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Chirps Construct on Sobolev Spaces and the Behavior of their $\| \quad \|_{L^2[-\varepsilon,\varepsilon]}$ and $\| \quad \|_{H^s[-\varepsilon,\varepsilon]}$ Norm

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Abstract: The main idea of this work is to characterize chirps construct on Sobolev spaces by studying the behavior of $\|f\|_{L^2[-\varepsilon,\varepsilon]}$ or $\|f\|_{H^s[-\varepsilon,\varepsilon]}$. We expect that the behavior is in order of $\varepsilon^{\alpha+(\beta+1)(|s|+\frac{1}{2})}$ when s tend to $-\infty$, and we believe that we have equivalence between this and the definition of chirps construct on Sobolev spaces. The formula of a chirp is given in the form $f(x) = |x|^{\alpha} g(|x|^{-\beta})$, where $\beta > 0$ and g is an indefinitely oscillating function on L^2 or H^s. Which means that g has for all m integer one primitive of the order m in the same space.

Keywords: Sobolev Spaces, Norm, Behavior of $\|f\|_{L^2[-\varepsilon,\varepsilon]}$.

1. Introduction

Chirps are known to exist in a variety of signals:

(1). First example gravitational waves [15]:

(a). By theoretical calculations based on the work of Einstein, Thibault Damour established the analytical form that should have a gravitational wave produced by the collapse on one another of two fast rotating neutrons stars.

(b). The gravitational waves have been measured and observed from September 14, 2015 at LIGO laboratory in California produced by the merger of two black holes in space and formalized in 2016 such phenomenon could never be observed.

(c). This first detection is a spectacular discovery : the gravitational waves were produced during the final fraction of a second of the merger of two black holes had been predicted but never observed. $[LIGO,\ Laser\ Interferometer\ Gravitational-wave\ Obervatory]$

(2). Ultra-sounds emitted by bats:

(a). The example considered here in is the cry of a bat. The signal is given by the formula: $F(x) = e^{\frac{-i}{x}} - 1$ which is a function of real variable.. We have then a discontinuity at the origin, this is obviously shown by the fact that $e^{\frac{-i}{x}} - 1 \sim \frac{-i}{x}$ at the infinity.

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(b). A second example is the emission of chirps to localize petroleum fields. It concerns signals with large band of frequency with short-lived. So that the detection possibility of a large range of objects by avoiding the interference thanks to short duration of those signals.

Let us take φ into $C_0^\infty(IR)$, and we suppose that $\varphi = 1$ in the neighborhood of 0. And we will give an estimation of $\left\|\varphi\left(\frac{x}{\varepsilon}\right)[f(x) - P(x-x_0)]\right\|_{H^s(IR)}$, where $H^s(IR)$ is a non homogeneous Sobolev space defined by usual conditions.

2. Main Results

Theorem 2.1. *The three following properties are equivalent:*

(1). f is chirps of type (α,β)

$$f(x) = x^\alpha g_\pm(x^{-\beta}) \tag{1}$$

$g_\pm$ are defined on $[T_0,+\infty)$. $g_\pm$ are indefinitely oscillating in H^s sense.

(2). $\forall \quad r \in IN$

$$\alpha_{j,k} = \int f(x)\psi_{j,k}(x)\,dx$$

where $\psi_{j,k}(x) = 2^j\psi\left(2^jx-k\right)$, Ψ has a compact support and $\psi \in C^r$ and $\int x^p\psi(x)\,dx = 0$ for $0 \le p \le r$

$$\sum_{j,k} |\alpha j,k|^2 2^{j(2\alpha+1)}(1+|k|)^{-2\alpha}\left[\frac{2^{j\beta}}{(1+|k|)^{\beta+1}}\right]^{1-2s} < \infty \tag{2}$$

for all real s such that $|s| < r$ and all couple (j,k) such that $2^{-j} + |k|\,2^{-j}$ is sufficiently small.

(3). By designing $H^s(IR)$ the non homogeneous Sobolev space we have:

$$\left\|x^m[f(x) - P(x-x_0)]\,\varphi\left(\frac{x}{\varepsilon}\right)\right\|_{H^s(IR)} \le C(m,s)\varepsilon^{m+\alpha+(\beta+1)(-s+\frac{1}{2})} \tag{3}$$

even then $m+\alpha+(\beta+1)\left(-s+\frac{1}{2}\right) \ge 0$, $m \in IN$ with $\varphi \in C_0^\infty$, $\varphi = 1$ in the neighborhood of 0.

Proof. (1) $\Rightarrow$ (2) is given in the Ph.D. of T.Elbouayachi. We will focus on the following problem.

(3) $\Rightarrow$ (2) We suppose that the support of $\psi_{j,k}$ is included in $[-\varepsilon,\varepsilon]$, we bigen by the case $m = 0$ and we suppose that

$$0 \le \alpha + (\beta+1)\left(-s+\frac{1}{2}\right) \le 1$$

Then we have

$$\alpha_{j,k} = \int f(x)\psi_{j,k}(x)\,dx$$
$$\Rightarrow \alpha_{j,k} = \int (f(x) - P(x-x_0))\psi_{j,k}(x)\,dx$$
$$|\alpha_{j,k}|^2 = \left|\int f(x)\psi_{j,k}(x)\,dx\right|^2$$
$$\|(f(x) - P(x-x_0)\|^2_{H^s(IR)} \ge \sum \sum_{S(j,k)\subset B_\varepsilon} |\alpha_{j,k}|^2 4^{js}$$

where $S(j,k)$ is the support of $\psi_{j,k}$. The condition $S(j,k) \subset B_\varepsilon$ is the consequence of $2^{-j} + \left|k2^{-j} - x_0\right| \le C_\varepsilon$ where C is a positive constant. So we have

$$\sum \sum_{S(j,k)\subset B_\varepsilon} |\alpha_{j,k}|^2 4^{js} \le C'\left(\varepsilon^{\boldsymbol{\alpha}+(\beta+1)(-s+\frac{1}{2})}\right)^2$$

And now as we have

$$\alpha + (\beta+1)\left(-s+\frac{1}{2}\right) \geq 0.$$

We take the minimum of ε with j,k fixed:

$$\sum_{j,k} |\alpha_{j,k}|^2 4^{js}\left(\frac{2^j}{1+|k|}\right)^{2\alpha+2(\beta+1)(-s+\frac{1}{2})} \leq C'$$

$$\Rightarrow \sum_{j,k} |\alpha_{j,k}|^2 2^{2j\alpha}(1+|k|)^{-2}\left[\frac{2^{j\beta}}{1+|k|^{\beta+1}}\right]^{(1-2s)} 2^j \leq C'$$

And this is the wanted estimation. Now if we have

$$-1 \leq \alpha + (\beta+1)\left(-s+\frac{1}{2}\right) \leq 0.$$

We compute the wavelets coefficients of

$$x\left[f(x) - P(x-x_0)\right]\varphi\left(\frac{x}{\varepsilon}\right)$$

With this hypothesis

$$2^{-j} + 2^{-j}|k| \leq \frac{\varepsilon}{C_0}$$

Thanks to the fact that ψ admits $r+1$ vanishing moments and $r>|s|$ we have:

$$\begin{aligned}
\alpha_{j,k} &= \langle f - P(x-x_0), \psi_{j,k}\rangle \\
&= \langle f, \psi_{j,k}\rangle \\
\int x\varphi\left(\frac{x}{\varepsilon}\right)\psi_{j,k}(x)\left[f(x) - P(x-x_0)\right]dx &= \int [f(x) - P(x-x_0)]\varphi\left(\frac{x}{\varepsilon}\right)\left(x-k2^{-j}\right)2^j\psi\left(x-k2^{-j}\right)dx \\
&+ k2^{-j}\int [f(x) - P(x-x_0)]\varphi\left(\frac{x}{\varepsilon}\right)\psi_{j,k}(x)\,dx
\end{aligned}$$

We know that $\widetilde{\psi} = x\psi(x)$ has the same qualitative properties than a wavelet. It results that

$$\begin{aligned}
\beta_{j,k} &= \int [f(x) - P(x-x_0)]\varphi\left(\frac{x}{\varepsilon}\right)\left(x-k2^{-j}\right)\psi_{j,k}(x)\,dx \\
&= \int [f(x) - P(x-x_0)]\varphi\left(\frac{x}{\varepsilon}\right)\left(x-k2^{-j}\right)2^j\psi\left(2^jx-k\right)dx \\
&= 2^{-j}\int [f(x) - P(x-x_0)]\varphi\left(\frac{x}{\varepsilon}\right)\left(2^jx-k\right)2^j\psi\left(2^jx-k\right)dx
\end{aligned}$$

We have then:

$$\int x\varphi\left(\frac{x}{\varepsilon}\right)\psi_{j,k}(x)\left[f(x) - P(x-x_0)\right]dx = 2^{-j}\int \left[f(x) - P(x-x_0)\right]\widetilde{\psi}_{j,k}(x)dx + k2^{-j}\alpha_{j,k}$$

So:

$$\int x\varphi\left(\frac{x}{\varepsilon}\right)f(x)\psi_{j,k}(x) = 2^{-j}\int f(x)\,\varphi\left(\frac{x}{\varepsilon}\right)\widetilde{\psi}_{j,k}(x)\,dx + k2^{-j}\int f(x)\,\varphi\left(\frac{x}{\varepsilon}\right)\psi_{j,k}(x)\,dx$$

We can use this new notation:

$$\gamma_{j,k} = 2^{-j}\beta_{j,k} + k2^{-j}\alpha_{j,k}$$

$$\Rightarrow \left|k2^{-j}\alpha_{j,k}\right| \leq \sqrt{\gamma_{j,k}^2 + 2^{-2j}\beta_{j,k}^2}$$

Then it's obvious to conclude:

$$(1+|k|)\left|2^{-j}\alpha_{j,k}\right| \le C_0\sqrt{\gamma_{j,k}^2+2^{-2j}\beta_{j,k}^2}$$

So

$$|\alpha_{j,k}|^2 \le C_0\left[\frac{2^j}{1+|k|}\right]^2\left[\gamma_{j,k}^2+2^{-2j}\beta_{j,k}^2\right]$$

The estimation on $\beta_{j,k}^2$ is given by the previous case $m=0$

$$\beta_{j,k}=2^{-j}\int f(x)\,\varphi\left(\frac{x}{\varepsilon}\right)\widetilde{\psi}_{j,k}(x)\,dx$$

So we have

$$\sum\sum_{S(j,k)\subset B_\varepsilon}|\beta_{j,k}|^2 4^{js} \le C'\left(\varepsilon^{\alpha+(\beta+1)(-s+\frac{1}{2})}\right)^2$$

Then

$$\beta_{j,k}2^{js} \le C_{j,k}^0\varepsilon^{\alpha+(\beta+1)(-s+\frac{1}{2})}$$

where

$$\sum_{j,k}\left|C_{j,k}^0\right|^2<\infty$$

and thanks to the fact that

$$2^{-j}+2^{-j}|k| \le \frac{\varepsilon}{C_0} \Rightarrow 2^{-j}\le C_\epsilon$$

We deduct then

$$\beta_{j,k}2^{js}2^j \le C_{j,k}^0\varepsilon^{1+\alpha+(\beta+1)(-s+\frac{1}{2})}$$

The estimation on $\gamma_{j,k}^2$ is given by:

$$\|x(f-P(x-x_0)\|_{H^s(IR)}^2 \ge \sum\sum_{S(j,k)\subset B_\varepsilon}|\gamma_{j,k}|^2 4^{js}$$

Where $S(j,k)$ is the support of $\psi_{(j,k)}$. The condition $S(j,k)\subset B_\varepsilon$ is the consequence of $2^{-j}+|k2^{-j}-x_0|\le C_\varepsilon$ where C is appositive constant. So we have

$$\sum\sum_{S(j,k)\subset B_\varepsilon}|\gamma_{j,k}|^2 4^{js} \le C'\left(\varepsilon^{1+\alpha+(\beta+1)(-s+\frac{1}{2})}\right)^2$$

which is equivalent to

$$\gamma_{j,k} \le C_{j,k}^1\varepsilon^{1+\alpha+(\beta+1)\left(-s+\frac{1}{2}\right)}$$

where

$$\sum_{j,k}\left|C_{j,k}^1\right|^2<\infty$$

Then it's obvious to conclude that:

$$(1+|k|)2^{-j}\left|\alpha_{j,k}\right|2^{js} \le C_{j,k}\epsilon^{1+\alpha+(\beta+1)(-s+\frac{1}{2})}$$

Where

$$\sum_{j,k}|C_{j,k}|^2<\infty$$

Optimizing by $2^{-j} + \left|k2^{-j}\right| \leq C\varepsilon$, we obtain

$$(1+|k|)2^{-j}\left|\alpha_{j,k}\right|2^{js} \leq C_{j,k}(1+|k|)2^{-j}\epsilon^{\alpha+(\beta+1)(-s+\frac{1}{2})}$$

And that is simplified in the wanted estimation. And by an analogue reasoning we can treat the case

$$-2 \leq \alpha + (\beta+1)\left(-s+\frac{1}{2}\right) \leq -1$$

and so on.

(1) ⇒ (3) Now let us give the definition of homogeneous space $\dot{H}^s(IR)$, we can definite it for $s>0$ by:

1. Fourier Transform

$$\|f\|^2_{\dot{H}^s(IR)} = \int_{IR} |\xi|^{2s}\left|\hat{f}(\xi)\right|^2 d\xi < \infty$$

2. Or by derivative

$$\|f\|_{\dot{H}^s(IR)} = \|D^s f\|_{L^2(IR)}$$

Now let us give the definition of $\dot{H}^s[-\varepsilon,\varepsilon]$: We define $\dot{H}^s[-\varepsilon,\varepsilon]$ by the following norm:

$$\|f\|^2_{\dot{H}^s[-\varepsilon,\varepsilon]} = \varepsilon^{-2s}\|f\|^2_{L^2[-\varepsilon,\varepsilon]} + \left\|\dot{f}\right\|^2_{\dot{H}^s(IR)}$$

where $\dot{f}$ is the prolongation of f on the whole real axe. And if $0<s<1$ then we have

$$\|f\|^2_{\dot{H}^s[-\varepsilon,\varepsilon]} \sim \varepsilon^{-2s}\|f\|^2_{L^2[-\varepsilon,\varepsilon]} + \iint\limits_{[-\varepsilon,\varepsilon]\times[-\varepsilon,\varepsilon]} \frac{|f(x)-f(y)|^2}{|x-y|^{2s+1}}dxdy$$

So for $0<s<1$ the first part of the right hand will become

$$\iint\limits_{[-\varepsilon,\varepsilon]\times[-\varepsilon,\varepsilon]} \frac{\left|x^\alpha g_\pm\left(|x|^{-\beta}\right) - y^\alpha g_\pm\left(|y|^{-\beta}\right)\right|^2}{|x-y|^{2s+1}}dxdy$$

$$\iint\limits_{\substack{-\varepsilon<x<\varepsilon\\ -\varepsilon<h+x<\varepsilon}} \frac{\left|(x+h)^\alpha g_\pm\left(|x+h|^{-\beta}\right) - x^\alpha g_\pm\left(|x|^{-\beta}\right)\right|^2}{|h|^{2s+1}}dxdh$$

For

$$0 < x \leq \varepsilon$$

$$0 < h \leq \delta|x|^{1+\beta}$$

We proceed the change of variable $X = x^{-\beta}$ and $Y = hx^{-\beta-1}$. Our integral become

$$\boldsymbol{I} = C(\beta)\iint\limits_{\substack{0<Y<\delta\\ X\geq\varepsilon^{-\beta}}} \frac{\left|\left(1+\frac{Y}{X}\right)^\alpha g_+\left(X\left(1+\frac{Y}{X}\right)^{-\beta}\right) - g_+(X)\right|^2}{|Y|^{2s+1}} X^{\frac{-2\alpha+(2s-1)(\beta+1)}{\beta}} dXdY$$

We pose again:

$$X\left(1+\frac{Y}{X}\right)^{-\beta} = X+Z$$

Hence we have:

$$Y = X\left[\left(1+\frac{Z}{X}\right)^{\frac{-1}{\beta}} - 1\right]$$

So

$$\boldsymbol{I} = C(\beta)\iint\limits_{\substack{0<Z<\tilde{\delta}\\ X\geq\varepsilon^{-\beta}}} \frac{\left|\left(1+\frac{Z}{X}\right)^{\frac{-\alpha}{\beta}} g_+\left(X+Z\right) - g_+\left(X\right)\right|^2}{|X|^{2s+1}\left[\left(1+\frac{Z}{X}\right)^{\frac{-1}{\beta}}-1\right]^{2s+1}}\left(1+\frac{Z}{X}\right)^{-1-\frac{1}{\beta}} X^{\frac{-2\alpha+(2s-1)(\beta+1)}{\beta}} dXdZ$$

and this integral is the same type than the following :

$$\begin{aligned}\boldsymbol{I} &= C(\beta)\iint\limits_{\substack{0<Z<\tilde{\delta}\\ X\geq\varepsilon^{-\beta}}} \frac{\left|g_+\left(X+Z\right) - g_+\left(X\right)\right|^2}{Z^{2s+1}} X^{\frac{-2\alpha+(2s-1)(\beta+1)}{\beta}} dXdZ \\ &\sim C(\beta)\varepsilon^{2\alpha-(2s-1)(\beta+1)}\iint\limits_{\substack{0<Z<\tilde{\delta}\\ X\geq\varepsilon^{-\beta}}} \frac{\left|g_+\left(X+Z\right) - g_+\left(X\right)\right|^2}{Z^{2s+1}} dXdZ\end{aligned}$$

For the second term we have

$$\varepsilon^{-2s}\|f\|^2_{L^2[-\varepsilon,\varepsilon]}$$

Which is equivalent to

$$\varepsilon^{-2s+\beta+1+2\alpha}\|g_+\|^2_{L^2[T,\infty)}$$

Another point of view is to specify the analysis of Sobolev spaces $H^s\left(IR\right)$ when $s-\frac{1}{2}$ is negative and the function to analyze has a support in $[-\varepsilon,\varepsilon]$. □

Lemma 2.2. *If $f\in H^s\left(IR\right)$ and f has a compact support then $f\left(x\right) = c_0\varphi\left(x\right)+c_1\varphi'(x)+\cdots+c_m\varphi^m\left(x\right)+r\left(x\right)$, where $\varphi\in C_0^\infty$ and $\varphi = 1$ in the neighborhood of 0. $r\left(x\right)\in\dot{H}^s\left(IR\right)$, $-m-1<s<m$. Indeed an appropriate choice of the coefficients: $c_0, c_1,\ldots,c_m$ we can have:*

$$0 = \int f\left(x\right)dx = \int xf\left(x\right)dx\cdots = \int x^m f\left(x\right)dx.$$

Lemma 2.3. *If f is a distribution of a compact support; and if $-m-1<s<-m$ and if $0=\int f\left(x\right)dx = \int xf\left(x\right)dx\cdots = \int x^m f\left(x\right)dx$ then*

$$f\in H^s\left(IR\right)\Leftrightarrow f\in\dot{H}^s\left(IR\right)$$

Remark 2.4. *For $s\geq 0$ the correction $c_0\varphi\left(x\right)+c_1\varphi'(x)+\cdots+c_m\varphi^m\left(x\right)$ is not necessary. We reason by recurrence on.*

- ***We begin by examining the case where*** $0<s<1$***:*** *Now let us give the definition of homogeneous space $\dot{H}^s\left(IR\right)$, we can definite it for $s>0$ by Fourier Transform:*

$$\|f\|^2_{\dot{H}^s(IR)} = \int_{IR}|\xi|^{2s}\left|\hat{f}(\xi)\right|^2 d\xi<\infty$$

and the definition of inhomogeneous space $H^s(IR)$ is given by:

$$\|f\|^2_{H^s(IR)} = \int_{IR}\left(1+|\xi|^s\right)^2\left|\hat{f}(\xi)\right|^2 d\xi<\infty$$

We will begin by remarking that for all $s > 0$ real we have $H^s(IR) \subset \dot{H}^s(IR)$. In our case we work on the interval $[-\varepsilon, \varepsilon]$, as a matter of fact we have:

$$\|f\|^2_{\dot{H}^s[-\varepsilon,\varepsilon]} = \varepsilon^{-2s}\|f\|^2_{L^2[-\varepsilon,\varepsilon]} + \left\|\dot{f}\right\|^2_{\dot{H}^s(IR)}$$
$$\|f\|^2_{H^s[-\varepsilon,\varepsilon]} = \|f\|^2_{L^2[-\varepsilon,\varepsilon]} + \|D^s f\|^2_{L^2[-\varepsilon,\varepsilon]}$$

So for > 0, if f is in $\dot{H}^s(IR) \Rightarrow f$ is in $H^s(IR)$. This is independent of the fact that f has a compact support. And in the other sense if f is in H^s then we have $D^s f$ belongs to $L^2[-\varepsilon, \varepsilon]$ and f belongs to $L^2[-\varepsilon, \varepsilon]$. So for $0 < s < 1$ we have:

$$\begin{aligned} \|f\|^2_{L^2[-\varepsilon,\varepsilon]} &= \int\limits_{[-\varepsilon,\varepsilon]\times[-\varepsilon,\varepsilon]} \frac{|f(x) - f(y)|^2}{|x-y|} dxdy \\ &\leq \varepsilon^{2s} \int\limits_{[-\varepsilon,\varepsilon]\times[-\varepsilon,\varepsilon]} \frac{|f(x) - f(y)|^2}{|x-y|^{2s+1}} dxdy \\ &= \varepsilon^{2s}\|D^s f\|^2_{L^2[-\varepsilon,\varepsilon]} \end{aligned}$$

- ***We will consider now the case $-1 < s < 0$:*** *So here it's about an integration and consequently we have not necessarily the fact that $D^s f$ is compactly supported. It's why we need that some moments of f are null. So if we have $\int f(x)\,dx = 0$ and if the support of f is included in $[-\varepsilon, \varepsilon]$ then $f(x) = \frac{d}{dx}g(x)$ where the support of g is included in $[-\varepsilon, \varepsilon]$. So if f belongs to H^s then $f = u + v'$ where u belongs to H^{s+1} and v belongs to H^{s+1}. Consequently $f = \frac{d}{dx}(g) = u + v'$ and $g = v + r$ where r is in H^{s+2} and v belongs to H^{s+1}. But we know that g is compactly supported and by using the previous case (By remarking that $0 < s+1 < 1$) we deduce that g belongs to the homogeneous Sobolev space H^{s+1}. And then f is in H^s.*

  Conversely if f is in $\dot{H}^s$ then $f = g'$ where g is in $\dot{H}^{s+1}$ and by the fact that f is compactly supported and that $\int f(x)\,dx = 0$ then the support of g is compact (because $\dot{H}^{s+1}$ is defined modulo constants). Finally g is inH^{s+1}.

Lemma 2.5. *We have*

$$\left|\int f(x)\varphi\left(\frac{x}{\varepsilon}\right)dx\right| = O\left(\varepsilon^{\alpha+N\beta+\frac{1}{2}}\right)$$

for all N integer.

Proof. Indeed

$$x^\alpha g\left(x^{-\beta}\right)\varphi\left(\frac{x}{\varepsilon}\right) = \frac{d}{dx}\left(x^{\alpha+\beta+1}g_1\left(x^{-\beta}\right)\varphi\left(\frac{x}{\varepsilon}\right)\right) \tag{4}$$
$$-x^{\alpha+\beta+1}g_1\left(x^{-\beta}\right)\frac{1}{\varepsilon}\varphi'\left(\frac{x}{\varepsilon}\right) - (\alpha+\beta+1)\,x^{\alpha+\beta}g_1\left(x^{-\beta}\right)\varphi\left(\frac{x}{\varepsilon}\right) \tag{5}$$

where g_1 is given by the formula $\frac{d}{dx}g_1 = -\frac{1}{\beta}g$. □

Now we can write the second term of the equation (4) like:

Remark 2.6.

$$-x^{\alpha+\beta+1}g_1\left(x^{-\beta}\right)\frac{1}{\varepsilon}\varphi'\left(\frac{x}{\varepsilon}\right) - (\alpha+\beta+1)\,x^{\alpha+\beta}g_1\left(x^{-\beta}\right)\varphi\left(\frac{x}{\varepsilon}\right) = -x^{\alpha+\beta}g_1\left(x^{-\beta}\right)\varphi_1\left(\frac{x}{\varepsilon}\right)$$

We have so

$$\int x^{\alpha} g\left(x^{-\beta}\right) \varphi\left(\frac{x}{\varepsilon}\right) dx = -\int x^{\alpha+\beta} g_1\left(x^{-\beta}\right) \varphi_1\left(\frac{x}{\varepsilon}\right) dx + x^{\alpha+\beta+1} g_1\left(x^{-\beta}\right) \varphi\left(\frac{x}{\varepsilon}\right)$$

And after N iterations and by using Cauchy's formula we obtain

$$\int x^{\alpha+N\beta} g_N\left(x^{-\beta}\right) \varphi_N\left(\frac{x}{\varepsilon}\right) dx = O\left(\varepsilon^{\alpha+N\beta+\frac{1}{2}}\right)$$

because g_N is in L^2.

Remark 2.7. *Of course for a good choice of N we assume that g_N is in L^2. Now the condition (??) in Theorem 2.1 becomes more precise and one can thus replace it by (6) and (7):*

$$\int x^m f(x) \varphi\left(\frac{x}{\varepsilon}\right) dx = O\left(\varepsilon^N\right) \tag{6}$$

for all m and all N, when ε tends to zero. Once corrected for having the right null moments (this correction is given by the formula $C_0\varphi\left(\frac{x}{\varepsilon}\right) + \cdots + C_q\varphi^q\left(\frac{x}{\varepsilon}\right)$ where $C_0 = O\left(\varepsilon^N\right), \ldots, C_q = O\left(\varepsilon^N\right)$. The function $x^m f(x)\varphi\left(\frac{x}{\varepsilon}\right)$ belongs to the homogeneous space $\dot{H}^s$ and her norm verifies

$$\left\| x^m f(x) \varphi\left(\frac{x}{\varepsilon}\right) \right\|_{H^s(IR)} \leq C(m,s)\varepsilon^{m+\alpha+(\beta+1)\left(-s+\frac{1}{2}\right)} \tag{7}$$

If we pose that $r_\varepsilon(x) = C_0\varphi\left(\frac{x}{\varepsilon}\right) + \cdots + C_q\varphi^q\left(\frac{x}{\varepsilon}\right)$, $q+s>0$ (this correction is not necessary for $s \geq 0$) and by using the fact that $\dot{H}^s$ are homogeneous we obtain:

$$\left\| x^{\alpha} g\left(x^{-\beta}\right) \varphi\left(\frac{x}{\varepsilon}\right) + r_\varepsilon(x) \right\|_{H^s(IR)} = \varepsilon^{\alpha-s+\frac{1}{2}} \left\| x^{\alpha} g\left(\varepsilon^{-\beta}x^{-\beta}\right) \varphi(x) + r_\varepsilon(\varepsilon x) \right\|_{H^s(IR)}$$

Taking into account the asymptotic behavior of the correction when ε tends to zero we have just to prove that

$$\left\| x^{\alpha} g\left(\varepsilon^{-\beta}x^{-\beta}\right) \varphi(x) \right\|_{H^s(IR)} \leq C\varepsilon^{\left(\frac{1}{2}-s\right)\beta}$$

if

$$\alpha + \left(\frac{1}{2} - s\right)(\beta+1) \geq 0.$$

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