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A Generalised Approach on Generation of Commutative Matrix

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Abstract We proposed a generalised approach on generating commutative matrix of any nonsingular matrix A (NxN) satisfying the condition $[A, B_i]$=0(i=1,2,3,4,.......∞)

Keywords: Inverse matrix, commutative matrix, continued fraction

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1 Introduction

Matrix analysis is a powerful tool in understanding many feature of mathematics having direct relavance with physical systems. However it is commonly known that matrix multiplication has two important relations[1]

$$Det(AB) = Det(A)Det(B) = Det(B)Det(A) \tag{1}$$

$$[A, B] \neq 0 \tag{2}$$

Interestingly it is possible to generate commutative matrices to a non-singular matrix[2,3]. In a recent paper any non-singular matrix A(NxN)(N=2,3) can possess commutative matrices B_L provided

$$B_L = \frac{1}{L + A} \tag{3}$$

Mathematically

$$[A, B_L] = 0 \tag{4}$$

By varying L one can generate infinite no of commutative matrices B_L. However in previous generation[3] ,the non-diagonal terms of entire B_L remain invariant with that of A. Hence it is felt that one can generate new matrices having different diagonal and non-diagonal elements. The procedure is as follows.

2 Commutative Matrices

2.1 Commutative Matrices: Series

Here we suggest a procedure[2-4] to generate infinite matrices B_L as Let B_L is

$$B_L = L + A + A^2 + A^3 + A^4 + ... = L + \sum_k A^k \tag{5}$$

where $k = 1, 2, 3, 4, ...\infty$ Then it is easy to show that

$$[A, B_L] = 0 \tag{6}$$

and

$$[B, B_L] = 0 \tag{7}$$

From matrix theory [1] that one can have

$$A, B_L, A^{-1}, B_L^{-1} \to \Psi \tag{8}$$

Let

$$A\Psi = \alpha\Psi \tag{9}$$

and

$$A^{-1}\Psi = \frac{1}{\alpha}\Psi \tag{10}$$

$$B_L\Psi = \beta\Psi \tag{11}$$

and

$$B_L^{-1}\Psi = \frac{1}{\beta}\Psi \tag{12}$$

Then

$$\alpha|\Psi >= A|\Psi > \tag{13}$$

multiplying both sides by B_L^{-1} we have

$$\alpha(B_L^{-1}|\Psi) >= \frac{\alpha}{\beta}|\Psi >= B_L^{-1}A|\Psi > \tag{14}$$

Similarly

$$B_L^{-1}|\Psi >= \frac{1}{\beta}|\Psi > \tag{15}$$

Multiplying A we have

$$AB_L^{-1}|\Psi >= \frac{1}{\beta}A|\Psi >= \frac{\alpha}{\beta}|\Psi > \tag{16}$$

Hence we have

$$AB^{-1} = B_L^{-1}A \to \frac{\alpha}{\beta} \tag{17}$$

In other words

$$[A, B_L^{-1}] = 0 \tag{18}$$

2.2 Commutative Matrices: Continued Fraction

Here we select B as

$$B_F = \frac{L}{L + \frac{A}{L+\frac{A}{L+\frac{A}{L+\frac{A}{L+A.......}}}}} \tag{19}$$

As in earlier ease it is easy to show that

$$[A, B_F^{-1}] = 0 \tag{20}$$

Hence we have two sets of commutative matrices B_L^{-1} and B_F^{-1}. corresponding to A. Below we consider simple matrices and find out the form of B_L and B_F as follows.

3 Infinite Generation of Commutative Matrices (B_L)

3.1 Infinite Generation of Commutative Matrices (B_L): Case Study (2x2)

Consider a simple (2x2) matrix A as [1-4]

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \tag{21}$$

(i) $B_1 = L + A$ as

$$B_1 = L + A = \begin{bmatrix} 2+L & 1 \\ 2 & 3+L \end{bmatrix} \tag{22}$$

Let us consider different values of L as follows **L=1** In this case we get the known matrix [1] i.e

$$B_1 = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \tag{23}$$

and

$$B_1^{-1} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix} \tag{24}$$

Then it easy to show that

$$AB_1^{-1} = B^{-1}A = \begin{bmatrix} 0.6 & 0.1 \\ 0.2 & 0.7 \end{bmatrix} \tag{25}$$

(ii)$B_2 = L + A + A^2$**(L=1)**
In this case we get the known matrix [1] i.e

$$B_2 = \begin{bmatrix} 9 & 6 \\ 12 & 15 \end{bmatrix} \tag{26}$$

and

$$B_2^{-1} = \begin{bmatrix} \frac{5}{21} & \frac{-2}{21} \\ \frac{-4}{21} & \frac{1}{7} \end{bmatrix} \tag{27}$$

Then it easy to show that

$$AB_2^{-1} = B_2^{-1}A = \begin{bmatrix} \frac{2}{7} & \frac{-1}{21} \\ \frac{-2}{21} & \frac{5}{21} \end{bmatrix} \tag{28}$$

(iii)$B_3 = L + A + A^2 + A^3$**(L=1)**
In this case we get the known matrix [1] i.e

$$B_3 = \begin{bmatrix} 31 & 27 \\ 54 & 58 \end{bmatrix} \tag{29}$$

and

$$B_3^{-1} = \begin{bmatrix} \frac{29}{170} & \frac{-27}{340} \\ \frac{-27}{170} & \frac{31}{340} \end{bmatrix} \tag{30}$$

Then it easy to show that

$$AB_3^{-1} = B_3^{-1}A = \begin{bmatrix} \frac{31}{170} & \frac{-23}{340} \\ \frac{-23}{170} & \frac{39}{340} \end{bmatrix} \tag{31}$$

(iv)$B_4 = L + A + A^2 + A^3 + A^4$**(L=1)**
In this case we get the known matrix [1] i.e

$$B_4 = \begin{bmatrix} 117 & 112 \\ 224 & 229 \end{bmatrix} \tag{32}$$

and

$$B_4^{-1} = \begin{bmatrix} \frac{229}{1705} & \frac{-112}{1705} \\ \frac{-224}{1705} & \frac{117}{1705} \end{bmatrix} \tag{33}$$

Then it easy to show that

$$AB_4^{-1} = B_4^{-1}A = \begin{bmatrix} \frac{234}{1705} & \frac{-107}{1705} \\ \frac{--214}{1705} & \frac{127}{1705} \end{bmatrix} \tag{34}$$

(v)$B_5 = L + A + A^2 + A^3 + A^4 + A^5$**(L=1)**
In this case we get the known matrix [1] i.e

$$B_5 = \begin{bmatrix} 459 & 453 \\ 906 & 912 \end{bmatrix} \tag{35}$$

and

$$B_5^{-1} = \begin{bmatrix} \frac{152}{1365} & \frac{-151}{2730} \\ \frac{-151}{1365} & \frac{51}{910} \end{bmatrix} \tag{36}$$

Then it easy to show that

$$AB_5^{-1} = B_5^{-1}A = \begin{bmatrix} \frac{51}{455} & \frac{-149}{2730} \\ \frac{-149}{1365} & \frac{157}{2730} \end{bmatrix} \tag{37}$$

3.2 Infinite Generation of Commutative Matrices (B_L): Case Study (3x3)

Here we just consider a simple example of (3x3) matrix[3] and generate suitable commutative counter part as follows. The explicit expression for A is [3]

$$A = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -2 & 1 \\ -1 & 3 & 3 \end{bmatrix} \tag{38}$$

Considering the value of L=1 we get $B_5 = L + A + A^2 + A^3 + A^4 + A^5$ as

$$B_5 = \begin{bmatrix} 23 & 177 & 344 \\ 341 & 73 & 227 \\ -77 & 491 & 858 \end{bmatrix} \tag{39}$$

$$B_5^{-1} = \begin{bmatrix} \frac{-227}{16398} & \frac{4259}{881614} & \frac{38}{8895} \\ \frac{-1363}{15504} & \frac{175}{13353} & \frac{883}{27785} \\ \frac{347}{7072} & \frac{-153}{21652} & \frac{-273}{14245} \end{bmatrix} \tag{40}$$

$$AB_5^{-1} = B_5^{-1}A = \begin{bmatrix} \frac{163}{4300} & \frac{-331}{30967} & \frac{-64}{6375} \\ \frac{518}{2825} & \frac{-158}{8411} & \frac{-346}{5135} \\ \frac{-362}{3525} & \frac{418}{31459} & \frac{191}{4641} \end{bmatrix} \tag{41}$$

4 Infinite generation of Commutative Matrices (B_F)

4.1 Infinite Generation of Commutative Matrices (B_F): Case Study (2x2)

Consider a simple (2x2) matrix A as [1-3]

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \tag{42}$$

Now consider the matrix **(i)** $B_1 = L + A$ as

$$B_1 = L + A = \begin{bmatrix} 2+L & 1 \\ 2 & 3+L \end{bmatrix} \tag{43}$$

Let us consider different values of L as follows **L=1** In this case we get the known matrix [1] i.e

$$B_1 = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \tag{44}$$

and

$$B_1^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \tag{45}$$

Then it easy to show that

$$AB_1^{-1} = B^{-1}A = \begin{bmatrix} 8 & 6 \\ 12 & 14 \end{bmatrix} \tag{46}$$

Now consider the matrix
(ii)$B_2 = L + \frac{A}{L+A}$**(L=1)**

$$B_2 = \begin{bmatrix} \frac{17}{27} & \frac{-1}{27} \\ \frac{-2}{27} & \frac{16}{27} \end{bmatrix} \tag{47}$$

$$B_2^{-1} = \begin{bmatrix} \frac{8}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{17}{10} \end{bmatrix} \tag{48}$$

Then it easy to show that

$$AB_2^{-1} = B_2^{-1}A = \begin{bmatrix} \frac{17}{5} & \frac{19}{10} \\ \frac{19}{5} & \frac{53}{10} \end{bmatrix} \tag{49}$$

Now consider the matrix
(iii)$B_3 = L + \frac{A}{L+\frac{A}{L+A}}$**(L=1)**

$$B_3 = \begin{bmatrix} \frac{73}{145} & \frac{-14}{145} \\ \frac{-28}{145} & \frac{59}{145} \end{bmatrix} \tag{50}$$

$$B_3^{-1} = \begin{bmatrix} \frac{59}{27} & \frac{14}{27} \\ \frac{28}{27} & \frac{73}{27} \end{bmatrix} \tag{51}$$

Then it easy to show that

$$AB_3^{-1} = B_3^{-1}A = \begin{bmatrix} \frac{146}{27} & \frac{101}{27} \\ \frac{202}{27} & \frac{247}{27} \end{bmatrix} \tag{52}$$

Now consider the matrix
(iv) $B_4 = L + \frac{A}{L+\frac{A}{L+\frac{A}{L+A}}}$**(L=1)** as

$$B_4 = \begin{bmatrix} \frac{147}{260} & \frac{-31}{520} \\ \frac{-31}{260} & \frac{263}{520} \end{bmatrix} \tag{53}$$

$$B_4^{-1} = \begin{bmatrix} \frac{243}{145} & \frac{31}{145} \\ \frac{62}{145} & \frac{294}{145} \end{bmatrix} \tag{54}$$

Then it easy to show that

$$AB_4^{-1} = B_4^{-1}A = \begin{bmatrix} \frac{588}{145} & \frac{356}{145} \\ \frac{712}{145} & \frac{944}{145} \end{bmatrix} \tag{55}$$

Now consider the matrix
$B_5 = L + \frac{A}{L+\frac{A}{L+\frac{A}{L+\frac{A}{L+A}}}}$**(L=1)** as

$$B_5 = \begin{bmatrix} \frac{1247}{2353} & \frac{-201}{2353} \\ \frac{-402}{2353} & \frac{521}{2353} \end{bmatrix} \tag{56}$$

$$B_5^{-1} = \begin{bmatrix} \frac{523}{260} & \frac{201}{520} \\ \frac{201}{260} & \frac{1247}{520} \end{bmatrix} \tag{57}$$

Then it easy to show that

$$AB_5^{-1} = B_5^{-1}A = \begin{bmatrix} \frac{1247}{260} & \frac{1649}{520} \\ \frac{1649}{260} & \frac{4143}{520} \end{bmatrix} \tag{58}$$

4.2 Infinite Generation of Commutative Matrices (B_F): Case Study (3x3)

Here we just consider a simple example of (3x3) matrix[3] and generate suitable commutative counter part as follows. The explicit expression for A is [3]

$$A = \begin{bmatrix} -2 & 1 & 2 \\ 3 & -2 & 1 \\ -1 & 3 & 3 \end{bmatrix} \tag{59}$$

Now consider the matrix $B_5 = L + \frac{A}{L+\frac{A}{L+\frac{A}{L+\frac{A}{L+A}}}}$(L=1) as

$$B_5 = \begin{bmatrix} \frac{932}{1207} & \frac{-217}{4866} & \frac{-781}{5230} \\ -\frac{-1339}{1049} & \frac{427}{398} & \frac{325}{1269} \\ \frac{323}{500} & \frac{502}{1341} & \frac{4140}{16661} \end{bmatrix} \tag{60}$$

$$B_5^{-1} = \begin{bmatrix} \frac{549}{440} & \frac{279}{1210} & \frac{2479}{4840} \\ \frac{731}{440} & \frac{1201}{1210} & \frac{-119}{4840} \\ \frac{-163}{220} & \frac{542}{605} & \frac{895}{337} \end{bmatrix} \tag{61}$$

$$AB_5^{-1} = B_5^{-1}A = \begin{bmatrix} \frac{-1019}{440} & \frac{2811}{1210} & \frac{2029}{467} \\ \frac{-141}{440} & \frac{-481}{1210} & \frac{1001}{236} \\ \frac{333}{220} & \frac{3288}{605} & \frac{2303}{312} \end{bmatrix} \tag{62}$$

5 Conclusion

This paper is the modified and more generalised version of the previous work of Rath[3] in generating commutative matrix of any non-sinular matrix. In fact one can generate infinite set of commutative matrix. Further one can check that matrices generated in this paper are different from that generated in previous approach [3] using the known matrix A. Here for the benefit of reader we have constructed considered simple cases (N×N)(N=2,3). However one can take any positive value of N.

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