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ON COMPARISON OF THE ROOT SETS OF FUNCTIONS BELONGING TO THE WEIGHT CLASSES BERGMAN

***Abstract**: This article provides a refinement of the theorem of the Israeli mathematician C. Horowitz on the properties of Bergman classes, based on a weight change.*

***Key words**: Bergman classes, analytic function, zero sets of functions of Bergman classes.*

***Language**: English*

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Introduction

Root sets of functions of the Hardy class H^p are well studied. The well-known theorem on the root sets of functions of the Hardy class says: a sequence $\{z_k\}_{k=1}^{+\infty}$ a sequence from the unit circle is the root set of a function from the class H^p for some $0 < p < +\infty$ if and only if the Blaschke condition $\sum_{k=1}^{+\infty}(1-|z_k|) < +\infty$ is satisfied. However, a complete description of the roots of functions from the classes A^p_α, $0 < p < +\infty$.

In 1974, the famous Israeli mathematician C. Horowitz obtained an unexpected property of these classes in [11]. It turns out that, in contrast to a class H^p, in Bergman classes A^p_α these sets differ significantly for different p and α : if $0 < p < +\infty$ and $\alpha > \beta > -1$, then there is a function $f(z) \in A^p_\alpha$ a sequence from the unit circle, such that $f(z_k) = 0$ $k = 1, 2\ldots$, $f(z) \neq 0$, at the same time for an arbitrary function $g(z) \in A^p_\beta$ from $g(z_k) = 0$ $k = 1, 2\ldots$, follows that $g(z) \equiv 0, \forall z \in D$.

To formulate the main result, we introduce the following notation:

Let C be a complex plane;

$D = \{z = x + iy \in C : |z| \leq 1\}$ - unit circle on a complex plane;

$H(D)$ - the set of all analytic functions in D.

Let us introduce the Bergman weight class (1) and (2).

$$A^p_{\omega_\alpha} = \left\{ f \in \mathrm{H}(D) : \|f\|^p_{A^p_{\omega_\alpha}} = \int_0^1 \int_{-\pi}^{\pi} \left| f\left(re^{i\varphi}\right)\right|^p (1-r)^\alpha \, \omega\left(\ln\frac{1}{1-r}\right) r dr d\varphi < +\infty \right\} \quad (1)$$

and

$$A^{\frac{p}{n}}_{\omega_\alpha} = \left\{ f \in \mathrm{H}(D) \left| \int_D \left| f(z) \right|^{\frac{p}{n}} (1-|z|)^{\alpha} \omega\left(\ln \frac{1}{1-|z|} \right) dm_2(z) < +\infty \right. \right\} \quad (2),$$

where $\omega(x)$ - is an arbitrary positive function monotonically increasing on $(0,+\infty)$, $0 < p < +\infty, \alpha > -1$.

Theorem. *Let $\omega(x)$ - be an arbitrary positive monotonically increasing function on $(0,+\infty)$ satisfying the following conditions:*

$$\sup_{1<x<+\infty} \frac{\omega(2x)}{\omega(x)} < +\infty ;$$

$$\int_1^{+\infty} \frac{dx}{x \cdot \omega(x)} < +\infty .$$

Let also $\alpha > -1$, $0 < p < +\infty$

Then:

1. *For large enough $a \in \mathbf{N}$ the function*

$$f_a(z) = \sum_{k=m}^{+\infty} \frac{1}{(k \cdot \omega(k))^{\frac{1}{p}}} a^{\lambda k} z^{a^k} \in A^p_\alpha .$$

2. *If $h \in H(D)$, $b \in H(D)$, $M_\infty(b,r) = \max_{|z|\le r} |b(z)|$,moreover $M_\infty(b,r) = o\left((1-r)^{-\lambda}\right)$, $r \to 1-0$, then $\left(f^n + b\right) h \notin A^{\frac{p}{n}}_{\omega_\alpha}$, $\forall n \in \mathbf{N}$.*

Evidence.

Let us prove the first item of the theorem.

Consider (3).

$$f_a(z) = \sum_{k=1}^{+\infty} \frac{1}{(k\cdot\omega(k))^{\frac{1}{p}}} a^{\lambda k} z^{a^k} = \sum_{j=1}^{k-1} \frac{1}{(j\cdot\omega(j))^{\frac{1}{p}}} a^{\lambda j} z^{a^j} + \frac{1}{(k\cdot\omega(k))^{\frac{1}{p}}} a^{\lambda k} z^{a^k} + \sum_{j=k+1}^{+\infty} \frac{1}{(j\cdot\omega(j))^{\frac{1}{p}}} a^{\lambda j} z^{a^j} \quad (3)$$

We denote

$$S_k(z) = \sum_{j=1}^{k-1} \frac{1}{(j\cdot\omega(j))^{\frac{1}{p}}} a^{\lambda j} z^{a^j} ,$$

$$U_k(z) = \frac{1}{(k\cdot\omega(k))^{\frac{1}{p}}} a^{\lambda k} z^{a^k} ,$$

$$T_k(z) = \sum_{j=k+1}^{+\infty} \frac{1}{(j\cdot\omega(j))^{\frac{1}{p}}} a^{\lambda j} z^{a^j} .$$

Let's put $r_k = e^{-\frac{\lambda}{a^k}}$. Obviously, for $r_k \to 1$ at $k \to \infty$, then, considering that$1 - r_k \sim -\log r_k$,

We get

$$-\log r_k = -\log(1-(1-r_k)) \sim -(-(1-r_k)) = 1 - r_k,$$

from where

$$-\log r_k = -\log e^{-\frac{\lambda}{a^k}} = \frac{\lambda}{a^k} \text{ и } 1 - r_k \sim \frac{\lambda}{a^k}.$$

Let us show that there exist positive c_1 and c_2, such that

$$c_1 |U_k(z)| \le |f_a(z)| \le c_2 |U_k(z)|, \ |z| = r_k$$

We estimate $|S_k(z)|$ and $|T_k(z)|$ in terms of $|U_k(z)|$ for $|z| = r_k$.

$$|U_k(z)| = \left| \frac{1}{(k\cdot\omega(k))^{\frac{1}{p}}} a^{\lambda k} z^{a^k} \right| = \frac{1}{(k\cdot\omega(k))^{\frac{1}{p}}} a^{\lambda k} \left| e^{-\frac{\lambda}{a^k}} \right|^{a^k} = \frac{1}{(k\cdot\omega(k))^{\frac{1}{p}}} \frac{a^{\lambda k}}{e^{\lambda}} \quad (4)$$

$$|S_k(z)| = \left| \sum_{j=1}^{k-1} \frac{1}{(j\cdot\omega(j))^{\frac{1}{p}}} a^{\lambda j} z^{a^j} \right| \le \sum_{j=1}^{k-1} \frac{1}{(j\cdot\omega(j))^{\frac{1}{p}}} a^{\lambda j} \left| z^{a^j} \right| \le$$

$$\le \sum_{j=1}^{k-1} \frac{1}{(j\cdot\omega(j))^{\frac{1}{p}}} a^{\lambda j} \left(e^{-\frac{\lambda}{a^k}} \right)^{a^j} = \sum_{j=1}^{k-1} \frac{1}{(j\cdot\omega(j))^{\frac{1}{p}}} a^{\lambda j} e^{-\lambda a^{j-k}} \le \sum_{j=1}^{k-1} \frac{1}{(j\cdot\omega(j))^{\frac{1}{p}}} a^{\lambda j}$$

$$\left|S_k(z)\right| = \sum_{j=1}^{k-1} \frac{1}{\left(j\cdot\omega(j)\right)^{\frac{1}{p}}} a^{\lambda j} = \sum_{j=1}^{\left[\frac{k-1}{2}\right]} \frac{1}{\left(j\cdot\omega(j)\right)^{\frac{1}{p}}} a^{\lambda j} + \sum_{j=\left[\frac{k-1}{2}\right]+1}^{k-1} \frac{1}{\left(j\cdot\omega(j)\right)^{\frac{1}{p}}} a^{\lambda j} = A_k + B_k$$

Estimate A_k and B_k separately

$$B_k = \sum_{j=\left[\frac{k-1}{2}\right]+1}^{k-1} \frac{1}{\left(j\cdot\omega(j)\right)^{\frac{1}{p}}} a^{\lambda j} \le \frac{1}{\left(\frac{k-1}{2}\,\omega\left(\frac{k-1}{2}\right)\right)^{\frac{1}{p}}} \sum_{j=\left[\frac{k-1}{2}\right]+1}^{k-1} a^{\lambda j} =$$

$$= \frac{1}{\left(\frac{k-1}{2}\,\omega\left(\frac{k-1}{2}\right)\right)^{\frac{1}{p}}} \cdot \frac{a^{\lambda\left(\left[\frac{k-1}{2}\right]+1\right)} - a^{\lambda k}}{1-a^{\lambda}} \cdot \frac{\left|U_k(z)\right|}{\frac{1}{\left(k\cdot\omega(k)\right)^{\frac{1}{p}}}\,\frac{a^{\lambda k}}{e^{\lambda}}} =$$

$$= \frac{\left(k\cdot\omega(k)\right)^{\frac{1}{p}}}{\left(\frac{k-1}{2}\,\omega\left(\frac{k-1}{2}\right)\right)^{\frac{1}{p}}} \cdot \frac{a^{\lambda k}\left(1-a^{\left(-\frac{k}{2}+\frac{1}{2}\right)}\right)}{a^{\lambda}-1} \cdot \frac{\left|U_k(z)\right|}{\frac{a^{\lambda k}}{e^{\lambda}}}$$

Considering that

$$\lim_{k\to\infty} \frac{\left(k\cdot\omega(k)\right)^{\frac{1}{p}}}{\left(\frac{k-1}{2}\,\omega\left(\frac{k-1}{2}\right)\right)^{\frac{1}{p}}} = 2^{\frac{1}{p}},$$

denote by

$$\varepsilon_{B_k}(a) = \frac{2^{\frac{1}{p}}\cdot\left(1-a^{\lambda\left(-\frac{k}{2}+\frac{1}{2}\right)}\right)}{e^{-\lambda}\left(a^{\lambda}-1\right)},$$

then $B_k \le \varepsilon_{B_k}(a)\left|U_k(z)\right|$.

We will do the same with

$$A_k = \sum_{j=1}^{\left[\frac{k-1}{2}\right]} \frac{1}{\left(j\cdot\omega(j)\right)^{\frac{1}{p}}} a^{\lambda j}$$

$$A_k = \sum_{j=1}^{\left[\frac{k-1}{2}\right]} \frac{1}{\left(j\cdot\omega(j)\right)^{\frac{1}{p}}} a^{\lambda j} \le \sum_{j=1}^{\left[\frac{k-1}{2}\right]} a^{\lambda j} \le \sum_{j=1}^{\left[\frac{k-1}{2}\right]} a^{\lambda\left[\frac{k-1}{2}\right]} = a^{\lambda\left[\frac{k-1}{2}\right]} \sum_{j=1}^{\left[\frac{k-1}{2}\right]} 1 \le a^{\lambda\frac{k}{2}} a^{-\frac{\lambda}{2}}\left(\left[\frac{k-1}{2}\right]-1\right) =$$

$$= a^{\lambda\frac{k}{2}} a^{-\frac{\lambda}{2}}\left(\left[\frac{k-1}{2}\right]-1\right) \cdot \frac{\left|U_k(z)\right|}{\frac{1}{\left(k\cdot\omega(k)\right)^{\frac{1}{p}}}\,\frac{a^{\lambda k}}{e^{\lambda}}} = \frac{\left(k\cdot\omega(k)\right)^{\frac{1}{p}}\cdot e^{\lambda}\cdot a^{-\frac{\lambda}{2}}\left(\left[\frac{k-1}{2}\right]-1\right)\cdot\left|U_k(z)\right|}{a^{\frac{\lambda k}{2}}} \quad (5)$$

Consider seperately

$$\frac{\left(k\cdot\omega(k)\right)^{\frac{1}{p}}\cdot e^{\lambda}\cdot a^{-\frac{\lambda}{2}}\left(\left[\frac{k-1}{2}\right]-1\right)}{a^{\frac{\lambda k}{2}}} = \exp\left(\frac{1}{p}\ln k + \frac{1}{p}\cdot\ln\left(\omega(k)\right) + \ln\left(\frac{k-1}{2}\right) - \right.$$

$$-\frac{\lambda}{2}\ln a-\frac{\lambda k}{2}\ln a)=\exp(-k(-\frac{\frac{1}{p}\ln k}{k}-\frac{\frac{1}{p}\ln\omega(k)}{k}-\frac{\ln\left(\frac{k-1}{2}\right)}{k}+\frac{\lambda}{2k}\ln a+\frac{\lambda}{2}\ln a)\ ;$$

$$\lim_{k\to\infty}\left(\exp(-k(-\frac{\frac{1}{p}\ln k}{k}-\frac{\frac{1}{p}\ln\omega(k)}{k}-\frac{\ln\left(\frac{k-1}{2}\right)}{k}+\frac{\lambda}{2k}\ln a+\frac{\lambda}{2}\ln a)\right)=0\ .$$

Denoting through

$$\varepsilon_{A_k}(a)=\frac{\left(k\cdot\omega(k)\right)^{\frac{1}{p}}\cdot e^{\lambda}\cdot a^{-\frac{\lambda}{2}}\left(\left[\frac{k-1}{2}\right]-m\right)}{a^{\frac{\lambda k}{2}}},$$

get

$$A_k\le\varepsilon_{A_k}(a)\left|U_k(z)\right|.$$

In this way,

$$\left|S_k(z)\right|\le A_k+B_k\le\varepsilon_{A_k}(a)\cdot\left|U_k(z)\right|+\varepsilon_{B_k}(a)\cdot\left|U_k(z)\right|=\varepsilon_1(a)\cdot\left|U_k(z)\right|,\ \ |z|=r_k\ ,$$

moreover, when $a\to+\infty$, $\varepsilon_{A_k}(a)$ and $\varepsilon_{B_k}(a)\to 0$.

Let us now show that

$$\left|T_k(z)\right|\le\varepsilon_{A_k}(a)\cdot\left|U_k(z)\right|,\ \ |z|=r_k\ .$$

$$\left|T_k(z)\right|=\left|\sum_{j=k+1}^{+\infty}\frac{1}{\left(j\cdot\omega(j)\right)^{\frac{1}{p}}}a^{\lambda j}z^{a^j}\right|\le\sum_{j=k+1}^{+\infty}\frac{1}{\left(j\cdot\omega(j)\right)^{\frac{1}{p}}}a^{\lambda j}\left|z\right|^{a^j}=$$

$$=\sum_{j=k+1}^{+\infty}\frac{1}{\left(j\cdot\omega(j)\right)^{\frac{1}{p}}}a^{\lambda j}e^{-\lambda a^{j-k}}=\left|\begin{matrix}j-k-1=m\\ j=m+k+1\end{matrix}\right|\le\frac{1}{\left(k\cdot\omega(k)\right)^{\frac{1}{p}}}\sum_{p=0}^{+\infty}a^{\lambda(m+k+1)}e^{-\lambda a^{(m+1)}}=$$

$$=\frac{1}{\left(k\cdot\omega(k)\right)^{\frac{1}{p}}}a^{\lambda(k+1)}\sum_{p=0}^{+\infty}a^{\lambda m}e^{-\lambda a^{(m+1)}}=\frac{1}{\left(k\cdot\omega(k)\right)^{\frac{1}{p}}}a^{\lambda(k+1)}\sum_{p=0}^{+\infty}\left(\frac{a^m}{e^{a^{(m+1)}}}\right)^{\lambda}\le$$

$$\le\frac{1}{\left(k\cdot\omega(k)\right)^{\frac{1}{p}}}a^{\lambda(k+1)}\sum_{p=0}^{+\infty}\left(\frac{a^m}{e^{a(m+1)}}\right)^{\lambda}\le\frac{1}{\left(k\cdot\omega(k)\right)^{\frac{1}{p}}}a^{\lambda(k+1)}\frac{e^{-\lambda a}}{1-\frac{a^{\lambda}}{e^{\lambda a}}}=$$

$$=\frac{1}{\left(k\cdot\omega(k)\right)^{\frac{1}{p}}}a^{\lambda(k+1)}\cdot\frac{e^{\lambda}}{e^{\lambda}}\cdot\frac{a^{\lambda}/e^{\lambda a}}{1-\frac{a^{\lambda}}{e^{\lambda a}}}=\varepsilon_2(a)\cdot\left|U_k(z)\right|\qquad(6)$$

Where in (6), $\varepsilon_2(a)=\dfrac{a^{\lambda}/e^{\lambda a}}{e^{-\lambda}\left(1-\frac{a^{\lambda}}{e^{\lambda a}}\right)}$, $|z|=r_k$.

Then

$$\left|f_a(z)\right|\le\left|S_k(z)\right|+\left|U_k(z)\right|+\left|T_k(z)\right|\le$$

$$\le\varepsilon_1(a)\cdot\left|U_k(z)\right|+\left|U_k(z)\right|+\varepsilon_2(a)\cdot\left|U_k(z)\right|,$$

$$|z|=r_k$$

Since $\varepsilon_1(a)\underset{a\to+\infty}{\to}0$, $\varepsilon_2(a)\underset{a\to+\infty}{\to}0$ then for sufficiently large a the estimate will be correct

$$\left|f_a(z)\right| \le \frac{3}{2}\left|U_k(z)\right|, \ |z| = r_k .$$

Let us show, that the estimate will be correct

$$\frac{1}{2}\left|U_k(z)\right| < \left|f_a(z)\right|, \ |z| = r_k .$$

Since $f_a(z) = S_k(z) + U_k(z) + T_k(z)$, then

$$f_a(z) - S_k(z) - T_k(z) = U_k(z),$$

from where

$$\left|U_k(z)\right| \le \left|f_a(z)\right| + \left|S_k(z)\right| + \left|T_k(z)\right| \text{ and}$$

$$\left|f_a(z)\right| \ge \left|U_k(z)\right| - \left|S_k(z)\right| - \left|T_k(z)\right|.$$

Considering that

$$\left|S_k(z)\right| \le \varepsilon_1(a)\cdot\left|U_k(z)\right|$$

and

$$\left|T_k(z)\right| \le \varepsilon_2(a)\cdot\left|U_k(z)\right|,$$

one can choose a, such that

$$\left|f_a(z)\right| \ge \frac{1}{2}\left|U_k(z)\right|.$$

In this way,

$$\frac{1}{2}\left|U_k(z)\right| \le \left|f_a(z)\right| \le \frac{3}{2}\left|U_k(z)\right|, \ |z| = r_k, \quad k = 1,2,\ldots \qquad (*)$$

Let us now estimate the integral

$$\int_D \left|f_a(z)\right|^p (1-|z|)^\alpha \, dm_2(z) = \int_0^1 \int_{-\pi}^{\pi} \left|f_a\left(re^{i\varphi}\right)\right|^p (1-r)^\alpha \, r dr d\varphi =$$

$$= \sum_{k=1}^{+\infty} \int_{r_{k-1}}^{r_k} \int_{-\pi}^{\pi} \left|f_a\left(re^{i\varphi}\right)\right|^p (1-r)^\alpha \, r dr d\varphi \le \sum_{k=1}^{+\infty} \int_{r_{k-1}}^{r_k} (1-r)^\alpha \int_{-\pi}^{\pi} \left|f_a\left(r_k e^{i\varphi}\right)\right|^p d\varphi dr \le$$

$$\sum_{k=1}^{+\infty} \int_{-\pi}^{\pi} \left|f_a\left(r_k e^{i\varphi}\right)\right|^p d\varphi \int_{r_{k-1}}^{r_k} (1-r)^\alpha \, dr \le \sum_{k=1}^{+\infty} \int_{-\pi}^{\pi} c\cdot\left|U\left(r_k e^{i\varphi}\right)\right|^p d\varphi \int_{r_{k-1}}^{r_k} (1-r)^\alpha \, dr =$$

$$= \sum_{k=1}^{+\infty} \int_{r_{k-1}}^{r_k} c\cdot\frac{1}{k\cdot\omega(k)} a^{\lambda pk} \left(e^{-\frac{\lambda p}{a^k}}\right)^{a^k} \cdot 2\pi (1-r)^\alpha \, dr == \sum_{k=1}^{+\infty} c_0 \cdot \frac{1}{k\cdot\omega(k)} a^{\lambda pk} \int_{r_{k-1}}^{r_k} (1-r)^\alpha \, dr$$

Because

$$\lim_{r_k\to 1} \frac{\int_{r_{k-1}}^{r_k} (1-r)^\alpha \, dr}{(1-r_k)^{\alpha+1}} = \lim_{r_k\to 1} \frac{(1-r_k)^\alpha}{(\alpha+1)(1-r_k)^\alpha} = const$$

and

$$\int_{r_{k-1}}^{r_k} (1-r)^\alpha dr \sim (1-r_k)^{\alpha+1} \sim \lambda a^{-(1+\alpha)k},$$

then equality (7) is true.

$$\int_0^1 \int_{-\pi}^{\pi} \left|f\left(re^{i\varphi}\right)\right|^p (1-r)^\alpha \, r dr d\varphi \le \sum_{k=1}^{+\infty} C\cdot\frac{1}{k\cdot\omega(k)} a^{\lambda pk} a^{-(1+\alpha)k} = \sum_{k=1}^{+\infty} C\cdot\frac{1}{k\cdot\omega(k)} a^{k(\lambda p-(1+\alpha))} = C\cdot\sum_{k=1}^{+\infty} \frac{1}{k\cdot\omega(k)} \quad (7)$$

Series $\sum_{k=1}^{+\infty} \frac{1}{k\cdot\omega(k)}$ - converges by condition. In this way, $f_a(z) \in A_\alpha^p$.

Let us prove the second part of the theorem, that is, we will show that

$$\left(f_a^n + b\right)h \notin A_{\omega_\alpha}^{\frac{p}{n}}, \ \forall n \in \mathbf{N}, \ \forall h \in \mathrm{H}(D),$$

$$b \in \mathrm{H}(D).$$

We introduce the function

$$g(z) = \left(f^n(z) + b(z)\right)h(z).$$

Consider

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|g\left(re^{i\varphi}\right)\right|^{\frac{1}{n}} d\varphi = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|g\left(re^{i\varphi}\right)\right|^{\frac{p}{n}} d\varphi\right) == \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|\left(\left(f^n + b\right)h\right)\left(re^{i\varphi}\right)\right|^{\frac{p}{n}} d\varphi\right) =$$

$$=\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left|\left(f^n+b\right)\left(re^{i\varphi}\right)\right|^{\frac{p}{n}}d\varphi+\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\left|h\left(re^{i\varphi}\right)\right|^{\frac{p}{n}}d\varphi\right)=$$

$$=\exp\left(\frac{p}{2\pi n}\int_{-\pi}^{\pi}\ln\left|\left(f^n+b\right)\left(re^{i\varphi}\right)\right|d\varphi+\frac{p}{2\pi n}\int_{-\pi}^{\pi}\ln\left|h\left(re^{i\varphi}\right)\right|d\varphi\right)\geq$$

$$\geq\exp\left(\frac{p}{2\pi n}\int_{-\pi}^{\pi}\ln\left|\left(f^n+b\right)\left(re^{i\varphi}\right)\right|d\varphi+\frac{p}{2\pi n}\ln h(0)\right)==c\cdot\exp\left(\frac{p}{n}\int_{-\pi}^{\pi}\ln\left|\left(f^n+b\right)\left(re^{i\varphi}\right)\right|d\varphi\right)$$

In the last inequality, we used the well-known Jensen inequality:

$$\ln\frac{1}{b-a}\int_a^b e^{f(x)}dx\geq\frac{1}{b-a}\int_a^b f(x)dx\Leftrightarrow\frac{1}{b-a}\int_a^b e^{f(x)}dx\geq\exp\left(\frac{1}{b-a}\int_a^b f(x)dx\right)$$

We use the estimate *(*)* and the inequality

$$\left|f^n\left(r_ke^{i\varphi}\right)+b\left(r_ke^{i\varphi}\right)\right|\geq\left\|f^n\left(r_ke^{i\varphi}\right)\right|-\left|b\left(r_ke^{i\varphi}\right)\right\|>\left|f^n\left(r_ke^{i\varphi}\right)\right|,$$

get

$$c\cdot\exp\left(\frac{p}{n}\int_{-\pi}^{\pi}\ln c_0\left|f^n\left(re^{i\varphi}\right)\right|d\varphi\right)\geq c\cdot\exp\left(\frac{p}{n}\int_{-\pi}^{\pi}\ln c_0\left|U_k\left(re^{i\varphi}\right)\right|^n d\varphi\right)=c\cdot\exp\cdot\ln\left|U_k\left(re^{i\varphi}\right)\right|^{\frac{p}{n}\cdot n}=c\cdot\left|U_k\left(re^{i\varphi}\right)\right|^p$$

Thus, we have shown that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|g\left(re^{i\varphi}\right)\right|^{\frac{p}{n}}d\varphi\geq c\cdot\left|U_k\left(re^{i\varphi}\right)\right|^p.$$

Taking into account the obtained estimate for the module g, we have:

$$\int_D\left|g(z)\right|^{\frac{p}{n}}(1-|z|)^\alpha\,\omega\left(\ln\frac{1}{1-r}\right)dm_2(z)=$$

$$=\int_0^1\int_{-\pi}^{\pi}\left|f\left(re^{i\varphi}\right)\right|^{\frac{p}{n}}(1-r)^\alpha\,\omega\left(\ln\frac{1}{1-r}\right)rdrd\varphi=$$

$$=\sum_{k=1}^{+\infty}\int_{r_k}^{r_{k+1}}\int_{-\pi}^{\pi}\left|g\left(re^{i\varphi}\right)\right|^{\frac{p}{n}}(1-r)^\alpha\,\omega\left(\ln\frac{1}{1-r}\right)rdrd\varphi\geq$$

$$\geq\sum_{k=1}^{+\infty}\int_{r_k}^{r_{k+1}}\left|U_k\left(r_ke^{i\varphi}\right)\right|^p(1-r)^\alpha\,\omega\left(\ln\frac{1}{1-r}\right)dr\geq$$

$$\geq\sum_{k=1}^{+\infty}\int_{r_k}^{r_{k+1}}\frac{1}{k\cdot\omega(k)}\frac{a^{\lambda pk}}{e^{p\lambda}}(1-r)^\alpha\,\omega\left(\ln\frac{1}{1-r}\right)dr\geq$$

$$\geq\sum_{k=1}^{+\infty}\frac{1}{k\cdot\omega(k)}\cdot\frac{a^{\lambda pk}}{e^{p\lambda}}\int_{r_k}^{r_{k+1}}(1-r)^\alpha\,\omega\left(\ln\frac{1}{1-r}\right)dr.$$

Let's estimate separately

$$\int_{r_k}^{r_{k+1}}(1-r)^\alpha\,\omega\left(\ln\frac{1}{1-r}\right)dr.$$

$$\int_{r_k}^{r_{k+1}}(1-r)^\alpha\,\omega\left(\ln\frac{1}{1-r}\right)dr\geq\int_{r_k}^{r_{k+1}}(1-r_{k+1})^\alpha\,\omega\left(\ln\frac{1}{1-r_k}\right)dr==(1-r_{k+1})^\alpha\,\omega\left(\ln\frac{1}{1-r_k}\right)(r_{k+1}-r_k)$$

Because

$$r_{k+1}-r_k=e^{-\frac{\lambda}{a^{k+1}}}-e^{-\frac{\lambda}{a^k}}=e^{-\frac{\lambda}{a^{k+1}}}\left(1-e^{-\frac{\lambda(a-1)}{a^{k+1}}}\right)$$ and

$$1-e^{-\frac{\lambda(a-1)}{a^{k+1}}}\sim\frac{\lambda(a-1)}{a^{k+1}},$$ then

$$r_{k+1}-r_k\sim e^{\frac{-\lambda}{a^{k+1}}}\cdot\frac{\lambda(a-1)}{a^{k+1}},$$

from where

$$(1-r_k)^\alpha\omega\left(ln\frac{1}{1-r_k}\right)(r_{k+1}-r_k)\sim\left(\frac{\lambda}{a^k}\right)^\alpha\cdot\omega\left(ln\frac{a^k}{\lambda}\right)\cdot\left(\frac{\lambda(a-1)}{a^{k+1}}\right)$$

Then

$$\sum_{k=1}^{+\infty} \frac{1}{k \cdot \omega(k)} \frac{a^{\lambda pk}}{e^{p\lambda}} \int_{r_k}^{r_{k+1}} (1-r)^{\alpha} \omega\left(ln \frac{1}{1-r}\right) dr \sim$$

$$\sim \sum_{k=1}^{+\infty} \frac{1}{k \cdot \omega(k)} \frac{a^{\lambda pk}}{e^{p\lambda}} \cdot \frac{\lambda^{\alpha+1}}{a^{k(\alpha+1)}} \cdot \omega\left(ln \frac{a^k}{\lambda}\right) \geq \sum_{k=1}^{+\infty} \frac{\omega\left(ln \frac{a^k}{\lambda}\right)}{k \cdot \omega(k)} \geq \sum_{k=1}^{+\infty} \frac{\omega(k)}{k \cdot \omega(k)} = \sum_{k=1}^{+\infty} \frac{1}{k} = +\infty$$

Consequently, $\left(f_a^n + b \right) h \notin A_{\omega_\alpha}^{\frac{p}{n}}$, $\forall n \in \mathbf{N}$, $\forall h \in \mathrm{H}(D)$, $b \in \mathrm{H}(D)$. The theorem is proved.

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