = fu$. On using (3) with $x = u$, $y = z = y_n$, we get

$$\mu(G(Pu, Qy_n, Qy_n)) = \mu(L(u, y_n, y_n)) - \varphi(L(u, y_n, y_n)), \tag{4}$$

where

$$L((u, y_n, y_n)) = \max\{G(fu, gy_n, gy_n), G(fu, Pu, Pu), G(gy_n, Qy_n, Qy_n), \frac{1}{3}\{G(fu, Qy_n, Qy_n) + G(gy_n, Pu, Pu)\}\}$$

So

$$\begin{aligned}\lim_{n\to\infty} L(u, y_n, y_n) &= \max\{G(t, t, t), G(t, Pu, Pu), G(t, t, t), \frac{1}{3}\{G(t, t, t) + G(t, Pu, Pu)\} \\ &= G(Pu, t, t), \quad \text{Since G is symmetric.}\end{aligned}$$

On letting $n \to \infty$ in (4), $\mu(G(Pu, t, t)) = \mu(G(Pu, t, t)) - \varphi(G(Pu, t, t))$ which implies that $\varphi(G(Pu, t, t)) = 0$ and so $G(Pu, t, t) = 0$, that is, $Pu = t = fu$, showing that u is a coincidence point of P and f. Since P and f are weakly compatible, we have $Pfu = fPu$ and so $Pt = ft$. Also, since $t \in gx$, there is a point v in X such that $t = gv$. From (3) with $x = x_n, y = z = v$, we get that

$$\mu(G(Px_n, Qv, Qv)) = \mu(L(x_n, v, v)) - \varphi(L(x_n, v, v)) \tag{5}$$

where

$$L((x_n, v, v)) = \max\{G(fx_n, gv, gv), G(fx_n, Px_n, Px_n), G(gv, Qv, Qv), \frac{1}{3}\{G(fx_n, Qv, Qv) + G(gv, Px_n, Px_n)\}\}$$

So

$$\lim_{n\to\infty} L((x_n, v, v)) = \max\{G(t, t, t), G(t, t, t), G(t, Qv, Qv), \frac{1}{3}\{G(t, Qv, Qv) + G(t, t, t)\}\}$$

$$= G\left(t, Qv, Qv\right)$$

Taking limit as $n \to \infty$ in (5), $\mu(G\left(t, Qv, Qv\right)) = \mu(G\left(t, Qv, Qv\right)) - \varphi(G\left(t, Qv, Qv\right))$ which implies that $\varphi(G\left(t, Qv, Qv\right)) = 0$ and so $G\left(t, Qv, Qv\right) = 0$, that is, $Qv = t = gv$, showing that v is a coincidence point of Q and g.Since Q and g are weakly compatible, we have $Qgv = gQv$ and so $Qt = gt$.

We claim that $Pt = ft = t$. From (3) with $x = t$, $y = z = v$, we have

$$\mu\left(G\left(Pt, Qv, Qv\right)\right) = \mu\left(L\left(t, v, v\right)\right) - \varphi\left(L\left(t, v, v\right)\right) \quad or$$
$$\mu\left(G\left(Pt, t, t\right)\right) = \mu\left(L\left(t, v, v\right)\right) - \varphi\left(L\left(t, v, v\right)\right),$$

where

$$\begin{aligned} L\left(t, v, v\right) &= \max\{G\left(ft, gv, gv\right), G\left(ft, Pt, Pt\right), G\left(gv, Qv, Qv\right), \frac{1}{3}\left\{G\left(ft, Qv, Qv\right) + G\left(gv, Pt, Pt\right)\right\}\} \\ &= \max\{G\left(Pt, t, t\right), G\left(Pt, Pt, Pt\right), G\left(t, t, t\right), \frac{1}{3}\left\{G\left(Pt, t, t\right) + G\left(t, Pt, Pt\right)\right\}\} \\ &= G(Pt, t, t) \end{aligned}$$

Hence we get, $\mu(G\left(Pt, t, t\right)) = \mu(G\left(Pt, t, t\right)) - \varphi(G\left(Pt, t, t\right))$ which implies that $\varphi\left(G\left(Pt, t, t\right)\right) = 0$, and hence $G\left(Pt, t, t\right) = 0$, that is, $Pt = t$. Therefore $Pt = ft = t$. Finally we prove that $Qt = gt = t$. We get from (3) with $x = u$, $y = z = t$, that

$$\mu\left(G\left(Pu, Qt, Qt\right)\right) = \mu\left(L\left(u, t, t\right)\right) - \varphi\left(L\left(u, t, t\right)\right) \quad or$$
$$\mu\left(G\left(t, Qt, Qt\right)\right) = \mu\left(L\left(u, t, t\right)\right) - \varphi\left(L\left(u, t, t\right)\right),$$

where

$$\begin{aligned} L\left(u, t, t\right) &= \max\{G\left(fu, gt, gt\right), G\left(fu, Pu, Pu\right), G\left(gt, Qt, Qt\right), \frac{1}{3}\left\{G\left(fu, Qt, Qt\right) + G\left(gt, Pu, Pu\right)\right\}\} \\ &= \max\{G\left(t, Qt, Qt\right), G\left(t, t, t\right), G\left(Qt, Qt, Qt\right), \frac{1}{3}\left\{G\left(t, Qt, Qt\right) + G\left(Qt, t, t\right)\right\}\} \\ &= G(t, Qt, Qt), \quad \text{since G is symmetric.} \end{aligned}$$

Hence $\mu(G\left(t, Qt, Qt\right)) = \mu(G\left(t, Qt, Qt\right)) - \varphi(G\left(t, Qt, Qt\right))$ which implies that $\varphi\left(G\left(t, Qt, Qt\right)\right) = 0$, and so $G\left(t, Qt, Qt\right) = 0$, that is, $Qt = t$. Therefore $Qt = gt = t$ and hence $Pt = ft = Qt = gt = t$, showing that t is a common fixed point of P, Q, f and g.

Uniqueness: Let s $(\neq t)$ be another common fixed point of P, Q, f and g. Then, we have

$$Ps = fs = Qs = gs = s.$$

Now from (3) with $x = t$, $y = z = s$, we obtain

$$\mu(G\left(Pt, Qs, Qs\right)) = \mu(L\left(t, s, s\right)) - \varphi\left(L(t, s, s)\right) \quad or$$
$$\mu(G\left(t, s, s\right)) = \mu(L\left(t, s, s\right)) - \varphi\left(M(t, s, s)\right),$$

where

$$L\left(t, s, s\right) = \max\{G\left(ft, gs, gs\right), G\left(ft, Pt, Pt\right), G\left(gs, Qs, Qs\right), \frac{1}{3}\left\{G\left(ft, Qs, Qs\right) + G\left(gs, Pt, Pt\right)\right\}\}$$

$$= \max\{G(t,s,s), G(t,t,t), G(s,s,s), \frac{1}{3}\{G(t,s,s) + G(s,t,t)\}\}$$

$$= G(t,s,s), \quad \text{since G is symmetric.}$$

Thus $\mu(G(t,s,s)) = \mu(G(t,s,s)) - \varphi(G(t,s,s))$ which implies that $\varphi(G(t,s,s)) = 0$, and hence $G(t,s,s) = 0$, that is, $t = s$. This completes the proof. □

If we put $Q = g = I_X$ (Identity mapping) in the Theorem 3.4., we get the following result.

Corollary 3.5. *Let (P,f) be a pair of weakly compatible self mappings of a symmetric G-metric space (X,G) satisfy $\mu(G(Px,y,z)) \leq \mu(L(X,y,z)) - \varphi(L(X,y,z))$ for all $x,y,z \in X$, where μ is an altering distance function and $\varphi : [0,\infty) \to [0,\infty)$ is lower semi continuous and non-decreasing function such that $\varphi(t) = 0$ if and only if $t = 0$ and $L(x,y,z) = \max\{G(fx,y,z), G(fx,Px,Px), G(y,y,z), \frac{1}{3}\{G(fx,y,z) + G(y,Px,Px)\}$. Further, if (P,f) satisfies CLR_f property, then the mappings P and f have a unique common fixed point in X.*

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