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MODELING AND COMPARISON OF MAXIMUM LIKELIHOOD AND MEDIAN RANK REGRESSION METHODS WITH FRÉCHET DISTRIBUTION

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A B S T R A C T

In reliability analysis researchers, commonly model failure times with the lifetime distributions. In this article, we have considered the Fréchet distribution as a lifetime model. Often risk managers must make decisions after only a few failures. Thus, an important question is how to estimate the parameters of Fréchet distribution for extremely small sample sizes and Type-I censored data. The life test data are almost always from Type-I censored tests because plan dictates the time at which the test will finish. The study compares the two methods of estimation maximum likelihood estimation and median rank regression by fitting Fréchet distribution parameters. Also, we have conducted a simulation study to empirically investigating the performance of the proposed methods and its properties. Moreover, the comparison of Maximum Likelihood and Median rank regression estimation methods are made by using Type-I censoring. Finally, the application of these methods is discussed by considering the real data for time to breakdown of an insulating fluid between electrodes at a voltage of 34kV. The reliability of insulating fluid is modelled through Fréchet distribution.

1. INTRODUCTION

There are numerous distributions are used to model failure data such as Normal, Exponential, Rayleigh, Weibull, Gamma, Fréchet, Lognormal, and others. In the current study, we concerned with (Fréchet, 1927) distribution introduced by French mathematician Maurice Fréchet (1878-1973) in 1927. It has widespread applications in reliability testing. Many different methods of estimating the parameters and important functions of the parameters are used but Maximum likelihood estimation (MLE) and median rank regression (MRR) methods are commonly used today. In previous studies the Weibull distribution is most commonly used for modeling and comparison of MLE and MRR methods. For example, the prediction on average tensile strength of 316L stainless steel is statistically analyzed by Weibull distribution method An et al. (2017). The MLE and MRR methods are commonly used today. Largely because of conflicting results from different studies that have been conducted to investigate the properties of these estimators there are

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sharp differences of opinion on which method should be used (Genschel & Meeker, 2010; Coria et al., 2016; Antonitsin, 2012; Nwobi & Ugomma, 2014). The comparison is made to evaluate the performance of two estimation methods MLE and MRR by considering highly right censored data and very small numbers of failures (Olteanu & Freeman, 2010). The transformer Weibull lifetime modelling which is known as essential for management within electric utilities. Two popular and widely adopted methods MLE and MRR are discussed and compared for their properties in estimating transformer lifetime data (Abbas & Firdos, 2018). An approach is proposed for modeling the life data for system components that have failure modes by different Weibull models. It is an efficient approach especially when the mixture is well mixed for moderate and large samples with a heavily censored data set and few exact failure times (Elmahdy, 2015). The discrimination between the Weibull and Lognormal distributions for complete samples are made. The MRR and MLE data fitting methods are described also goodness of fit using maximum likelihood ratio and most powerful invariant tests is presented (Pasha et al., 2006; Elmahdy, 2017).

In present study, we concerned with the modeling and evaluation of MLE and MRR methods by using Fréchet distribution as a lifetime model. This article begins with the introductory section. The rest of the article is arranged as follows Section 2 consists of the methodology of the present study, Section 2.1 demonstrates the MRR estimators of underlying parameters for complete and Type-I censored data. Section 2.2 comprises MLE method for both the complete and Type-I censoring cases. Section 3 contains the numerical example that demonstrate the comparison between MLE and MRR estimates. Section 4 presents the results of a simulation study to compare the two procedures that are MLE and MRR in terms of their ability to estimate the individual parameters. The results for complete data is presented in Section 4.1 and, Section 4.2 consists of the results for Type-I censored data. Theoretical modeling is illustrated in Section 5. Finally, conclusions are reported in Section 6.

2. METHODOLOGY

The probability density function (PDF), cumulative distribution function (CDF) and Reliability function of a random variable \`t' that has the two parameter Fréchet distribution is given by:

$$f(t,\alpha,\theta) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{-(\beta+1)} \exp\left[-\left(\frac{t}{\theta}\right)^{-\beta}\right], \beta > 0 \quad t > 0, \qquad (1)$$

$$(t) = \exp\left[-\left(\frac{t}{\theta}\right)^{-\beta}\right], \quad \theta, t > 0, \qquad (2)$$

$$R(t) = 1 - exp[-(t/\theta)^\wedge(-\beta)], \quad \theta, t > 0, \qquad (3)$$

here, β is slope or shape parameter and θ is scale parameter or characteristic life (time when 63.2% of the units will fail), and t > 0 is the number of cycles to failure.

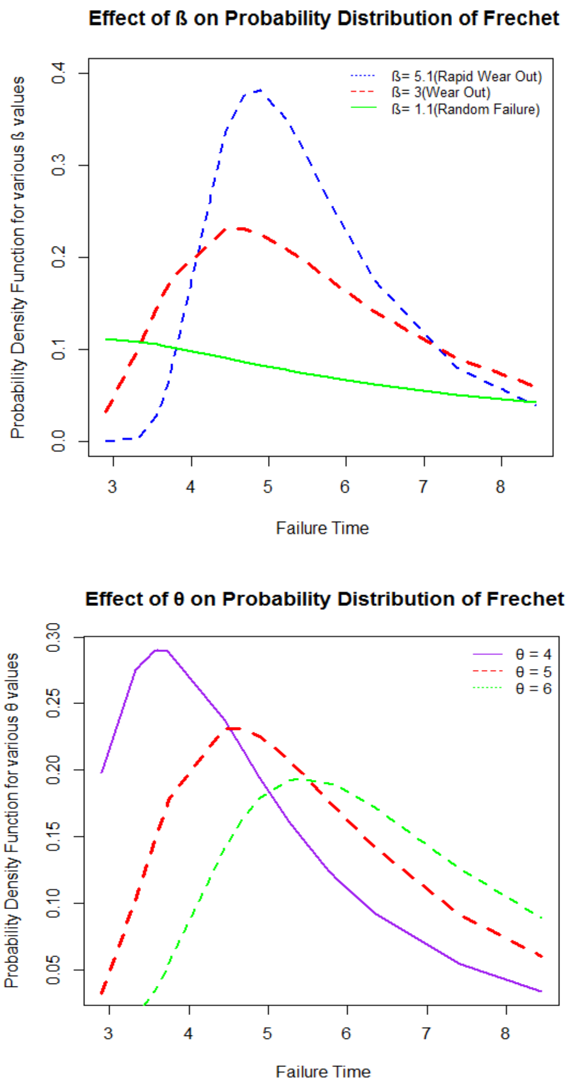

Figure 1. Fréchet curves for different values of β and θ

The slope parameter β determines the shape of the Fréchet curve (See Figure 1).The effect of β can be explained into several modes of failures, at β=1.1 curve indicates the random failure means that failures are independent of time, atβ=3 curve demonstrates the wear out behavior can be due to common failure modes, such as erosion, rust etc., and at β=5.1 steep curve shows fast wear out. Also, Figure 1.b. shows the width of distribution peaks for various values of θ.In the current study, we have utilized the two methods of estimation which are MRR and MLE methods for parameter estimation of Fréchet distribution.

2.1 Median Rank Regression

First, we will examine the fitting of two-parameter Fréchet distribution by using MRR method. This method proceeds as follows:

By considering equation 2 representing Fréchet CDF. This equation can be linearized by taking natural logarithm on both sides,

$$\left[\ln\left\{\ln\left(\frac{1}{1-F(t)}\right)\right\}\right] = \beta\ln(t) - \beta\ln(\theta), \quad (4)$$

By rearranged equation 4, the linear regression model is obtained as:

$$Y = a + bX, \quad (5)$$

Where, $Y = \ln\{\ln[1/1 - F(x)]\}$,
$a = -\beta ln\theta, b = \beta$ and $X = ln(x)$.
and

$$\theta = exp\left(-\frac{a}{\beta}\right). \quad (6)$$

Median ranks can be calculated by using equation 6:

$$\sum_k \binom{n}{k}(MR)^k(1-MR)^{n-k} = 0.50 = 50\%. \quad (7)$$

Bernard used an approximation of it which is:

$$MR = \frac{i-0.3}{n+0.4} \quad (8)$$

Where, i is failure order number and n is total sample size.

For an incomplete dataset, which is also the case for the present study, the rank value i in (8) is replaced by the adjusted rank defined as:

Adjusted Rank= ((Reverse Rank)*(Previous Adjusted Rank)+(N+1))/(Reverse Rank+1). (9)

2.2 Median Rank Regression

Another popular method for Fréchet parameter estimation is MLE. The likelihood function for Fréchet parameters for n failed items is given by:

$$\boldsymbol{L}(\theta,\beta) = \prod_{i=1}^{n}\frac{\beta}{\theta}\left(\frac{t_i}{\theta}\right)^{-(\beta+1)} exp\left[-\left(\frac{t_i}{\theta}\right)^{-\beta}\right], \quad (10)$$

The log-likelihood function is:

$$logL(\theta,\beta) = nlog(\beta) - n\log(\theta) - \sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{-\beta} - (\beta+1)\left[\sum_{i=1}^{n} log\left(\frac{t_i}{\theta}\right)\right], \quad (11)$$

The first derivative of log- likelihood function with respect to β and θ are given by:

$$\frac{\partial \log L}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} log\left(\frac{t_i}{\theta}\right) + \sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{-\beta} log\left(\frac{t_i}{\theta}\right), \quad (12)$$

$$\frac{\partial \log L}{\partial\theta} = \frac{n}{\theta} - \frac{\beta}{\theta}\sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{-\beta} - (\beta+1)\sum_{i=1}^{n}\left(\frac{\theta}{t_i}\right). \quad (13)$$

The above equations of MLE for β and θ does not have close form solution and are obtained by optim function BFGS in R software (version. 3.4.2).

2.3 Maximum Likelihood Estimation with Type I censoring

To account for the Type-I censoring the complete likelihood function becomes:

$$\boldsymbol{L}(\theta,\beta) = \prod_{i=1}^{n}\frac{\beta}{\theta}\left(\frac{t_i}{\theta}\right)^{-(\beta+1)} exp\left[-\left(\frac{t_i}{\theta}\right)^{-\beta}\right]\prod_{i=1}^{n}\left[1 - exp\left\{-\left(\frac{t}{\theta}\right)^{-\beta}\right\}\right], \quad (14)$$

The log-likelihood function is:

$$logL(\theta,\beta) = nlog(\beta) - n\log(\theta) - \sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{-\beta} - (\beta+1)\left[\sum_{i=1}^{n} log\left(\frac{t_i}{\theta}\right)\right] + cu\left(\log\left[1 - exp\left\{-\left(\frac{t_i}{\theta}\right)^{-\beta}\right\}\right]\right). \quad (15)$$

The partial derivative of log-likelihood function with respect to β and θ are given by:

$$\frac{\partial \log L}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} log\left(\frac{t_i}{\theta}\right) + \sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{-\beta} log\left(\frac{t_i}{\theta}\right) - cu\left[\frac{1}{1-exp\left\{-\left(\frac{t_i}{\theta}\right)^{-\beta}\right\}} exp\left(-\left(\frac{t_i}{\theta}\right)^{-\beta}\right)\left(\frac{t_i}{\theta}\right)^{-\beta} log\left(\frac{t_i}{\theta}\right)\right], \quad (16)$$

$$\frac{\partial \log L}{\partial\theta} = \frac{n}{\theta} - \frac{\beta}{\theta}\sum_{i=1}^{n}\left(\frac{t_i}{\theta}\right)^{-\beta} - (\beta+1)\sum_{i=1}^{n}\left(\frac{\theta}{t_i}\right) + cu\left[\frac{1}{1-exp\left\{-\left(\frac{t_i}{\theta}\right)^{-\beta}\right\}} exp\left(-\left(\frac{t_i}{\theta}\right)^{-\beta}\right)\left(\frac{t_i}{\theta}\right)^{-\beta}\left(\frac{\beta}{\theta}\right)\right]. \quad (17)$$

3. NUMERICAL EXAMPLE

To fit the parameters of Fréchet distribution and perform goodness of fit analysis the following steps are involved.

1. Generate 20 values from Fréchet distribution with β = 3 and θ= 5.
2. Sort and rank them from lowest to highest.
3. Apply Bernard’s approximation to compute new ranks.
4. Fit least squares linear regression to get the slope (b) and intercept (a) estimates.

The simulated data for n =20, β = 3 and θ= 5 is presented in Table 1.

Table 1. Simulated data of Fréchet Distribution

2.9040	**3.3255**	**3.5614**	**3.6008**
4.6771	**4.8886**	**5.2235**	**5.2661**
5.3978	**5.7665**	**5.9076**	
6.3560	**7.4234**	**8.4428**	**9.0431**
9.1362	**14.0444**		

The Figure 2 demonstrates the linear regression model with the regression line by using MRR to see how close the least squares line is to the data. The Figure 2 shows that fit is acceptable.

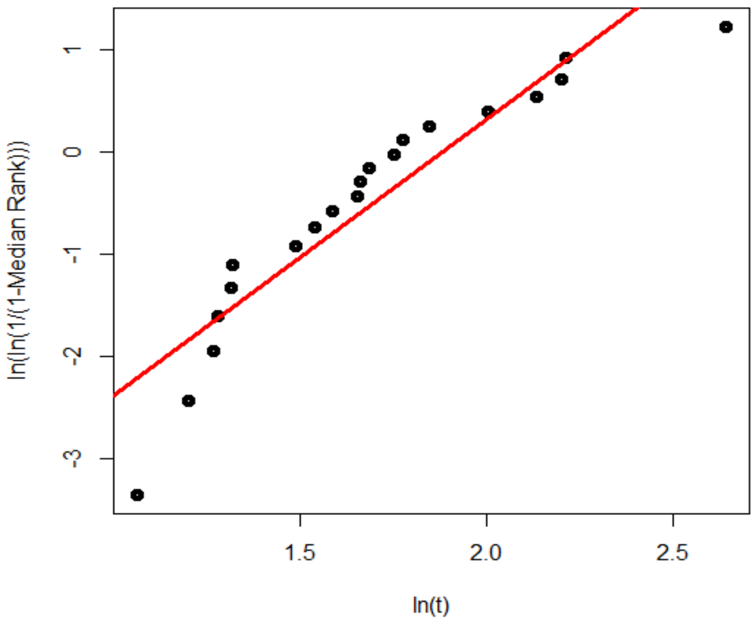

Figure 2. Regression line for simulated data of Fréchet distribution

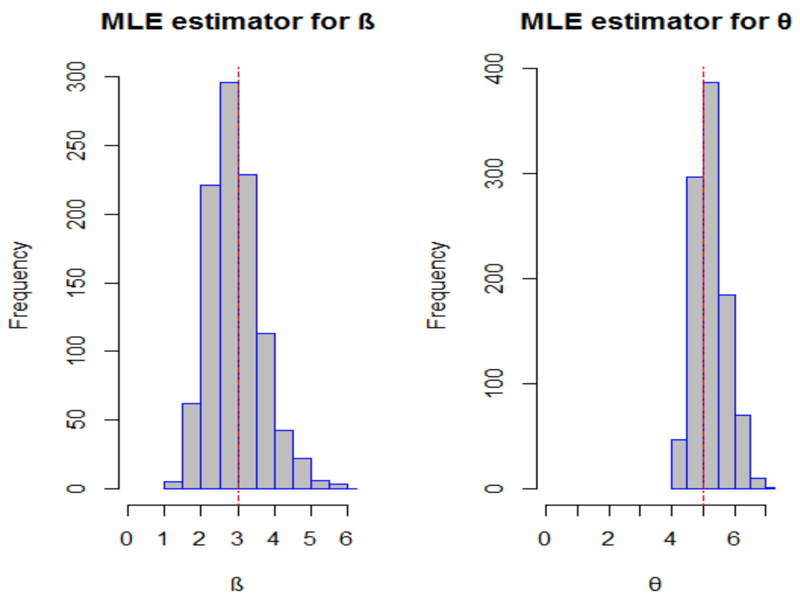

Figure 3. Simulation study for estimating β and θusing MLE.

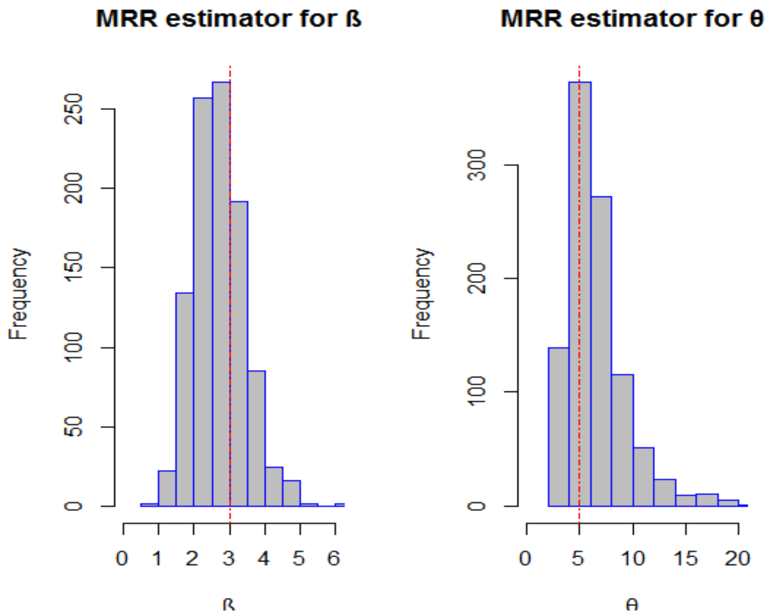

Figure 4. Simulation study for estimating β and θusing MRR

Table 3. Comparison of MLE against MRR estimates by simulation study

	Estimates		Variance		95% C. I	
	β	θ	β	θ	β	θ
MLE	3.2314	5.0579	0.2959	0.1761	1.9339, 4.0661	4.2353, 5.8806
MRR	2.7125	6.5487	0.4976	1.5210	1.3299, 4.0951	2.5827, 7.4173

We can see from Table 3 that MLE produces lower variance for both parameters as compared to MRR. Also, the width of 95% CI is too narrow of MLE method for both the parameters which indicates that estimates of MLE method are stable then the MRR method.

4. SIMULATION STUDY

Furthermore, study is extended for different sample sizes. In order to obtain the ML and MRR estimates (β, θ) and study their properties through the variance a simulation study is performed for complete and Type-I censored data.

4.1 Simulation Details of complete data

For this purpose, several data sets with sample sizes$n =$ (15,25,75,100,150) with β = 3 and θ= 5 are generated from two parameter Fréchet distribution and the process in replicated with 1000 times. The results are presented in Table 4. Graphs for variances of β and θ for both methods are presented in Figure 5.

Table 4. Comparison of MLE and MRR estimates by simulation study with different sample sizes

Sample size (n)		Estimates		Variance		95% $C.I$	
		β	θ	β	θ	β	θ
15	MLE	3.3035	5.0611	0.7235	0.0866	2.1552, 3.8448	3.5868, 6.4132
	MRR	2.7243	7.6057	0.7011	1.5804	1.3589, 4.6411	2.5360, 7.4640
25	MLE	3.1766	5.0291	0.5422	0.0793	1.5567, 4.4433	4.5786, 5.4214
	MRR	2.6791	7.5970	0.3655	0.8917	1.8150, 4.1850	3.1491, 6.8509
50	MLE	3.0794	5.0272	0.0957	0.0624	2.3936, 3.6064	4.4586, 5.5414
	MRR	2.6620	7.5552	0.2063	0.4446	2.1097, 3.8903	3.6931, 6.3069
75	MLE	3.0519	5.0132	0.0635	0.0498	2.5061, 3.4939	4.5627, 5.4373
	MRR	2.6586	7.5443	0.1487	0.2935	2.2441, 3.7559	3.9381, 6.0619
100	MLE	3.0394	5.0107	0.0569	0.0290	2.5324, 3.4676	4.6660, 5.3340
	MRR	2.6566	7.5481	0.1449	0.2894	2.2540, 3.7460	3.9455, 6.0545
150	MLE	3.0299	5.0023	0.0445	0.0182	2.5866, 3.4134	4.7357, 5.2643
	MRR	2.6425	7.5568	0.1389	0.2711	2.2695, 3.7305	3.9795, 6.0205

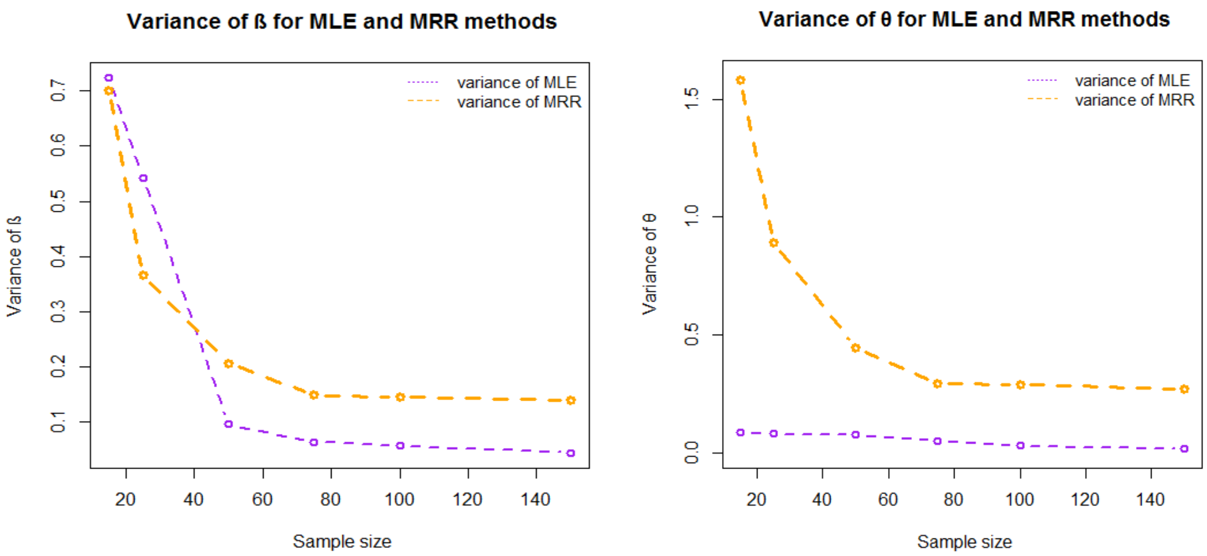

Figure 5. Variances for β and θ with MLE and MRR

From results, which are presented in (Table. 4) we can see that MLE method showed better results. The Figure 5 displays that variance of parameter β with MRR method demonstrates better result for small sample size but as we increase sample size than MLE gives small variance for parameter β. The variance of parameter θ for MLE is lesser as compared to MRR method for all sample sizes. The reason MRR is still used in industry is its simple methodology and ability to visualize the fit. Also, a serious drawback for using MLE is its "optimism", overestimating lifecycle of the item, for small samples which is not desirable in many industry applications.

4.2 Simulation Details of Type I Censoring

This section has purpose of illustrating the analysis through simulation study and to give a sense of the differences between MLE and MRR estimates. The present study also focused on showing that the Type-I censoring under which estimators are compared can have an effect on the comparison. Life test data are almost always from Type-I censored tests because plan dictates the time at which the test will end. The simulation study for $n =$ (15,25,50,75,100,150), $\beta = 3$ *and* $\theta = 5$ and censoring time (τ)= 10 with 1000 replication is conducted. The results are reported in Table 5.

Table 5. Comparison of MLE and MRR estimates by simulation study with Type-I censoring for different sample sizes

Sample size (n)		**Estimates**		**Variance**		**95% $C.I$**	
		β	θ	β	θ	β	θ
15	MLE	4.2216	4.7378	0.9401	0.0780	1.0996,4.9004	4.4526, 5.5474
	MRR	3.6104	6.7502	0.5289	0.4742	1.5745, 4.4255	3.6503, 6.3497
25	MLE	3.9693	4.7123	0.4476	0.0787	1.6887, 4.3113	4.4499, 5.5500
	MRR	3.6491	6.7253	0.2611	0.2718	1.9984, 4.0016	3.9781, 6.0219
50	MLE	3.8315	4.7109	0.2306	0.0344	2.0587, 3.9413	4.6360, 5.3640
	MRR	3.7275	6.6961	0.1265	0.1402	2.3027, 3.6973	4.2660, 5.7331
75	MLE	3.7958	4.7137	0.1111	0.0275	2.3466, 3.6534	4.6750, 5.3250
	MRR	3.7240	6.7023	0.1192	0.1256	2.3232, 3.6768	4.3052, 5.6948
100	MLE	3.7536	4.6895	0.0679	0.0228	2.4134, 3.5866	4.7280, 5.2711
	MRR	3.7609	6.6951	0.0601	0.0664	2.5194, 3.4805	4.4949, 5.5051
150	MLE	3.7340	4.6883	0.0399	0.0192	2.6080, 3.3920	4.7283, 5.2718
	MRR	3.7846	6.6833	0.0435	0.0445	2.5908, 3.4092	4.5862, 5.4138

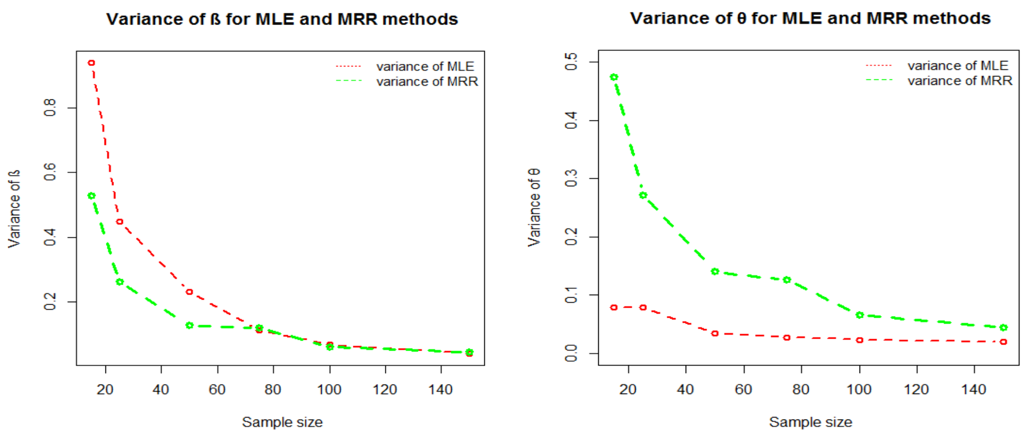

Figure 6. Variances for θ and β with MLE and MRR with Type-I censoring.

Notice that from Figure 6 the MLE has the smallest variance for parameter θ. An interesting attribute of the MRR estimator is that for small sample sizes the variance for parameter β is the smallest but for large sample size both MLE and MRR have almost same variances. However, we see again that the MLE is the best estimator, according to the variance.

5. THEORETICAL MODELING

Fréchet distribution is a theoretical basis of reliability analysis and life testing evaluation and it is widely used in reliability engineering. In the present study, we have considered the real-life data which is taken from Nelson (1982) about time to breakdown of an insulating fluid between electrodes at a voltage of 34Kv presented in Table. 5 for theoretical application of statistical methods in reliability analysis.

Table 5. Time to breakdown of an insulating fluid

0.19	**0.78**	**0.96**	**1.31**	**2.78**	**3.16**	**4.15**
4.67	**4.85**	**6.50**	**7.35**	**8.01**	**8.27**	**12.06**
31.75	**32.52**	**33.91**	**36.71**	**72.89**		

The graphical investigation is made through MRR to see how close the least squares line is to the data. The linear regression model with the regression line for failure time of an insulating fluid is demonstrated through Figure 7. The Figure 5 shows that fit is good. Then the estimates of β and θ are obtained by using both MLE and MRR method and are presented in Table. 6.

Table. 6. MLE and MRR estimates for real data set of time to breakdown of an insulating fluid

Parameters	MRR	MLE
$\boldsymbol{\beta}$	0.7550	0.6434
θ	2.1965	2.7729

From the results, it can be observed that the slope of the line is 0.6434 and 0.7550 from MLE and MRR methods respectively, which is the value of the shape parameter β. As the value of shape parameter is $\beta < 1$ which indicates a decreasing failure rate. The value of scale parameter is 2.7729 and 2.1965 from MLE and MRR methods respectively.

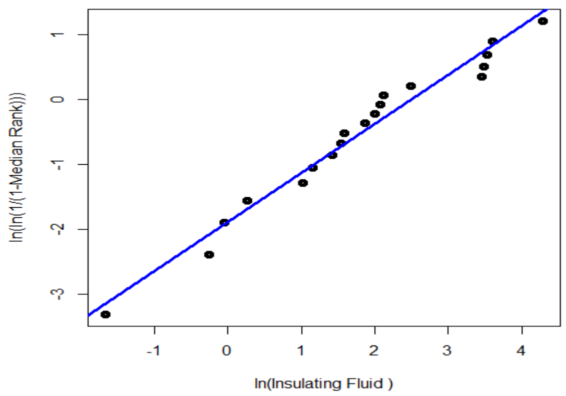

Figure 7. Regression line for time to breakdown of an insulating fluid.

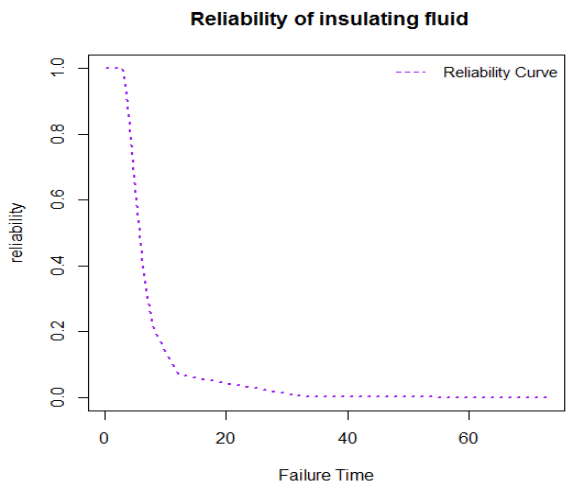

Figure 8. Fréchet Reliability distribution

The plot of reliability is shown in Figure 8. For certain assessment, consider 0.10 reliability levels. When this value is put as R (t) in Equation 3 and the equation is solved for t, the insulating fluidvalue 43.27 is obtained.

In another word, breakdown of an insulating fluid between electrodes will crush with 0.10 probability for a tension of 43.27 kV.

6. CONCLUSION

Fréchet distribution is the most commonly used distribution to approximate life data in reliability testing. It has widespread applications in the field of reliability and handles complex circuits very easily also it is used for Opto-electronic device such as solar cell, photo diodes, photo transistor, light emitting devices, etc. The purpose of this study was to focus on a very practical problem that experts in the field of reliability face on a regular basis. Therefore, the study considered complete and Type-I censored data in an order to compare the performance of two estimation methods, MLE and MRR. According to the results of numerical example, it can be said that the Fréchet parameters to fit and fitting method should be chosen based on the data and purpose of the analysis.

Findings shows that for complete data if there are small datasets with, we can use two-parameter Fréchet fitted by MLE method. From results, of simulation study we can say that for the cases studied, there is no clear-cut winner between the two estimation methods. However, some general conclusions can be drawn about in what cases which method should be used. In the case of complete data, MRR provides better estimates for the Fréchet shape parameter for small sample sizes but MLE performs well for large sample sizes according to variances. MLE offers better estimates of the Fréchet scale parameter with smallest variances. In the Type-I censored case, the results demonstrate the almost same behavior as in case of complete data accept that for large sample size MLE and MRR behave alike for shape parameter of Fréchet distribution.

Moreover, time to breakdown of an insulating fluid is modeled using Fréchet distribution. The Fréchet distribution allows researchers to describe the insulating fluid strength of a material in terms of a reliability function. The time to breakdown of an insulating fluid follows the Fréchet distribution with scale parameter is 2.7729 and 2.1965 from MLE and MRR methods respectively and shape parameter or slope of the line is 0.6434 and 0.7550 from MLE and MRR methods respectively. The value of shape parameter $\beta < 1$ which indicates a decreasing failure rate. From the reliability curve, we found that breakdown of an insulating fluid between electrodes will crush with 0.10 probability for a tension of 43.27 kV.

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