The Review of Finance and Banking
Volume 07, Issue 2, Year 2015, Pages 007– 015
S print ISSN 2067-2713, online ISSN 2067-3825

ON A CLASS OF ALPHA-STABLE DISTRIBUTIONS AND ITS APPLICATIONS IN ESTIMATING MARKET RISK

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Abstract. This paper uses a straightforward application of alpha-stable distributions for Romanian Stock Market, showing how a relatively simple implementation in the real world of a complex mathematical tool can be much more reliable in risk management than the classical Gaussian or log-normal distributions. In this paper we use a SAS macro for estimating the parameters of an alpha-stable distribution, using the time-series regression method from Kogon and Williams (1998). Using the Fast Fourier Transform, we estimate the probability density function, the cumulative distribution function and consequently, the VaR (99.5%) and TVaR (99%). For numerical illustration we are using daily logreturns of the BET Index; the measures of market risk, estimated on rolling windows using alpha-stable distributions and Gaussian distribution, are then compared to the actual logreturns of the BET Index. Numerical experiments show that using alpha-stable distributions for estimating VaR and TVaR can be a better alternative for managing the risk of financial assets.

1. Introduction

Stable distributions are an alternative to the Gaussian distribution, allowing for more robust estimates for the probability of extreme events.

Stable distributions are a class of distributions having the property of being invariant under linear transformations; Gaussian distribution is a special case of such stable distributions. There are numerous studies regarding the application of stable distributions in financial modelling. Thus, Rachev (2003) and Rachev and Mittnik (2000) describe in their papers, in detail, the methods of estimation for such distributions.

Nolan (2011) proposes an efficient method for estimating parameters of stable distributions, using the maximum likelihood method with numerical approximations for the probability density function.

In this paper we use a macro developed under SAS 9.3 (Pele, 2014) for estimating the parameters of an alpha-stable distribution, using the time-series regression method from Kogon and Williams (1998).

Using the Fast Fourier Transform, we estimate both the probability density function and the cumulative distribution function and consequently, the VaR (99.5%) and TVaR (99%).

For numerical illustration we are using daily logreturns of the BET Index; the measures of market risk, estimated on rolling windows using alpha-stable distributions and Gaussian distribution, are then compared to the actual logreturns of the BET Index.

Numerical experiments show that using alpha-stable distributions for estimating VaR and TVaR can be a better alternative for managing the risk of financial assets.

Received by the editors December 6, 2015. Accepted by the editors December 28, 2015.
Keywords: stable distribution, Value at Risk, Tail Value at Risk, Solvency II.
JEL Classification: G01, G15.
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This paper is in final form and no version of it will be submitted for publication elsewhere.

Although there are numerous papers about alpha-stable distributions in finance, very few of them are dealing with Romanian Stock Market (see for instance Emmeneger & Serbinenko, 2007).

This paper uses a straightforward application of alpha-stable distributions for Romanian Stock Market, showing how a relatively simple implementation in the real world of a complex mathematical tool can be much more reliable in risk management than the classical Gaussian or log-normal distributions.

The paper is organized as follows: a brief introduction in the formalism of stable distribution, a review of simulation and estimation methods and the results of the numerical experiments.

2. Stable distributions

The difficulty that occurs for stable distributions is that in most situations an explicit form of the probability density function does not exist, but only the expression of the characteristic function is known.

Thus, a random variable X follows a stable distribution with parameters $(\alpha, \beta, \gamma, \delta)$ (Nolan, 2011) if exists $\gamma > 0, \delta \in R$ such as X and $\gamma Z + \delta$ have the same distribution, where Z is a random variable with the characteristic function

$$\phi(t) = E[e^{itZ}] = \begin{cases} \exp(-|t|^{\alpha}[1 - i\beta \tan(\frac{\pi\alpha}{2}) sign(t)]), \alpha \neq 1 \\ \exp(-|t|[1 + i\beta t \frac{2}{\pi} sign(t)(\ln(|t|)]), \alpha = 1 \end{cases} .$$

In the above notations $\alpha \in (0.2]$ is the stability index, controlling for probability in the tails (for Gaussian distribution $\alpha = 2$), $\beta \in [-1, 1]$ is the skewness parameter, $\gamma \in (0, \infty)$ is the scale parameter and $\delta \in R$ is the location parameter.

A random variable X follows a stable distribution $S(\alpha, \beta, \gamma, \delta; 0)$ if his characteristic function has the form

$$\phi(t) = E[e^{itX}] = \begin{cases} \exp(-\gamma^{\alpha}|t|^{\alpha}[1 + i\beta \tan(\frac{\pi\alpha}{2}) sign(t)(|\gamma t|^{1-\alpha} - 1)] + i\delta t), \alpha \neq 1 \\ \exp(-\gamma |t|[1 + i\beta t \frac{2}{\pi} sign(t)(\ln(|\gamma t|)] + i\delta t), \alpha = 1 \end{cases} .$$

A random variable X follows a stable distribution $S(\alpha, \beta, \gamma, \delta; 1)$ if his characteristic function has the form

$$\phi(t) = E[e^{itX}] = \begin{cases} \exp(-\gamma^{\alpha}|t|^{\alpha}[1 - i\beta \tan(\frac{\pi\alpha}{2}) sign(t)] + i\delta t), \alpha \neq 1 \\ \exp(-\gamma |t|[1 + i\beta t \frac{2}{\pi} sign(t)(\ln(|t|)] + i\delta t), \alpha = 1 \end{cases} .$$

The parameterisation $S(\alpha, \beta, \gamma, \delta; 1)$ has the advantage that is more suitable for algebraic manipulations, although his characteristic function is not continuous for all parameters.

The parameterisation $S(\alpha, \beta, \gamma, \delta; 0)$ is suitable for numerical simulations and statistical inference, although the expression of characteristic function is more difficult to utilise in algebraic calculus.

Nolan (2011) shows that the two parameterisations $S(\alpha, \beta, \gamma, \delta_1; 1)$ and $S(\alpha, \beta, \gamma, \delta_0; 0)$ are equivalent

$$\delta_0 = \begin{cases} \delta_1 + \beta\gamma \tan\frac{\pi\alpha}{2}, \alpha \neq 1 \\ \delta_1 + \beta\frac{2}{\pi}\gamma \ln\gamma, \alpha = 1 \end{cases} .$$

The behavior of stable distributions is driven by the values of stability index α: small values are associated to higher probabilities in the tails of the distribution.

2.1. Estimating the parameters of an alpha- stable distribution using McCulloch method.

McCulloch method (1986) involves the following steps for estimating the parameters of a $S(\alpha, \beta, \gamma, \delta; 0)$ random variable:

- estimate α and β, using the quintiles of the empirical distribution (for more details, see Racheva-Iotova, 2010);

- define $v_\alpha = \frac{x_{0.95}-x_{0.05}}{x_{0.75}-x_{0.25}}$ and $v_\beta = \frac{x_{0.95}+x_{0.05}-2x_{0.25}}{x_{0.95}-x_{0.05}}$, where x_p is the p-quintile of the empirical distribution, having thus $\nu_\alpha = \phi_1(\alpha,\beta)$ and $\nu_\beta = \phi_2(\alpha,\beta)$ or, by inversion, $\alpha = \psi_1(v_a, v_\beta)$ and $\beta = \psi_2(v_a, v_\beta)$.

More, $\alpha = \psi_1(v_a, v_\beta) = \psi_1(v_a, -v_\beta)$ and $\beta = \psi_2(v_a, v_\beta) = -\psi_2(v_a, -v_\beta)$.

The functions $\psi_1(\cdot)$ and $\psi_2(\cdot)$ are tabulated for different values of ν_a and ν_b, so the estimates of α and β can be obtained using a bi-linear interpolation.

In a quite similar manner, the location parameter δ and the scale parameter γ can be estimated using the corresponding tabulated functions and the previous estimations for α and β.

2.2. **Estimating parameters of an alpha stable distribution using the Kogon-Williams method.** In order to estimate the parameters of a stable distribution in parameterisation S1, the following algorithm can be applied (following Kogon and Williams, 1998 and Pele, 2014):

Step 1. Define the maximum error for convergence *error* and the maximum number of iterations *maxiter*;

Step 2. Use the initial estimates $\alpha_0, \beta_0, \gamma_0, \delta_0$ from McCulloch method and normalize the sample: $x_j \to \frac{x_j - \delta_0}{\gamma_0}$;

Step 3. Estimate the regression model $y_k = b + \alpha_1 w_k + \varepsilon_k$, with $k = 0,..,9$, $y_k = \ln[-Re[\ln(\hat{\phi}(u_k)]]$, $w_k = \ln|u_k|$, $u_k = 0.1 + 0.1k$, $k = 0,..,9$, and $\hat{\phi}(\cdot)$ is the empirical characteristic function of the normalized sample. If $\hat{b}$ and $\hat{\alpha}_1$are the estimates of the regression model, then the estimate of the scale parameter is $\hat{\gamma}_1 = \exp(\hat{b}/\hat{\alpha})$.

Step 4. Estimate the regression model $z_k = \delta_{11} + \beta_1 v_k + \eta_k$, with $k = 0,..,9$, $z_k = Im[\ln(\hat{\phi}(u_k)]$, $w_k = \hat{\gamma}_1 u_k(|\hat{\gamma}_1 u_k|^{\hat{\alpha}_1 - 1} - 1)\tan(\pi\hat{\alpha}_1/2)$, $u_k = 0.1 + 0.1k$, $k = 0,..,9$.

Step 5. The final estimates are the following: $(\alpha_1, \beta_1, \gamma_1, \delta_1) = (\hat{\alpha}_1, \hat{\beta}_1, \hat{\gamma}_1, \hat{\delta}_{11} - \hat{\gamma}_1\hat{\beta}_1\tan(\pi\hat{\alpha}_1/2))$.

Step 6. Compute the relative error as $err = (\alpha_0 - \alpha_1)^2 + (\beta_0 - \beta_1)^2 + (\gamma_0 - \gamma_1)^2 + (\delta_0 - \delta_1)^2$. If $err > error$ or number of iterations reaches *maxiter*, repeat step 1 to 5, renormalizing the sample with the new values $(\alpha_0, \beta_0, \gamma_0, \delta_0) \to (\alpha_1, \beta_1, \gamma_1, \delta_1)$.

This algorithm is implemented as a SAS macro in Pele (2014) and can be used to obtain estimates for the parameters of stable distributions.[1]

2.3. **Numerical methods.** Using the Fast Fourier Transform we know that the probability density function (PDF) can be written as

$$\rho(x;\, \alpha,\, \beta,\, \gamma,\, \delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi(t;\, \alpha,\, \beta,\, \gamma,\, \delta)dt$$

The above integral can be calculated for N equally-spaced points with distance h such that $x_k = (k - 1 - \frac{N}{2})h$, $k = 1, ..., N$. Setting $t = 2\pi\omega$ implies

$$\rho((k - 1 - \frac{N}{2})h;\, \alpha,\, \beta,\, \gamma,\, \delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i2\pi\omega(k-1-\frac{N}{2})h}\varphi(2\pi\omega;\, \alpha,\, \beta,\, \gamma,\, \delta)dt \tag{2.1}$$

The integral in (2.1) can be approximated by

$$\begin{aligned}&\rho((k - 1 - \tfrac{N}{2})h;\, \alpha,\, \beta,\, \gamma,\, \delta)\_ \approx \sum_{n=1}^{N} \varphi(2\pi s(n - 1 - \tfrac{N}{2});\, \alpha,\, \beta,\, \gamma,\, \delta)\times \\ &\_ \times \exp(-i2\pi\omega(k - 1 - \tfrac{N}{2})(k - 1 - \tfrac{N}{2})hs)\end{aligned} \tag{2.2}$$

or by setting $s = \frac{1}{hN}$:

[1]The SAS program implementing this algorithm can be found on the website of Quantnet, a scientific project developed by Humboldt University of Berlin, Ladislaus von Bortkiewicz Chair of Statistics: http://www.quantlet.de.

$$\begin{aligned} &\rho((k-1-\tfrac{N}{2})h;\, \alpha,\, \beta,\, \gamma,\, \delta) \approx (-1)^{k-1-\frac{N}{2}} \sum_{n=1}^{N} (-1)^{n-1} \times \\ &\_ \times \varphi(2\pi s(n-1-\tfrac{N}{2});\, \alpha,\, \beta,\, \gamma,\, \delta) \times \\ &\_ \times \exp(-i\pi(n-1)(k-1)/N) \end{aligned} \tag{2.3}$$

The sum in (2.3) can be efficiently computed by applying Fast Fourier transform to the sequence $(-1)^{n-1}\varphi(2\pi s(n-1-\frac{N}{2});\, \alpha,\, \beta,\, \gamma,\, \delta)$, $n = 1, ..., N$.

Normalizing the k^{th} element of this sequence by $s(-1)^{k-1-N/2}$ we obtain the PDF value for each grid point. By substituting the Fourier transform into (2.2) with $t = 2\pi s(n-1-N/2)$, standardized values for PDF values can be calculated.

Interpolation can be used to evaluate PDF values data points falling between the equally-spaced grid points. It is to be noted that linear interpolation suffices in most practical applications.

The cumulative distribution function (CDF) can be estimated the same way, taking into account the fact that CDF can be express as $F(x; \alpha,\, \beta,\, \gamma,\, \delta) = \int_{-\infty}^{x} \rho(u; \alpha,\, \beta,\, \gamma,\, \delta)du$.

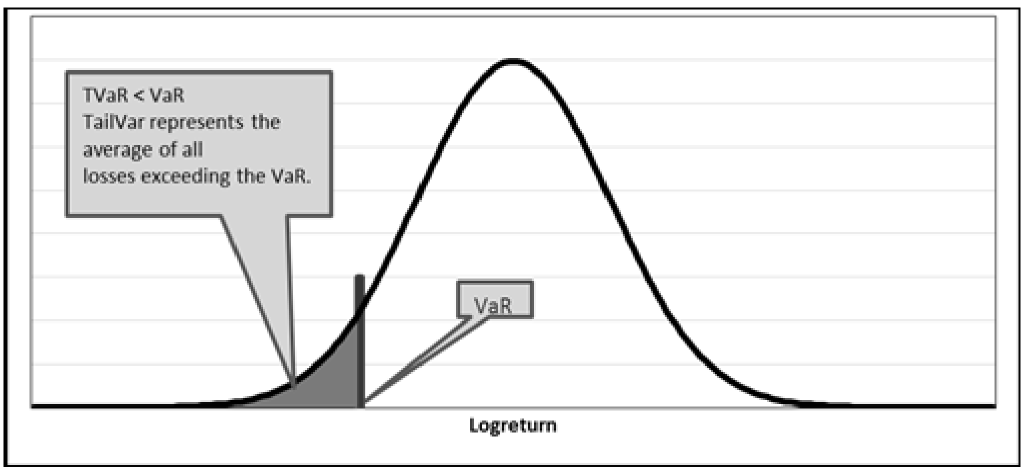

Figure 1. Illustration of Var and TVaR

Based on the estimated cumulative distribution function the two indicators of market risk can be defined:

- Value at Risk at significance level $\alpha \in (0, 1)$: $F(VaR_\alpha; \alpha,\, \beta,\, \gamma,\, \delta) = \Pr(X < VaR_\alpha) = \alpha$.
- Tail Value at Risk at significance level $\alpha \in (0, 1)$: $TVaR_\alpha = E[X|X <= VaR_\alpha]$ or

$$TVaR_\alpha(X) = VaR_\alpha + \frac{P[X < VaR_\alpha]}{1-\alpha} E\left[(VaR_\alpha - X|X < VaR_\alpha))\right].$$

3. Numerical experiments

We use daily logreturns of the main index of Bucharest Stock Exchange – BET Index, for the period 01/03/2001 – 04/22/2015 (3552 daily observations).

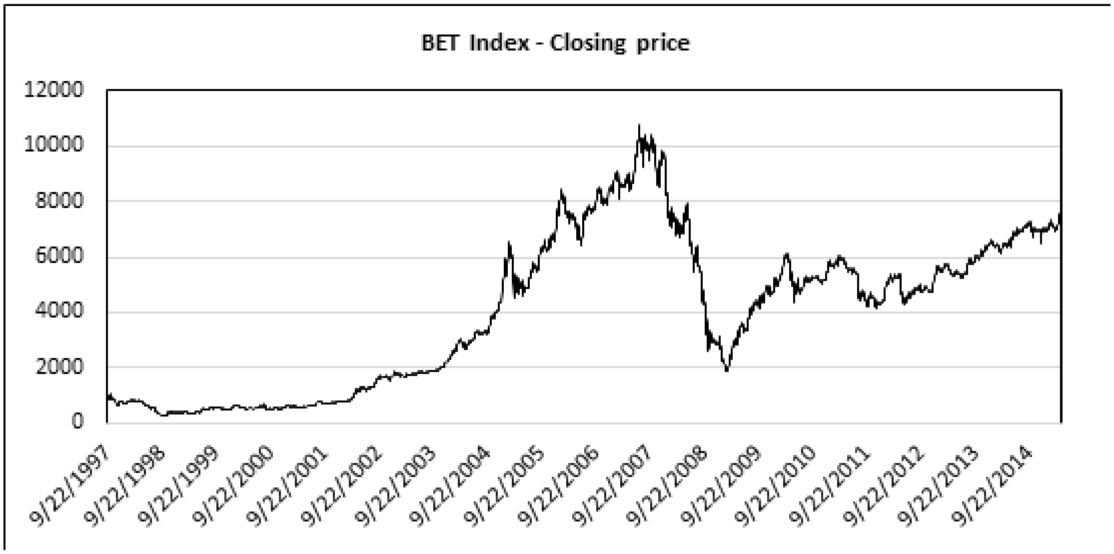

Figure 2. Closing prices of the BET Index

The strategy for estimating Value at Risk is the following:

- Use a rolling window of w trading days for estimate the parameters of the corresponding alpha-stable distribution and the Gaussian distribution of logreturns: $[t-w,t]$, $t=T-1,\ T-2,...$, where T is the number of trading days in the sample and w is 250, 500 and 750 trading days, consequently.

- Based on the probability density function (or the cumulative distribution function), estimate $VaR^{Stable}_{t+1,\alpha}$ and $VaR^{Gauss}_{t+1,\alpha}$, as the α-quantile of the distribution, where $\alpha = 0.005$, corresponding to a 99.5% probability.

- Estimate $TVaR^{Stable}_{t+1,\alpha}$, as the expected value of the logreturns below the α-quantile of the distribution, where $\alpha = 0.01$, corresponding to a 99% probability.

- Compare the estimated Value at Risk at time *t+1* with the actual logreturn $r_{t+1} = \log P_{t+1} - \log P_t$ at time *t+1*.

- Apply the VaR forecasting test (Kupiec, 1995) to evaluate the quality of the forecasting:

- ***The LR Test of Unconditional Coverage:***

Step 1. Let $\underset{t+1}{\widehat{VaR}}$ denote the forecasted VaR and let r_{t+1} the observed logreturn

Step 2. Define I

$$_{t+1} = \begin{cases} 1, \ if \ \underset{t+1}{\widehat{VaR}} < r_{t+1} \\ 0, \ otherwise \end{cases};$$

Step 3. Hypothesis:

$$\begin{cases} H_0 : E(I_{t+1}) = 1-\alpha \\ H_A : E(I_{t+1}) > 1-\alpha \end{cases};$$

Step 4. LR Test:

$$LR_{uc} = -2\log\frac{L(\alpha)}{L(\hat{\alpha})} = -2\log\frac{\alpha^{n_0}(1-\alpha)^{n-n_0}}{\hat{\alpha}^{n_0}(1-\hat{\alpha})^{n-n_0}} \approx \chi^2 \tag{3.1}$$

where $\hat{\alpha} = \frac{n_0}{n} = \Pr(I_{t+1}=0)$.

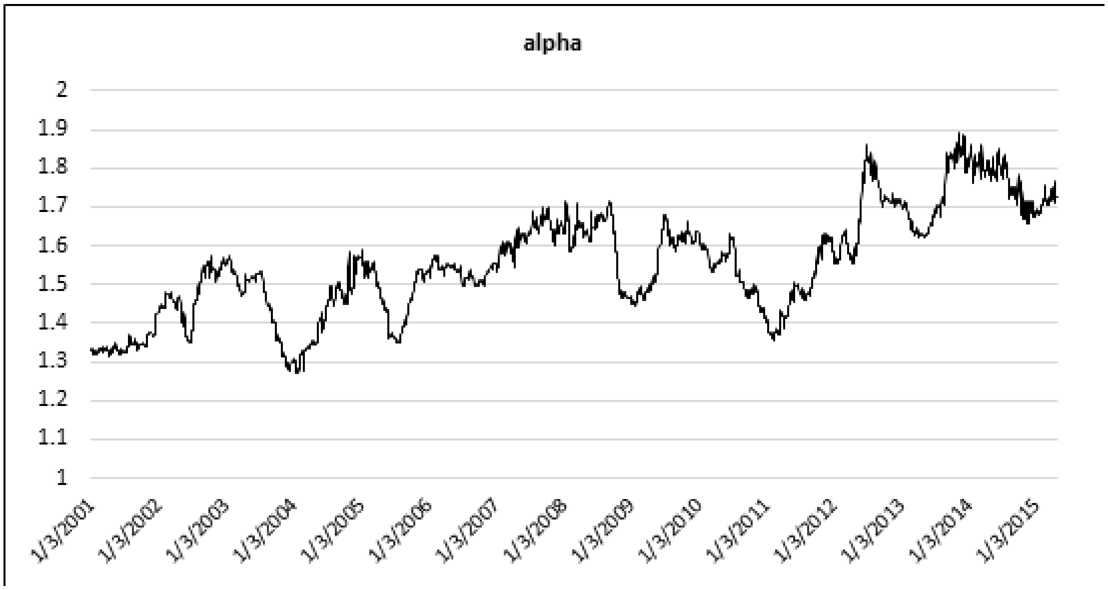

Figure 3. Stability index α for 500 trading days rolling windows

During the analysed time-frame, the BET Index exhibits periods with large departures from normality, when the values of stability index is significantly lower than 2, the case of Gaussian distribution. In this case, the probability of risky events may be seriously under evaluated, and this translates into sever losses.

Table 1. Summary of the Value at Risk estimation at 99.5% probability

Distribution	Number of	Window length	$\Pr(\widehat{VaR}_{t+1} < r+1)$	$\#(\widehat{VaR}_{t+1} > r+1)$	Kupiec test LR
Stable distribution	3552	250	99.63%	13	76.04**
Gaussian distribution	3552	250	98.56%	51	#N/A
Stable distribution	3552	500	99.75%	9	115.76**
Gaussian distribution	3552	500	98.76%	44	#N/A
Stable distribution	3552	750	99.86%	5	158.64**
Gaussian distribution	3552	750	98.82%	42	#N/A

** - significant at 99%.

We estimated Value at Risk (99.5%) for daily logreturns of BET index, using rolling windows of 250, 500 and 750 trading days, results being summarized in table 1.

In all the cases the forecasting based on stable distribution outperforms the estimates based on Gaussian distribution.

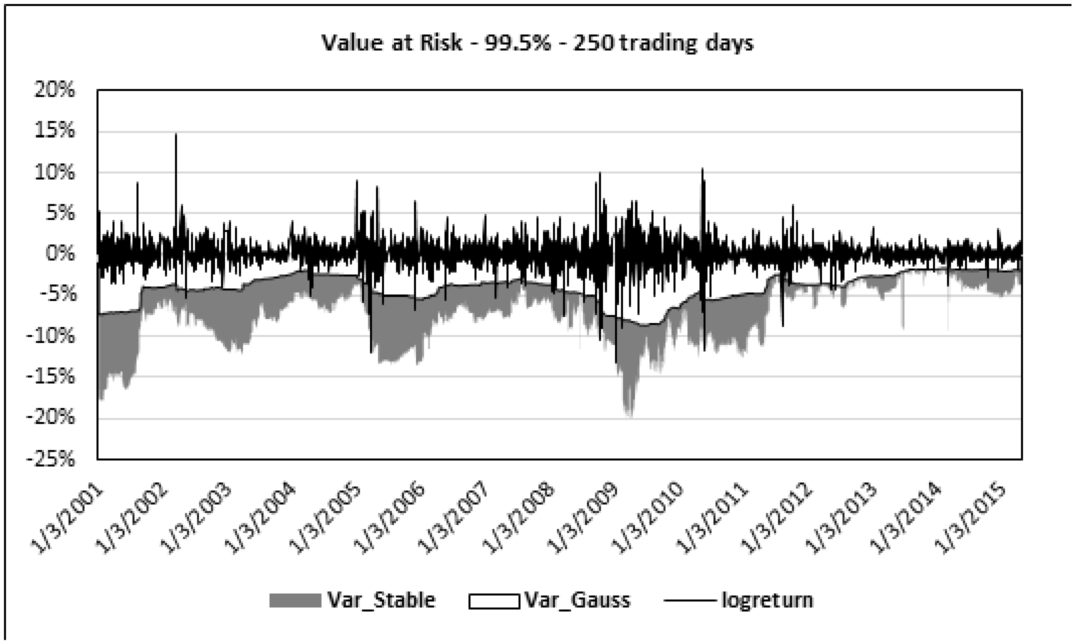

Figure 4. Value at Risk for 250 trading days rolling windows

For example, when VaR is estimated using 750 trading days rolling window, the stable distribution is accurate for 99.86% of the cases, while the Gaussian distribution is accurate only for 98.92% of the cases, lower than 99.5%.

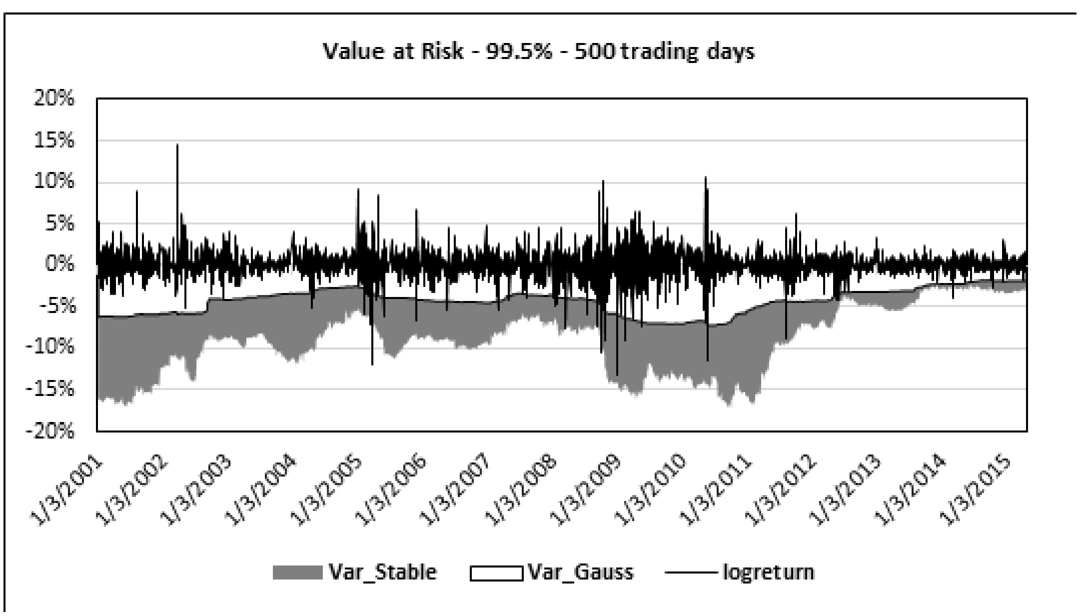

Figure 5. Value at Risk for 500 trading days rolling windows

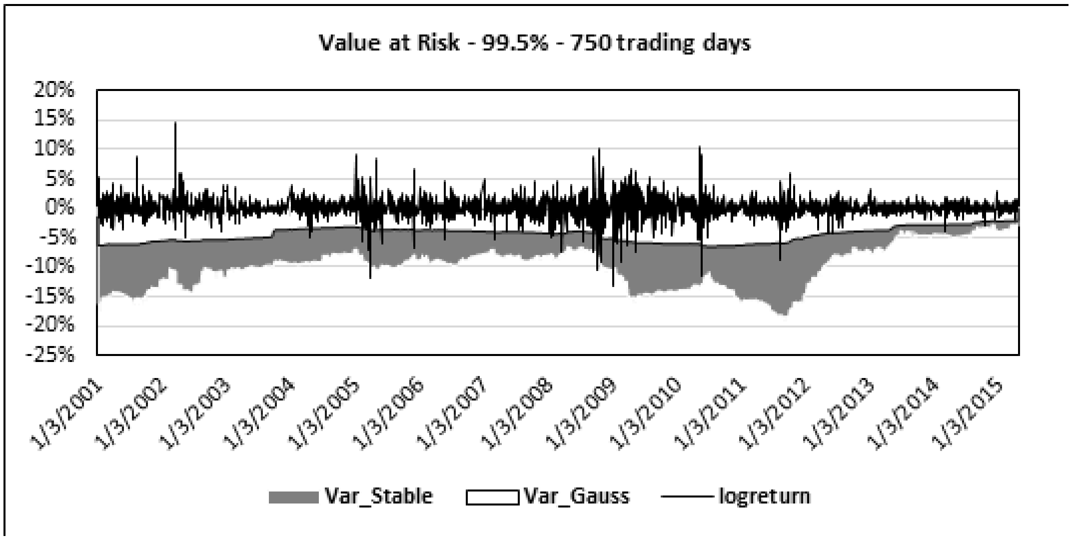

Figure 6. Value at Risk for 750 trading days rolling windows

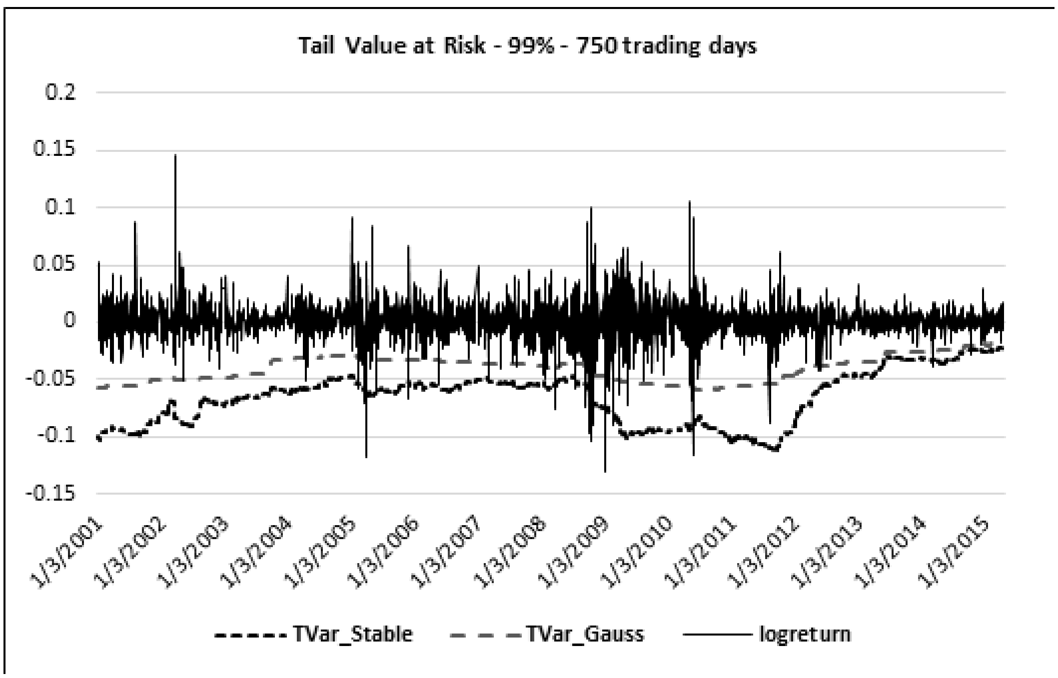

Figure 7. Tail Value at Risk for 750 trading days rolling windows

Tail Value at Risk was also estimated for both distributions, using a 99% probability and rolling windows of 500 and 750 trading days, results being summarized in Table 2.

Table 2. Summary of the Tail Value at Risk estimation at 99% probability

Distribution	Number of daily logreturns	Window length (trading days)	$Pr(\hat{TVaR}_{t+1} < r+1)$	$\#(\hat{TVaR}_{t+1} > r+1)$
Stable distribution	3552	500	99.27%	26
Gaussian distribution	3552	500	98.17%	65
Stable distribution	3552	750	99.49%	18
Gaussian distribution	3552	750	98.48%	54

In both cases, Tail Value at Risk estimated using stable distributions outperforms the estimations based on the Gaussian distribution.

4. Conclusion

Estimating the parameters of a stable distribution is a highly intensive task in terms of computational effort, yet the applications of the stable distributions justify this effort, especially in the stock markets, in order to have a proper calibration of risk indicators.

In this paper we proposed a very intuitive approach for estimating the market risk (Value at Risk and Tail Value at Risk) using the alpha-stable distributions. The parameters of a stable distribution using McCulloch method and Kogon-Williams method; further developments are required for implementing a procedure for estimating the parameters of a stable distribution using maximum likelihood method.

The two values of market risk, Value at Risk and Tail Value at Risk, estimated using stable distributions, outperforms the classical Gaussian distribution for daily logreturns of BET Index.

Moreover it might be more appropriate to use Fast Fourier Transform instead of Monte Carlo method in the context of internal models framework in order to get VaR or TVaR.

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