Curran's Method for Approximating Arithmetic Average Option for Several Futures

Hagop Kechejian [a,*], Victor K. Ohanyan [b], Vardan G. Bardakhchyan [b]

[a] Freepoint Commodities, Stamford, CT, USA
[b] Yerevan State University, Yerevan, Armenia

Abstract

In this paper we calculate the price of the arithmetic average Asian option on several consecutive future contracts. The calculations are made using the method introduced by Curran, and the underlying
model for future prices proposed by Andersen. His model describes future prices by Stochastic Differential Equations with several coefficients, which are to be evaluated for each case, generally by model calibration. We use least squares principle to do that, taking the sum of squared differences of real values and the values suggested by model to minimum. Curran's method is based on the order of geometric and arithmetic means, and to calculate value of options takes expectation of conditional expectation of the considered derivative. Splitting the integral into two parts it evaluates explicitly one of them, and approximates the other. For our case new structure of the options and the underlying assets, requires review of the formulas. In the paper we derive the formulas for this case, and use them for calculating the value of Asian option. In the considered example derivative is based on 4 future contracts with 15, 30, 30, 15 days of engagement in Asian option.

Keywords: Curran's method, Arithmetic Asian option, future price.

1. Introduction

The paper continuous the investigation in (Kechejian, Ohanyan, 2012), and (Kechejian et al., 2015 and 2016b). We consider Asian options based on multiple futures contracts in the averaging period. We use the Markov model on futures prices introduced in (Andersen, 2008), as the underlying price process. We have introduced an explicit formula for geometric average option in (Kechejian et al., 2016b). Next we used Monte Carlo method with control variance for approximating Asian options with arithmetic mean (see Kechejian et al., 2015). The latter method is well covered in (Rubinstein, Kroese, 2017). Another widely used approach for approximating the arithmetic average options was introduced by Curran (see Curran, 1994), where conditioning of arithmetic average with geometric is employed. But first let's revise some formulas used in Andersen's paper (Andersen, 2008).

$$\ln F(t,T) = \ln F(0,T) + e^{a(T)}(z_1(t)e^{-k(T-t)+d(T)} + z_2(t)) -$$
$$-\frac{e^{2a(T)}}{4k}[e^{2d(T)-2kT}(e^{2kt}-1)(h_1^2+h_2^2) + 4h_1h_\infty e^{d(T)-kT}(e^{kt}-1) + 2h_\infty^2 tk], \qquad (1.1)$$

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[*] Corresponding author
E-mail addresses: hkechejian@hotmail.com (H. Kechejian), victoohanyan@ysu.am (V.K. Ohanyan),
vardanbardakchyan@gmail.com (V.G. Bardakhchyan)

where $d(T) = b(T) - a(T)$,
$$dz_1(t) = -kz_1(t)dt + h_1 dW_1(t) + h_2 dW_2(t)\,; \quad dz_2(t) = h_\infty dW_1(t);$$
with $z_1(0) = z_2(0) = 0$ and $\ln F(0,T)$ is given.

So for geometric average options we have to compute the following

$$\Phi(F) = E\left[\exp\left\{\frac{1}{T_n - t_n}\left(\int_0^{T_1} \ln F(u, T_1)\,du + \int_{T_1}^{T_2} \ln F(u, T_2)\,du + \cdots + \int_{T_{n-1}}^{T_n - t_n} \ln F(u, T_1)\,du\right)\right\} - K\right]^+$$

where t_n is the number of days of last futures contract used in the averaging period. Therefore, we obtain

$$\Phi(F) = E\left[\exp\left\{\frac{1}{T_n - t_{m_n,n}}\left(\sum_{i=1}^{m_1} \ln F(t_{i,1}, T_1) + \sum_{i=1}^{m_2} \ln F(t_{i,2}, T_2) \cdots + \sum_{i=1}^{n} \ln F(t_{i,n}, T_n)\right)\right\} - K\right]^+$$

(1.2)

We note that we do not use all the daily prices up to the expiration time T_i of futures prices in the averaging period. This means that it is not necessarily to consider the case $m_i = T_i - T_{i-1}$.

The reason for the latter in that futures prices may be erratic close to expiration dates, hence we omit these days from the averaging periods. Therefore, we have

$$z_1(t) = e^{-kt}\left(\int_0^t h_1 e^{ks}dW_1(s) + \int_0^t h_2 e^{ks}dW_2(s)\right)$$

and

$$z_2(t) = \int_0^t h_\infty\, dW_1(s).$$

We can note that $\ln F(t,T)$-s for $T > t$ and $t \neq 0$ are normally distributed, hence their sum is also normally distributed. Therefore, we get

$$\sum_{i=1}^{m_1} \ln F(t_{i,1}, T_1) + \sum_{s=2}^{n-1}\sum_{i=1}^{m_s} \ln F(t_{i,2}, T_s) + \sum_{i=1}^{m_n} \ln F(t_{i,n}, T_n) \sim N(\mu, \sigma^2),$$

where μ and σ^2 are respectively

$$\mu = \sum_{s=1}^{n} m_s \ln F(0, T_s)$$

$$- \sum_{s=1}^{n}\sum_{i=1}^{m_s} \frac{e^{2a(T_s)}}{4k}\left((h_1^2 + h_2^2)e^{2d(T_s)-2kT_s}\left(e^{2kt_{i,s}} - 1\right) + 4h_1 h_\infty e^{d(T_s)-kT_s}\left(e^{kt_{i,s}} - 1\right)\right.$$
$$\left. - 2h_\infty^2 kt_{i,s}\right)$$

and

$$\sigma^2 = \sum_{s=1}^{n}\sum_{i=1}^{m_s}\left(e^{2a(T_s)+2d(T_s)-2kT_s}(h_1^2+h_2^2)\frac{1}{2k}\left(e^{2kt_{i,s}}-1\right)+e^{2a(T_s)}h_\infty^2 t_{i,s}\right.$$

$$\left.+2h_1 h_\infty e^{2a(T_s)+d(T_s)-kT_s}\frac{1}{k}\left(e^{kt_{i,s}}-1\right)\right)$$

$$+2\sum_{s=1}^{n}\sum_{i=1}^{m_s}(m_s-i)\left(e^{2a(T_s)+2d(T_s)-2kT_s}(h_1^2+h_2^2)\frac{1}{2k}\left(e^{2kt_{i,s}}-1\right)+e^{2a(T_s)}h_\infty^2 t_{i,s}\right.$$

$$\left.+2h_1 h_\infty e^{2a(T_s)+d(T_s)-kT_s}\frac{1}{k}\left(e^{kt_{i,s}}-1\right)\right)$$

$$+2\sum_{l=1}^{n-1}\sum_{s=l+1}^{n}e^{a(T_l)+a(T_s)}m_s\sum_{i=1}^{m_l}\left((h_1^2+h_2^2)e^{d(T_l)+d(T_s)-k(T_l+T_s)}\frac{1}{2k}\left(e^{2kt_{i,l}}-1\right)+h_\infty^2 t_{i,l}\right.$$

$$\left.+h_1 h_\infty\left(e^{d(T_l)-kT_l}+e^{d(T_s)-kT_s}\right)\frac{1}{k}\left(e^{kt_{i,l}}-1\right)\right)$$

These formulas were presented in (Kechejian et al., 2015 and 2016b). Now let's turn to Curran's method.

2. Results
Curran's method

Curran's method (see Curran, 1994), is based on the fact that expectation of conditional expectation is always equal to expectation. That is, if we denote by A and G, the arithmetic average and geometric average of some sequence of random variables (in our case it will be the collection of future prices), then can say that $E(E((A-K)^+|G)) = E(A-K)^+$, where the right hand side represents the Asian options price with arithmetic mean at time 0. We have

$$E(A-K)^+ = \int_{-\infty}^{\infty}E((A-K)^+|x)g(x)\ dx, \tag{2.1}$$

where $g(x)$ is the density function of geometric average.

Next the inequality between arithmetic and geometric mean is used in the second integral. Using conditional expectation we can rewrite (2.1) in the following form:

$$E(A-K)^+ = \int_0^K E((A-K)^+|x)\ g(x)d + \int_K^\infty E((A-K)^+|x)\ g(x)dx$$

$$= \int_0^K E((A-K)^+|x)\ g(x)d + \int_K^\infty E\big((A|x)-K\big)g(x)dx \tag{2.2}$$

$$= \int_0^K E((A-K)^+|x)\ g(x)dx + \int_K^\infty E(A|x)\ g(x)dx - K\int_K^\infty g(x)\ dx$$

$$= \int_{-\infty}^{\ln K} E((A-K)^+|x)\ f(x)dx + \int_{\ln K}^\infty E(A|x)\ f(x)dx - K\int_{\ln K}^\infty f(x)\ dx.$$

In (2.2) $f(x)$ is the density of $\ln G$, that is the density of normal distribution. The last integral can be calculated explicitly. The second still needs some simplification, however it can also be found. The first integral needs to be approximated explicitly.

In our case $\ln G$ has normal distribution with mean and variance respectively

$$\mu(G) = \frac{\mu}{\sum_{s=1}^{n}m_s}, \quad \sigma^2(G) = \frac{\sigma^2}{(\sum_{s=1}^{n}m_s)^2}\ .$$

So the last integral will be

$$K\int_{\ln K}^\infty f(x)\ dx = K\left(1-\Phi\left(\frac{\ln K-\mu(G)}{\sigma(G)}\right)\right),$$

where $\Phi(x)$ is the distribution function of the standard normal distribution.

To calculate the second integral we use the fact that $(\ln F(t_{j,l},T_l),\ln G)$ has bivariate normal distribution, as $\ln F(t_{j,l},T_l)$ itself is normally distributed. So the conditional $(\ln F(t_{j,l},T_l)\,|\,\ln G = x)$ will be normally distributed with the following mean and variance:

$$N\left(\mu_{j,l} + \frac{\sigma_{j,l,G}}{\sigma^2(G)}(x - \mu(G)), \sigma_{j,l}^2 - \frac{\sigma_{j,l,G}^2}{\sigma^2(G)}\right),$$

where $\mu_{j,l}$ and $\sigma_{j,l}^2$ are the mean and variance of $\ln F(t_{j,l}, T_l)$ and $\sigma_{j,l,G} = cov\,(\ln F(t_{j,l}, T_l), \ln G)$. Moreover, we have

$$\mu_{j,l} = \ln F(0, T_l) - \frac{e^{2a(T_l)}}{4k}\left((h_1^2 + h_2^2)e^{2d(T_l)-2kT_l}\left(e^{2kt_{j,l}} - 1\right) + \right. \tag{2.3}$$
$$\left. + 4h_1 h_\infty e^{d(T_l)-kT_l}(e^{kt_{j,l}} - 1) - 2h_\infty^2 kt_{j,l}\right)$$

$$\sigma_{j,l}^2 = e^{2a(T_l)+2d(T_l)-2kT_l}(h_1^2 + h_2^2)\frac{1}{2k}\left(e^{2kt_{j,l}} - 1\right) + e^{2a(T_l)}h_\infty^2 t_{j,l} + $$
$$+ 2h_1 h_\infty e^{2a(T_l)+d(T_l)-kT_l}\frac{1}{k}\left(e^{kt_{j,l}} - 1\right) \tag{2.4}$$

$$\sigma_{j,l,G} = \frac{1}{\sum_{r=1}^n m_r}\left(2(m_l - j)\left(e^{2a(T_l)+2d(T_l)-2kT_l}(h_1^2 + h_2^2)\frac{e^{2kt_{j,l}} - 1}{2k} + e^{2a(T_l)}h_\infty^2 t_{j,l}\right.\right.$$
$$\left. + 2h_1 h_\infty e^{2a(T_l)+d(T_l)-kT_l}\frac{1}{k}(e^{kt_{j,l}} - 1)\right)$$
$$+ 2\sum_{i=1}^{j}\left(e^{2a(T_l)+2d(T_l)-2kT_l}(h_1^2 + h_2^2)\frac{1}{2k}(e^{2kt_{i,l}} - 1) + e^{2a(T_l)}h_\infty^2 t_{i,l} + 2h_1 h_\infty e^{2a(T_l)+d(T_l)-kT_l}\frac{1}{k}\right.$$
$$(e^{kt_{i,l}} - 1)\right) + 2\sum_{s=1}^{l-1} e^{a(T_s)+a(T_l)}\sum_{i=1}^{m_s}(e^{d(T_l)+d(T_s)-k(T_l+T_s)}$$
$$(h_1^2 + h_2^2)\frac{e^{2kt_{i,s}}-1}{2k} + h_\infty^2 t_{i,s} + h_1 h_\infty(e^{d(T_s)-kT_s} + e^{d(T_l)-T_l})\frac{1}{k} \tag{2.5}$$
$$(e^{kt_{i,s}} - 1) + 2\sum_{s=l+1}^{n} e^{a(T_s)+a(T_l)}\, m_s(e^{d(T_l)+d(T_s)-k(T_l+T_s)}(h_1^2 + h_2^2)$$
$$\frac{e^{2kt_{j,l}}-1}{2k} + h_\infty^2 t_{j,l} + h_1 h_\infty(e^{d(T_s)-kT_s} + e^{d(T_l)-kT_l})\frac{1}{k}(e^{kt_{j,l}} - 1))).$$

So the distribution of $\left(\ln F(t_{j,l}, T_l) \mid \ln G = x\right)$ is normal with already determined coefficient. Using log-normality of $F(t_{j,l}, T_l)$ we obtain

$$E(A|x) = \sum E\left(F(t_{j,l}, T_l)|e^x\right) = \sum \exp\{\mu_{j,l} + \frac{\sigma_{j,l,G}}{\sigma^2(G)}(x - \mu(G)) + \frac{1}{2}(\sigma_{j,l}^2 - \frac{\sigma_{j,l,G}^2}{\sigma^2(G)})\}.$$

So the first integral in (2.2) becomes

$$\int_{\ln K}^{\infty} E(A|x)f(x)dx =$$

$$\frac{1}{\sum_{s=1}^n m_s}\sum_{l=1}^n\sum_{j=1}^{m_l}\int_{\ln K}^{\infty}\exp\{\mu_{j,l} + \frac{\sigma_{j,l,G}}{\sigma^2(G)}(x - \mu(G)) + \frac{1}{2}(\sigma_{j,l}^2 - \frac{\sigma_{j,l,G}^2}{\sigma^2(G)})\}f(x)dx$$

$$= \frac{1}{\sum_{s=1}^n m_s}\sum_{l=1}^n\sum_{j=1}^{m_l}\exp\{\mu_{j,l} + \frac{\sigma_{j,l}^2}{2}\}\left(1 - \Phi\left(\frac{\ln K - \mu(G) - \sigma_{j,l,G}}{\sigma(G)}\right)\right),$$

where $\mu_{j,l}$, $\sigma_{j,l}^2$ and $\sigma_{j,l,G}$ are as in (2.3) – (2.5).

Evaluation of the first integral

We approximate the second integral in (2.2) exactly as it is done in (Curran, 1994). This technique needs the covariance matrix of the vector random variable consisting of only the $\ln F(t_{j,l}, T_l)$-s. So if we denote this covariance matrix with C, then it would be $\sum_{r=1}^n m_r \times \sum_{r=1}^n m_r$ matrix, with entries

$$C_{i,j} = \prod_{k=0}^{n-1} I_{i\in (m_k, m_{k+1}]}\prod_{s=0}^{n-1} I_{j\in (m_s, m_{s+1}]}\, cov\,(\ln F(t_{i-m_k,k+1}, T_{k+1}), \ln F(t_{j-m_s,s+1}, T_{s+1}))$$
$$\tag{3.1}$$

with $m_0 = 0$.

This is the precise formula of the covariance matrix. Further, denote the covariance by $\sigma_{i,j,l,s}$, then we would have the following formula

$$\sigma_{i,j,l,s} = e^{a(T_l)+a(T_s)}((h_1^2 + h_2^2)e^{d(T_l)+d(T_s) - k(T_l+T_s)}$$
$$\frac{1}{2k}(e^{2k(t_{i,l}\wedge t_{j,s})} - 1) + h_\infty^2 (t_{i,l} \wedge t_{j,s}) + h_1 h_\infty (e^{d(T_l-kT_l)} - e^{d(T_s)-kT_s})\frac{1}{k}(e^{k(t_{i,l}\wedge t_{j,s})} - 1))$$

(3.2)

and the conditional covariance matrix with respect to $\ln G$, have that and the random vector and $\ln G$ itself are normally distributed, will be of the following form:

$$\hat{C} = C - \frac{1}{\sigma^2(G)}\frac{1}{(\sum_{r=1}^n m_r)^2}C^2.$$

(3.3)

One can see, that this does not depend explicitly on the value of $\ln G$.

Remark 1. In all above formulas (3.1) − (3.3) we assume that our data is daily, with no days omitted. So when data is taken for unequal periods, a slight change should be done in the formulas, i.e. $\frac{1}{(\sum_{r=1}^n m_r)^2}$ formula will not be the same for the cases when data are not equidistant.

Having the conditional covariance matrix, we can get the first integral. The only case where we can get rid of the maximum sign is when $\ln G = \ln K$. So the idea is to approximate the integral near this value of geometric average.

Curran achieves this by approximations the sum of lognormals with lognormal. The technique assumes $(A - G|G)$ to be lognormal, near $\ln G = \ln K$. So let's first find the mean and variance at that point. Note that this can be done explicitly. The following formulas can be found in the (Curran, 1994) (just a little bit adjusted to our situation).

$$\hat{\mu}_A = \frac{1}{\sum_{r=1}^n m_r} \sum_{l=1}^n \sum_{i=1}^{m_l} e^{\widehat{\mu}_{i,l}+\frac{1}{2}\hat{C}_{m_{l-1}+i;\,m_{l-1}+i}}$$

with

$$\hat{\mu}_{i,l} = \mu_{j,l} + \frac{\sigma_{j,l,G}}{\sigma^2(G)}\left(\ln K - \mu(G)\right)$$

and

$$\widehat{\sigma_A^2} = \frac{1}{(\sum_{r=1}^n m_r)^2}\sum_{l=1}^n \sum_{s=1}^n \sum_{i=1}^{m_l} \sum_{j=1}^{m_s}$$

$$\left(\exp\{\widehat{\mu_{i,l}} + \widehat{\mu_{j,s}}\} \exp\left\{\frac{1}{2}(\hat{C}_{m_{l-1}+i;\,m_{l-1}+i} + \hat{C}_{m_{s-1}+j;\,m_{s-1}+j} + 2\hat{C}_{m_{l-1}+i;\,m_{s-1}+j})\right\}\right.$$

$$\left. - \exp\left\{\hat{\mu}_{i,l} + \frac{1}{2}\hat{C}_{m_{l-1}+i;\,m_{l-1}+i}\right\}\exp\left\{\hat{\mu}_{j,s} + \frac{1}{2}\hat{C}_{m_{s-1}+j;\,m_{s-1}+j}\right\}\right)$$

$(A - G|G)$ with G near K is then lognormally distributed with parameters $\gamma^2 = \ln(\frac{\widehat{\sigma_A^2}}{\widehat{\mu_A}-K} + 1)$ and $\beta = \ln(\widehat{\mu_A} - K) - \frac{\gamma^2}{2}$. As per Curran we then use the approximation:

$$\int_{-\infty}^K E((A - K)^+|x)\, g(x)dx \approx h \sum_{p=1}^m BS(ph)\, g(K - ph).$$

For some suitable m. $BS(ph)$, is Black-Scholes formula for strike price ph. g has lognormal distribution with parameters $\mu(G)$, $\sigma^2(G)$.

Numerical results

Using Anderson's model for futures, and doing preliminary model calibration with implied volatilities and the least squares method (see Andersen, 2008) we get the following values of coefficients in this model

h_1	-0.176014365
h_2	0.354977735
h_∞	-0.11036869
k	0.445339952

As a numerical example we consider an Asian option with 90 day averaging period. Four consecutive futures contracts are used in the averaging period with 15, 30, 30 and 15 days used from each contract respectively.

Valuation date for option was taken to be 31.12.2016. And for futures we obtain.

i	Initial Price of future $F(0, T_i)$	Maturity of future T_i	Days in averaging period
1	43.91	01.20.17	15
2	44.37	02.21.17	30
3	44.98	03.21.17	30
4	45.68	04.20.17	15

And we got the following results for Asian options prices calculated with Curran's method.

Strike	Curran
10	34.80801207
32	12.97993474
34	11.23442471
37	9.00816554
40	7.30159877
41.5	6.60293169
43	5.96866820
100	0.00001051

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