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On Essential (Complement) Submodules with Respect to an Arbitrary Submodule

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Abstract
In this paper we Proved other properties of essential and complement submodules to an arbitrary submodule of an R-module M .We prove that for a family $\{M_\alpha\}_{\alpha \in \Lambda}$ of modules . If T_α and N_α are submodules of M_α with $N_\alpha + T_\alpha \leq_{T_\alpha\text{-}e} M_\alpha$,$\forall$ α, then $\oplus_{\alpha\in\Lambda} (N_\alpha + T_\alpha) \leq_{\oplus_{\alpha\in\Lambda} T_\alpha - e} \oplus_{\alpha\in\Lambda} M_\alpha$. Also we show that for submodules T, A, B and C of a module M such that $T \leq A \leq C$. If B is T- c for A in M and C is T- c for B in M , then C is maximal T- essential extension of A in M .

Keywords: Essential submodules, T-essential submodules.

حول المقاسات الجزئية الجوهرية (المكملة) نسبة إلى مقاس جزئي عشوائي

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الخلاصة
في هذا البحث نحن نطور الخصائص للمقاسات الجزئية الجوهرية والمكملة بالنسبة إلى مقاس جزئي اختياري. نثبت ذلك لعائلة $\{M_\alpha\}_{\alpha\in\Lambda}$من الموديولات . إذا كانت T_α وN_αمقاسات جزئية من M_α و $N_\alpha + T_\alpha \leq_{T\alpha-e} M_\alpha$ لكل α ، فان $\oplus_{\alpha\in\Lambda} (N_\alpha + T_\alpha) \leq_{\oplus_{\alpha\in\Lambda} T\alpha - e} \oplus_{\alpha\in\Lambda} M_\alpha$. كما نبين انه بالنسبة للمقاسات الجزئية T, A, B و C من المقاس M بحيث إن $T \leq A \leq C$. إذا كانت B مكملة من النمط T ل A في M و C هي مكملة من النمط T ل B في M،فان C هو اكبر توسيع جوهري من النمط T ل A في M.

1. Introduction

In this paper, all rings are. Associative with. identity and all modules are unitary left R-modules. Recall that a submodule A of an R-module M is essential submodule of M{denoted by $A \leq_e M$},,if for every $B \leq M$, $A \cap B= 0$ implies that B = 0.

A submodule B of a module M is called complement for a submodule A of M if it is maximal. with respect to. the property that $A \cap B = 0$. More details about essential submodules and complement can be found in [1-4].In [5], the authors introduced the definition of T – essential (complement) submodules as follows:

Let $T \lneqq M$, a submodule A of M is called T – essential submodule of M {denoted by $A \leq_{T\text{-}e} M$}, provided that $A \nleq T$ and for each submodule B of M,$A\cap B \leq T$ implies that $B \leq T$.A submodule B of M is called a T –.complement for a submodule A in M if B is maximal with respect to the property that $A \cap B \leq T$.

In section 2,we develop the properties of T- essential submodules and we introduce the definition T – essential monomorphism. We show that, $A \leq_{T\text{-}e} M$ iff $\forall$ $Rx \leq M$, $Rx \nleq T$ implies that $A \cap Rx \nleq T$, see Proposition 2.5.Also we prove that, if every T –essential submodule A of M with $T \leq A$ is finitely

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generated, then $\frac{M}{T}$is Noetherian, seeTheorem2-15.In section3, we develop the properties of T-complement submodules. We prove that. For submodules A, B and C of a module M. If $A \leq_e M$ and C is a complement for B in M, then $A+C \leq_{C-e} M$, see proposition 3.9 .

2.The T – essential submodules

In ,this section, we proved basic properties of the essential submodules with respect to an arbitrary submodule T of an R-module Mand we introduced the definition ofT- essential monomorphism.

Definition2.1.[5]Let T be a proper. submodule of a module M. A submodule A of a module M is called .T-essential submodule , denoted by $A\leq_{T-e} M$, provided that $A\nleq T$ and for each submodule B of a module M, $A\cap B \leq T$ implies that $B\leq T$. Clearly that when T = 0,then $A \leq_{T-e} M$iff $A \leq_e M$.

The following .proposition gives the basic properties of T – essential submodules, see [5].

Proposition 2.2 [5]

Let T,A and B be submodules. of a module M. Then

1- If $A \leq_{T-e} M$, then$\frac{(A+T)}{T}\leq_e\frac{M}{T}$.
2- If $T \leq A$, then $A\leq_{T-e} M$ iff $\frac{A}{T}\leq_e\frac{M}{T}$.
3- $A\leq_{T-e} M$ iff $\forall\, x\in M-T$, $\exists r\in R$ such that $r x\in A-T$.
4- If A and B are T – essential submodules of M , then $A\cap B\leq_{T-e} M$.
5- Let $A\leq B \leq M$ such that $T \leq B$. Then $A\leq_{T-e} M$ iff $A\leq_{T-e} B$ and $B \leq_{T-e} M$.6- Let $h: M_1\rightarrow M_2$ be epimorphism. If $A\leq_{T-e}M_2$, then $f^{-1}(A)\leq_{f^{-1}(T)-e}M_1$, where $M_1$and M_2 are left R-modules .

Remark2.3. Let T and Abe submodules of a module M.

1- Let T = M . Then $A \cap B \leq T$ and $B \leq T$, $\forall B \leq M$.
2- Let $T \lneqq M$. Then $A \leq_{T-e} M$ iff $\forall B \leq M$, $A \cap B \leq T$ implies $B \leq T$.

Proof:1-clear.

2-clear.For the converse, we only need to show that $A \nleq T$. Assume $A\leq T$ and let B = M. Then $A\cap B = A\leq T$, but $B = M\nleq T$, which is a contradiction, thus $A \nleq T$.

Hence we see that the condition T is a proper submodule of M is not necessary. Thus, in this paper by a T – essential submodule we mean let T be a submodule of M (not necessary proper) and let A be a submodule of M. A is T – essential submodule of M if$\forall B \leq M$, $A \cap B \leq T$ implies that $B \leq T$.

Cleary that when T = M, then every submodule of M is T – essential in M .

Propositio2.4

Let T and A be submodules of a module M. Then $A\leq_{T-e} M$ iff for every submodule B of M,$B\nleq T$ implies that $A\cap B\nleq T$.

Proof: The proof is clear and hence is omitted .

Proposition 2.5

Let T and Abe submodules of a module M. Then $A\leq_{T-e} M$ iff $\forall\, Rx\leq M$,$Rx \nleq T$ implies that $A \cap Rx \nleq T$.

Proof: clear by proposition 2.4.For the converse, let $B\leq M$ such that $B\nleq T$. We want to show that $A \cap B \nleq T$. Let $x \in B- T$,then $Rx\nleq T$, By our assumption $A \cap Rx \nleq T$ and hence $A \cap B\nleq T$ .Thus $A \leq_{T-e} M$.

Proposition 2.6

Let T, A, A_1, B and B_1 be submodules. of a module M such that $A \leq_{T-e}A_1$,and $B \leq_{T-e}B_1$, then $A \cap B \leq_{T-e} A_1\cap B_1$.

Proof: Let $A \leq_{T-e}A_1$ and $B \leq_{T-e}B_1$. To show that $A \cap B \leq_{T-e}A_1\cap B_1$,let$x\in (A_1\cap B_1)-T$. Since $A \leq_{T-e}A_1$, then $\exists\, r\in R$ such that $r x \in A -T$.

But $r x\in B_1- T$ and$B \leq_{T-e}B_1$, then $\exists\, r_1\in R$ such that $r_1(r x) \in B -T$.

Hence $r_1rx \in (A \cap B)-T$. Thus$A \cap B \leq_{T-e}A_1 \cap B_1$.

Proposition 2.7

Let T,A be ideals of a ring R. If T is a prime ideal of Rand $A\nleq T$ then $A\leq_{T-e} R$.

Proof: Let $x \in R -T$ and $y \in A-T$. Clearly that $x . y \in A$. Claim that $y. x \notin T$. To show that assume $y. x \in T$. But T is a prim ideal , then either $y \in T$ or $x \in T$ which is a contradiction . Thus $y. x \in A-T$ and $A\leq_{T-e} R$.

Before we give next proposition ,we will recall the following definition.

Let M be 'an R– module. Recall that Z (M) ={x ∈ M ; ann (x) $\leq_e$ R}is called ;the singular submodule of M . If Z (M) = M then M is called singular module . If Z(M) = 0, then M is called a nonsingular module , [6] .

Proposition 2.8

Let T and Abe submodules of a module M . If A+ T $\leq_{T\text{-}e}$ M , then $\frac{M}{A+T}$is singular .

Proof: Since A + T$\leq_{T\text{-}e}$ M, then by proposition 2.2 – 2,$\frac{A+T}{T}\leq_e\frac{M}{T}$.By[6, p.32] $\frac{(M/T)}{((A+T)/T)}$is singular . By third isomorphic theorem $\frac{M/T}{(A+T)/T}\cong\frac{M}{A+T}$. Then $\frac{M}{A+T}$is singular .

We1 introduce, the following ,,definition

Definition 2.9. Let M_1 and M_2 be two modules and let T be a submodule of a module M_2 . A homomorphism h : $M_1 \rightarrow M_2$ is called T –. essential monomorphism if h (M_1) $\leq_{T\text{-}e} M_2$.

Proposition2.10

For submodules T and A of a module M. The, following statement are equivalent.
1- A$\leq_{T\text{-}e}$ M .
2-The inclusion map I_A:A→ M is a T –essential monomorphism;
3-for each module M_1 and f ∈ Homomorphism (M , M_1) such that Ker (f) ∩ A≤T ,then Ker (f) ≤ T.

Proof:1→ 2)Let B ≤ M such that I_A(A) ∩ B≤ T .To show B≤ T, since I_A(A) ∩B = A ∩ B≤ T ,and since A $\leq_{T\text{-}e}$ M .Then B ≤ T .

2→ 1) It's clear.

1→ 3) Let A $\leq_{T\text{-}e}$ M and f :M →M_1 be a homomorphism such that Ker (f) ∩ A≤ T. To show Ker (f) ≤ T, since A $\leq_{T\text{-}e}$ M. Then Ker (f) ≤ T.

3→1) To show A $\leq_{T\text{-}e}$ M, let B ≤ M such that A ∩ B≤ T, To show B≤ T.

Define ∏ : M → $\frac{M}{B}$be a natural epimorphism , ∏∈ Homomorphism (M , $\frac{M}{B}$). ThenKer ∏ ∩ A= A ∩ B≤ T, hence Ker ∏= B ≤ T.

Remark 2.11.The sum of T – essential submodules need not be T – essential. As shown in the following example

Example 2.12 Let R =Z , M=Z ⨁Z_2 and let T = {0} , A_1=A_2=2Z⨁($\bar{0}$) ≤ M, B_1= Z⨁($\bar{0}$) andB_2=Z(1,$\bar{1}$) ≤ M. One can easily show that $A_1 \leq_{\{0\}\text{-}e} B_1$ and$A_2 \leq_{\{0\}\text{-}e} B_2$. But$A_1+A_2$ = A_1=2Z ⨁($\bar{0}$) andB_1+B_2= Z⨁ ($\bar{0}$) + Z(1,$\bar{1}$)= M ,and (2Z⨁ ($\bar{0}$))∩(0⨁Z_2)=0 . So A_1+A_2 is not T - essential in M .

Theorem 2.13.Let { M_α , α∈Λ }be a family of modules and T_α and N_α be submodules of a module M_α,∀ α∈Λ .If N_α+ $T_\alpha \leq_{T\alpha\text{-}e} M_\alpha$∀ α∈Λ .Then $\bigoplus_{\alpha\in\Lambda}$ (N_α + T_α) $\leq_{\bigoplus_{\alpha\in\Lambda} T\alpha\text{-}e} \bigoplus_{\alpha\in\Lambda} M_\alpha$.

Proof:-Assume that N_α + $T_\alpha \leq_{T\alpha\text{-}e} M_\alpha$∀ α∈ Λ. Then by proposition 2.2 -2$\frac{N\alpha+T\alpha}{T\alpha} \leq_e \frac{M\alpha}{T\alpha}$, ∀α∈ Λ . By [2, corollary 5.1.7, p. 110]$\bigoplus_{\alpha\in\Lambda}(\frac{N\alpha+T\alpha}{T\alpha}) \leq_e \bigoplus_{\alpha\in\Lambda}(\frac{M\alpha}{T\alpha})$. Hence$\frac{\bigoplus_{\alpha\in\Lambda}(N\alpha+T\alpha)}{\bigoplus_{\alpha\in\Lambda}T\alpha}=\frac{[(\bigoplus_{\alpha\in\Lambda}N\alpha)+(\bigoplus_{\alpha\in\Lambda}T\alpha)]}{\bigoplus_{\alpha\in\Lambda}T\alpha} \leq_e \frac{\bigoplus_{\alpha\in\Lambda}M\alpha}{\bigoplus_{\alpha\in\Lambda}T\alpha}$. Therefore ,by proposition 2.2 – 2 ,$\bigoplus_{\alpha\in\Lambda}$ (N_α + T_α) $\leq_{\bigoplus_{\alpha\in\Lambda} T\alpha\text{-}e} \bigoplus_{\alpha\in\Lambda} M_\alpha$.

Corollary 2.14. Let { M_α, α∈Λ}be a family of modules and T_α, N_α be submodules of M_α with $T_\alpha \leq N_\alpha$, ∀α∈Λ. If $N_\alpha \leq_{T\alpha\text{-}e} M_\alpha$ ∀α∈ Λ, then $\bigoplus_{\alpha\in\Lambda} N_\alpha \leq_{\bigoplus_{\alpha\in\Lambda} T\alpha\text{-}e} \bigoplus_{\alpha\in\Lambda} M_\alpha$.

Theorem 2.15. Let T be a submodule of a module M . If every T –essential submodule A of M with T≤A is finitely generated , then $\frac{M}{T}$is Noetherian.

Proof:- Let$\frac{A}{T}\leq \frac{M}{T}$, to show $\frac{A}{T}$is finite generated. By Zorn's lemma$\frac{A}{T}$hasa complement say,$\frac{B}{T}$in$\frac{M}{T}$.By[6,proposition.1.3,p.17]then$\frac{A}{T}\oplus\frac{B}{T}\leq_e\frac{M}{T}$, and then $\frac{A+B}{T}\leq_e\frac{M}{T}$. ByProposition2.2- 2,then A+B $\leq_{T\text{-}e}$ M . Then A+ B is finite generated, and then$\frac{A}{T}\oplus\frac{B}{T}$is finite generated. Let $\frac{A}{T}\oplus\frac{B}{T}$= R($a_1$+$b_1$+T) +⋯+R($a_n$+ b_n+ T), a_i∈A, b_i∈B ∀ i=1,2,⋯, n .

Claim that $\frac{A}{T}$= R(a_1+T)+ ⋯ +R(a_n+ T).Let x+ T∈$\frac{A}{T}$. Then x+ T =r_1(a_1+b_1+ T) +⋯ +r_n (a_n+ b_n+ T), a_i∈A, b_i∈B∀ i=1,2,⋯, n.Therefore [x−(r_1a_1+⋯ +$r_n a_n$)]+ T =(r_1b_1+⋯+$r_n b_n$)+ T ∈($\frac{A}{T}$)∩ ($\frac{B}{T}$) = T . Then[x−(r_1a_1+⋯+ $r_n a_n$)]+ T= T, therefore x− (r_1a_1+⋯ +$r_n a_n$) ∈T. Hence x +T= (r_1a_1+⋯+$r_n a_n$)+T , hence x +T ∈ R(a_1+ T) +⋯ +R(a_n+ T), thus $\frac{A}{T}$is finite generated.

3. TheT–complement submodules

In, this section, we proved properties of the complement submodule with respect to an arbitrary submodule T of an R-module M

Definition3.1[5] Let T be a proper ,submodules 'of a module M and let Abe a submodule of M . A submodule B of M is called a T –.complement to A in M { denoted by B is a T – c to A in M },if B is maximal with, respect to the property that $A \cap B \leq T$.

Let M be a module and let T = 0 . For a submodules A and B of M . Clearly that B is a T – c to A in M iff B is a complement for A in M .

Theorem 3.2. Let T and Abe submodules of, a module M , then A has a T –.complement in M .

Proof: Let T and $A \leq M$. We want to show A has a T–complement. Let $\digamma = \{B \leq M \mid A \cap B \leq T\}$. $\digamma \neq \emptyset$, since $0 \in \digamma$, let$\{C\}_{\alpha \in \Lambda}$ be a chain in $\digamma$. To show that($\cup_{\alpha \in \Lambda} C_\alpha$)$\in \digamma$. Clearly $\cup_{\alpha \in \Lambda} C_\alpha \leq M$. Since $A \cap (\cup_{\alpha \in \Lambda} C_\alpha) = \cup_{\alpha \in \Lambda} (A \cap C_\alpha) \leq T$. Then $\cup_{\alpha \in \Lambda} C_\alpha \in \digamma$. By Zorn's lemma $\digamma$ has a. maximal element say H .Claim H is a T – c to A in M. To show that ,let $H \lneq L \leq M$ such that $A \cap L \leq T$, therefore $L \in \digamma$ which is contradiction . Thus H=L.

Remark 3.3 Let T and A be submodules of a module M. Then a T –complement of A in M need not be unique as the following example shows : Consider Z_{12} as Z-module . Let $A=\{\bar{0},\bar{3},\bar{6},\bar{9}\}$ and $T=\{\bar{0},\bar{2},\bar{4},\bar{6},\bar{8},\overline{10}\}$. Let $B=\{\bar{0},\bar{6}\}$ and $C=\{\bar{0},\bar{4},\bar{8}\}$, one can easily show that each of B , C is a T – complement to A in Z_{12}.

Proposition3.4.

Let T, A and B be submodules of a module M, if$\frac{B}{T}$ is a complement for$\frac{A}{T}$ in$\frac{M}{T}$, then B is a, T – c to A .in M. The converse is true if $T \leq A \cap B$.

Proof: Let$\frac{B}{T}$ is a complement for $\frac{A}{T}$in$\frac{M}{T}$, then$\frac{B}{T}$is maximal with, respect to the ,property $(\frac{A}{T}) \cap (\frac{B}{T})=0$. Hence B is. maximal with respect to the property $A \cap B= T$. To show that B is a T – c to A in M, let $B \leq N \leq M$ such that $A \cap N \leq T$. Now $A \cap N \leq T = A \cap B$. But $B \leq N$, therefore $A \cap B \leq A \cap N$. Thus $A \cap B = A \cap N$. Therefore $(\frac{A}{T}) \cap (\frac{B}{T}) = \frac{(A \cap B)}{T} = \frac{(A \cap N)}{T} = \frac{T}{T} = 0$ .But$\frac{B}{T}$is a complement for $\frac{A}{T}$in $\frac{M}{T}$, so $\frac{N}{T} = \frac{B}{T}$and hence N = B .Thus B is a T – c to A in M .For the converse **,** let B is a T – c to A in M and $T \leq A \cap B$.

Then $T = A \cap B$.$(\frac{A}{T}) \cap (\frac{B}{T}) = \frac{(A \cap B)}{T} = \frac{T}{T} = 0$. Now let $\frac{B}{T} \leq \frac{N}{T} \leq \frac{M}{T}$ such that $(\frac{A}{T}) \cap (\frac{N}{T})=0$. Then$\frac{(A \cap N)}{T}=0$, and hence $A \cap N= T$. But B is a T – c to A in M, therefore N=B. Thus $\frac{N}{T} = \frac{B}{T}$.

Corollary 3.5. Let T ,A and B be submodules of a module M such that $\frac{M}{T} = (\frac{A}{T}) \oplus (\frac{B}{T})$. Then B is a T – c to A in M.

Proof:-Let $\frac{M}{T} = (\frac{A}{T}) \oplus (\frac{B}{T})$.Then $(\frac{A}{T}) \cap (\frac{B}{T}) = 0$. Claim that$\frac{B}{T}$ is a complement for$\frac{A}{T}$in $\frac{M}{T}$. Let$\frac{B}{T} \leq \frac{N}{T} \leq \frac{M}{T}$such that$(\frac{A}{T}) \cap (\frac{N}{T})=0$.Since $\frac{M}{T} = (\frac{A}{T}) + (\frac{B}{T})$ and $\frac{B}{T} \leq \frac{N}{T}$. Then$\frac{M}{T} = \frac{A}{T} + \frac{N}{T}$.So$\frac{B}{T} = \frac{N}{T}$and hence$\frac{B}{T}$is a complement for$\frac{A}{T}$in $\frac{M}{T}$.By Proposition 3.4,Bis a T – c to A in M .

Proposition3.6.

Let T, A, B and C be submodules of an module M with $A \leq C$. If B is an T – c to A in M and C is a T – c to B in M. Then B is a T – c to C in M.

Proof: Let B is a T – c to A in M and C is a T – c to Bin M and $A \leq C$. Then $B \cap C \leq T$. To show that B is a T – c to C in M, let $B \leq L \leq M$ such that $L \cap C \leq T$. Since $(A \leq C)$, then $A \cap L \leq C \cap L \leq T$, implies that $A \cap L \leq T$. But B is maximal with respect to property $A \cap B \leq T$, therefore B= L .Thus B is a T – c to C in M.

Proposition3.7

Let T, A, B and C be submodules of a module M such that $T \leq A \leq C$. If B is a T – c to A in M and C is a T – c to B in M. Then C is a maximal T –essential extension of A in M.

Proof:-Let B is a T – c to A in M ,C is a T – c to B in M and $T \leq A \leq C$ .First, we prove that $A \leq_{T-e} C$, let $K \leq C$ such that $A \cap K \leq T$ .Claim that $A \cap (B + K) \leq T$. To show that, let a = b +k such that $a \in A$, $b \in B$,$k \in K$. Thus $b = a - k \in B \cap C \leq T \leq A$. Hence $a - b = k \in A \cap K \leq T$ and hence $a \in T$. But B is maximal with respect .to the property $A \cap B \leq T$, therefore B+K = B .Then $K \leq B$. Hence $K = K \cap C \leq B \cap C \leq T$. Thus $A \leq_{T-e} C$. Now to show C is maximal T –essential extension of A in M. Let $C \leq N \leq M$ with

$A \leq_{T-e} N$. Since $A\cap B\leq T$, then $(A \cap B)\cap N\leq T \cap N = T$, and hence $A \cap(B\cap N) \leq T$ .Since $A \leq_{T-e} N$, then $B\cap N \leq T$. But C is maximal with respect to the property $B \cap C \leq T$, therefore N= C.

Proposition3.8

Let A and B be submodules of a module M then B is a complement for A in M iff $A\oplus B\leq_{B-e} M$.

Proof:

.→) Let $A \leq M$ and B is an complement for A in M .Then by [6,prop.1.3,p.17]$A\oplus B \leq_e M$. But B is closed in M, by[6, prop.1.4,p.18] therefore$\frac{(A\oplus B)}{B}\leq_e\frac{M}{B}$,by[6,prop.1.4,p.18]. By proposition 2.2 – 2, $A\oplus B\leq_{B-e} M$.

←) Let $A \oplus B \leq_{B-e} M$, then $A\cap B =0$. By proposition 2.2 – 2, $\frac{(A\oplus B)}{B} \leq_e\frac{M}{B}$.Now, let $B\leq H$ and $A \cap H = 0$. Now,$\frac{H}{B}\leq \frac{M}{B}and\frac{(A\oplus B)}{B}\cap \frac{H}{B}=\frac{(A\oplus B)\cap H}{B} = \frac{(A\cap H)\oplus B}{B}$by modular law $\frac{0\oplus B}{B}$=0 .But $\frac{(A\oplus B)}{B} \leq_e\frac{M}{B}$, therefore $\frac{H}{B}= 0$. Hence B = H. Thus B is a complement for A in M.

Proposition3.9

Let A , B and C be submodules of a module M . If $A \leq_e M$ and C is a complement for B in M , then $A+ C\leq_{C-e} M$.

Proof: Let $A \leq_e M$ and C is a complement for B in M. Claim that $\frac{(A+C)}{C}\leq_e\frac{M}{C}$. First we prove that, let $\frac{N}{C}\leq\frac{M}{C}$ such that $\frac{(A+C)}{C}\cap \frac{N}{C}=0$.Then$\frac{(A+C)\cap N}{C}=0$. By modular law$\frac{(A\cap N)+C}{C}=0$. Implies that $(A \cap N) + C=C$ .Therefore $A \cap N\leq C$.

Hence $(A \cap N) \cap B\leq C\cap B=0$, then $A \cap (N \cap B) = 0$. Since $A \leq_e M$, then $N\cap B = 0$. But C is maximal with respect to, the property $B\cap C = 0$, so N = C . Thus $\frac{(A+C)}{C} \leq_e\frac{M}{C}$. By proposition 2.2- 2, $A+ C \leq_{C-e} M$.

Proposition3.10

Let T , A, B and C be submodules of a module M such that $T \leq A$. If $A\leq_{(T+C)-e} M$ and C is a T – c to B in M, then$\frac{(A+C)}{C} \leq_{(T+C)-e}\frac{M}{C}$.

Proof : Let $A \leq_{(T+C)-e} M$ and C is a T –c to B in M . To show$\frac{(A+C)}{C} \leq_{(T+C)-e}\frac{M}{C}$.Let $\frac{N}{C} \leq\frac{M}{C}$such that $\frac{(A+C)}{C} \cap \frac{N}{C}\leq\frac{(T+C)}{C}$.Since$\frac{(A+C)}{C} \cap \frac{N}{C}=\frac{(A+C)\cap N}{C}= \frac{(A\cap N)+C}{C}$, then $(A\cap N) + C \leq T +C$, and hence $A \cap N\leq T +C$. But $A \leq_{(T+C)-e} M$, therefore $N \leq T +C$ and hence$\frac{N}{C}\leq\frac{(T+C)}{C}$.

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